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# ON THE BRUALDI-LI MATRIX AND ITS PERRON EIGENSPACE*

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**Abstract.** The $n \times n$ Brualdi-Li matrix $\mathcal{B}_n$ has recently been shown to have maximal Perron value (spectral radius) $\rho$ among all tournament matrices of even order $n$, thus settling the conjecture by the same name. This renews our interest in estimating $\rho$ and motivates us to study the Perron eigenvector $x$ of $\mathcal{B}_n$, which is normalized to have 1-norm equal to one. It follows that $x$ minimizes the 2-norm among all Perron vectors of $n \times n$ tournament matrices. There are also interesting relations among the entries of $x$ and $\rho$, allowing us to rank the teams corresponding to a Brualdi-Li tournament according to the Kendall-Wei and Ramanajucharyula ranking schemes.

**Key words.** Tournament matrix, Perron value, Perron vector, Brualdi-Li matrix, Team ranking.

**AMS subject classifications.** 05C20, 15A18, 15B34, 15B48.

**1. Introduction.** A $n \times n$ tournament matrix is a $(0,1)$-matrix $T$ such that $T+T^t = J-I$, where $T^t$ denotes the transpose of $T$, $J$ the all-ones matrix and $I$ the identity matrix. A tournament matrix $T=[t_{ij}]$ can serve as a model for a round-robin tournament among $n$ teams by recording $t_{ij}=1$ (and $t_{ji}=0$) when team $i$ defeats team $j$. No ties are allowed.

Given a tournament matrix $T$, the question of ranking the teams arise. A survey of ranking methods can be found in [13]. As $T$ is an entrywise nonnegative matrix, the Perron-Frobenius theorem implies that its spectral radius $\rho$ is an eigenvalue, having a corresponding eigenvector $x$ with nonnegative entries. Naturally, the row sums of $T$ (i.e., total number of wins by each team), and also the eigenvector $x$ have been used to rank the teams. For example, the Kendall-Wei method (see [7, 15]) ranks the relative strengths of the teams corresponding to an irreducible tournament matrix in increasing order of the entries of $x$. As a consequence, there has been considerable attention paid to the eigenspace of a tournament matrix corresponding to its spectral radius.

When $n$ is odd and each team records $(n-1)/2$ wins, the corresponding tournament $T$ is called *regular*, $\rho(T)=(n-1)/2$ and all teams are ranked equally. A particularly interesting situation arises when $n$ is even and each team defeats the

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maximum number of competing teams possible. This means one half of the teams defeat $n/2$ teams and the other half defeat $(n-2)/2$ teams. We then refer to the corresponding tournament matrix as *almost regular*. An intriguing conjecture associated with almost regular tournament matrices and their spectral radii was posed by Brualdi and Li [2], stating that among all tournament matrices of a given even order $n$, the maximal spectral radius is attained by the Brualdi-Li matrix,

$$\mathcal{B}_n = \left[\begin{array}{c|c} L & L^t \\ \hline L^t + I & L \end{array}\right],$$

where $L$ denotes the $\frac{n}{2} \times \frac{n}{2}$ strictly lower triangular (tournament) matrix all of whose entries below the main diagonal are equal to one[1]. Notice that $\mathcal{B}_n$ is almost regular. This conjecture has now been confirmed by Drury [3] who shows that $\rho(\mathcal{B}_n) \geq \rho(T)$ for every $n \times n$ tournament $T$.

The settlement of the Brualdi-Li conjecture and the desire to compute $\rho(\mathcal{B}_n)$ provide us with renewed interest in the subject. Furthermore, the ordering of the spectral radii of tournament matrices can be cast as an ordering problem of the 2-norms of the corresponding (normalized) eigenvectors (see Theorem 4.1). This motivates us to study the eigenspace of the Brualdi-Li matrix corresponding to its spectral radius.

Due to the special structure of almost regular tournament matrices and of $\mathcal{B}_n$, there are interesting relations between $\rho(\mathcal{B}_n)$ and its corresponding eigenvectors. In this paper, we aim to develop and explore these relations. Specifically, the following results will be shown and their consequences considered in light of the validation of the Brualdi-Li conjecture.

THEOREM 1.1. *Let* $\lambda$ *and* $x = \left[\begin{array}{c} v \\ \hline w \end{array}\right]$, *where* $v = [v_j] \in \mathbb{R}^m$ *and* $w = [w_j] \in \mathbb{R}^m$, *be an eigenpair of the Brualdi-Li matrix* $\mathcal{B}_{2m}$, *and* $m \geq 2$. *Then*

$$\lambda = \frac{\mathbf{1}^t x - w_k}{v_k + w_k} \quad (k = 1, 2, \ldots, m;\ \mathbf{1} \textit{ is the all ones column vector}). \tag{1.1}$$

*If, in addition,* $\lambda = \rho$ *is the spectral radius of* $\mathcal{B}_{2m}$ *and* $x$ *is the corresponding positive eigenvector normalized so that its entries add up to one, then*

$$v_m < v_{m-1} < \cdots < v_1 < w_1 < w_2 < \cdots < w_m,$$

$$v_1 = \frac{(m-1)(\rho+1) - \rho^2}{\rho^2}, \quad w_1 = \frac{\rho + 1 - m}{\rho},$$

[1] The original Brualdi-Li matrix as defined in [2] has $L$ replaced by $L^t$ and is permutationally similar to $\mathcal{B}_n$.

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$$v_m = \frac{\rho + 1 - m}{\rho + 1}, \quad w_m = \frac{1 + m\rho - \rho^2}{(\rho + 1)^2},$$

$$v_{k+1} = \frac{v_k(\rho + 1)^2 - 1}{\rho^2}, \quad w_{k+1} = \frac{w_k(\rho + 1)^2 - 1}{\rho^2} \ (k = 1, 2, \ldots, m - 1).$$

This paper is organized as follows. Section 2 contains most definitions and global notation. In Section 3, an explicit formula for the inverse of the Brualdi-Li matrix is provided and selective results from the literature that are important to our developments are quoted in the interest of self-containment. In Section 4, we examine direct and recursive relationships among the spectral radius and the entries of a normalized corresponding eigenvector of the Brualdi-Li matrix. Section 5 contains ranking results for the Brualdi-Li tournament according to the Kendall-Wei and Ramanajucharyula ranking schemes.

**2. Definitions and notation.** Consider an $n \times n$ real matrix $A$ and denote its *spectrum* by $\sigma(A)$ and its *spectral radius* by $\rho(A) = \max\{|\lambda| : \lambda \in \sigma(A)\}$.

Let $\mathbf{1}$ denote an all-ones vector and $I$ the identity matrix (whose sizes are determined by the context or a subscript), and denote an all-ones matrix by $J = \mathbf{1}\mathbf{1}^t$.

We assume certain familiarity with the theory of (entrywise) nonnegative matrices, specifically the Perron-Frobenius Theorem and the notion of irreducibility; see, e.g., [6, Chapter 8]. Given an (entrywise) nonnegative square matrix $A$, by the Perron-Frobenius theorem, we know that $\rho(A) \in \sigma(A)$, having (entrywise) nonnegative right and left eigenvectors $x$ and $y$, respectively. We refer to $\rho(A)$ as the *Perron value* of $A$. When $x$ and $y$ are normalized so that their 1-norms equal one ($x^t\mathbf{1} = 1 = y^t\mathbf{1}$), we refer to them, respectively, as the (*right*) *Perron vector* and the *left Perron vector* of $A$.

We continue by reviewing some basic facts and terminology about tournament matrices. Recall that an $n \times n$ *tournament matrix* $T$ is a $(0, 1)$-matrix such that $T + T^t = J - I$. We refer to $T\mathbf{1}$ as the *score vector* of $T$. For odd $n$, $T$ is called *regular* if all its row sums equal $(n - 1)/2$, i.e., its score vector is $[(n - 1)/2]\mathbf{1}$. For even $n$, $T$ cannot have all row sums equal; in this case, $T$ is called *almost regular* if half its row sums equal $n/2$ and the rest equal $(n - 2)/2$. The spectrum of a tournament matrix is well-studied and most basic facts can be found e.g., in [1].

It is noted that each almost regular tournament matrix of order at least 4 is irreducible [5], and thus, its spectral radius is a simple eigenvalue. As a consequence, the results herein will be stated for Brualdi-Li matrices $\mathcal{B}_{2m}$, where $m \geq 2$.

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**3. The inverse of the Brualdi-Li matrix and other preliminaries.** The Brualdi-Li matrix $\mathcal{B}_{2m}$ is invertible for all $m \geq 2$ since, according to [4, Theorem 5.10], its determinant equals $1-m$. Next, we offer an explicit formula for the inverse of the Brualdi-Li matrix.

THEOREM 3.1. *Let $L_k$, $I_k$, and $J_k$ denote the $k \times k$ strictly lower tournament matrix, identity matrix, and all-ones matrix, respectively. Let also $m \geq 2$, $0_{m-1}$ denote the $(m-1) \times 1$ zero vector and consider the $m \times m$ anti-diagonal matrix $P = [p_{ij}]$ defined by*

$$p_{ij} = \begin{cases} 1 & \text{if } i+j = m+1, \\ 0 & \text{otherwise.} \end{cases}$$

*Also consider the matrices*

$$A = \left[\begin{array}{c|c} J_{m-1} - (m-1)I_{m-1} & \mathbf{1}_{m-1} \\ \hline 0_{m-1}^t & 1-m \end{array}\right], \quad C = \left[\begin{array}{c|c} 0_{m-1} & J_{m-1} - (m-1)I_{m-1} \\ \hline m-1 & 0_{m-1}^t \end{array}\right].$$

*Then*

$$\mathcal{B}_{2m}^{-1} = \frac{1}{m-1}\left[\begin{array}{c|c} A & C \\ \hline J_m - (m-1)I_m & PA^tP \end{array}\right].$$

*Proof.* First notice that

$$AL_m^t = A \cdot \left[\begin{array}{c|c} L_{m-1}^t & \mathbf{1}_{m-1} \\ \hline 0_{m-1}^t & 0 \end{array}\right] = \left[\begin{array}{c|c} J_{m-1}L_{m-1}^t + (1-m)L_{m-1}^t & 0_{m-1} \\ \hline 0_{m-1}^t & 0 \end{array}\right],$$

$$AL_m = A \cdot \left[\begin{array}{c|c} L_{m-1} & 0_{m-1} \\ \hline \mathbf{1}_{m-1}^t & 0 \end{array}\right] = \left[\begin{array}{c|c} J_{m-1}L_{m-1} + (1-m)L_{m-1} + J_{m-1} & 0_{m-1} \\ \hline (1-m)\mathbf{1}_{m-1}^t & 0 \end{array}\right],$$

$$CL_m^t = C \cdot \left[\begin{array}{c|c} 0 & \mathbf{1}_{m-1}^t \\ \hline 0_{m-1} & L_{m-1}^t \end{array}\right] = \left[\begin{array}{c|c} 0 & J_{m-1}L_{m-1}^t - (m-1)L_{m-1}^t \\ \hline 0_{m-1} & (m-1)\mathbf{1}_{m-1}^t \end{array}\right],$$

$$CL_m = C \cdot \left[\begin{array}{c|c} 0 & 0_{m-1}^t \\ \hline \mathbf{1}_{m-1} & L_{m-1} \end{array}\right] = \left[\begin{array}{c|c} 0 & J_{m-1}L_{m-1} + (1-m)L_{m-1} \\ \hline 0 & 0_{m-1}^t \end{array}\right].$$

Furthermore, $PA^tP = \left[\begin{array}{c|c} 1-m & \mathbf{1}_{m-1}^t \\ \hline 0_{m-1} & J_{m-1} - (m-1)I_{m-1} \end{array}\right]$, which gives

$$\begin{aligned} PA^tPL_m &= \left[\begin{array}{c|c} 1-m & \mathbf{1}_{m-1}^t \\ \hline 0_{m-1} & J_{m-1} - (m-1)I_{m-1} \end{array}\right] \cdot L_m \\ &= \left[\begin{array}{c|c} (m-1) & \mathbf{1}_{m-1}^tL_{m-1} \\ \hline 0_{m-1} & J_{m-1}L_{m-1} - (m-1)L_{m-1} \end{array}\right], \end{aligned}$$

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$$PA^tPL_m^t = \left[\begin{array}{c|c} 1-m & \mathbf{1}_{m-1}^t \\ \hline 0_{m-1} & J_{m-1}-(m-1)I_{m-1} \end{array}\right] \cdot L_m^t$$
$$= \left[\begin{array}{c|c} 0 & (1-m)\mathbf{1}_{m-1}^t + \mathbf{1}_{m-1}^t L_{m-1}^t \\ \hline 0_{m-1} & J_{m-1}L_{m-1}^t-(m-1)L_{m-1}^t \end{array}\right].$$

Denoting $v = \mathbf{1}_{m-1}L_{m-1}^tP_{m-1} = [m-2,\dots,1,0]$, we have

$$J_mL_m + (1-m)L_m = \left[\begin{array}{c|c} m-1 & v \\ \hline 0_{m-1} & J_{m-1}L_{m-1}+(1-m)L_{m-1} \end{array}\right],$$

$$J_mL_m^t + (1-m)L_m^t = \left[\begin{array}{c|c} J_{m-1}L_{m-1}^t+(1-m)L_{m-1}^t & 0_{m-1} \\ \hline \mathbf{1}_{m-1}^tL_{m-1}^t & m-1 \end{array}\right].$$

Using the above relationships, it is straightforward to verify that

$$\begin{aligned} AL_m^t + CL_m &= 0 \in \mathbb{C}^{m\times m}, \\ AL_m + CL_m^t + C &= (m-1)I_m, \\ J_mL_m + (1-m)L_m + PA^tPL_m^t + PA^tP &= 0 \in \mathbb{C}^{m\times m}, \\ J_mL_m^t + (1-m)L_m^t + PA^tPL_m &= (m-1)I_m. \end{aligned}$$

Therefore, if $F = \frac{1}{m-1}\left[\begin{array}{c|c} A & C \\ \hline J_m-(m-1)I_m & PA^tP \end{array}\right]$, we have

$$\begin{aligned} F\mathcal{B}_{2m} &= \frac{1}{m-1}\left[\begin{array}{c|c} A & C \\ \hline J_m-(m-1)I_m & PA^tP \end{array}\right] \cdot \left[\begin{array}{c|c} L_m & L_m^t \\ \hline L_m^t+I_m & L_m \end{array}\right] \\ &= \frac{1}{m-1}\left[\begin{array}{c|c} AL_m+CL_m^t+C & AL_m^t+CL_m \\ \hline J_mL_m+(1-m)L_m+PA^tPL_m^t+PA^tP & J_mL_m^t+(1-m)L_m^t+PA^tPL_m \end{array}\right] \\ &= \frac{1}{m-1}\left[\begin{array}{c|c} (m-1)I_m & 0 \\ \hline 0 & (m-1)I_m \end{array}\right] = I_{2m} \quad \square \end{aligned}$$

Example 3.2. According to Theorem 3.1,

$$\mathcal{B}_6 = \left[\begin{array}{ccc|ccc} 0&0&0&0&1&1 \\ 1&0&0&0&0&1 \\ 1&1&0&0&0&0 \\ \hline 1&1&1&0&0&0 \\ 0&1&1&1&0&0 \\ 0&0&1&1&1&0 \end{array}\right], \quad \mathcal{B}_6^{-1} = \frac{1}{2}\left[\begin{array}{rrr|rrr} -1&1&1&0&-1&1 \\ 1&-1&1&0&1&-1 \\ 0&0&-2&2&0&0 \\ \hline -1&1&1&-2&1&1 \\ 1&-1&1&0&-1&1 \\ 1&1&-1&0&1&-1 \end{array}\right].$$

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In our subsequent discussion, we shall also refer to the following facts about the Perron value and the Perron vectors of tournament matrices.

PROPOSITION 3.3. [11, Corollary 1.1] *Let* $v = [v_j] \in \mathbb{R}^m$ *and* $w = [w_j] \in \mathbb{R}^m$ $(m \geq 2)$ *so that* $x = \left[\begin{array}{c} v \\ \hline w \end{array}\right] \in \mathbb{R}^{2m}$ *is the Perron vector of an almost regular* $2m \times 2m$ *tournament matrix* $T$ *whose top* $m$ *row sums equal* $m-1$. *Then for all* $i, j = 1, 2, \ldots, m$, *we have* $v_i < w_j$.

PROPOSITION 3.4. [9, Theorem 1] *The Perron value* $\rho$ *of* $\mathcal{B}_{2m}$ $(m \geq 2)$ *satisfies*

$$2\rho^2 - 2\rho(m-1) - (m-1) = \left( \left( \frac{\rho+1}{\rho} \right)^{2m} + 1 \right)^{-1}.$$

THEOREM 3.5. [3, Theorem 1] *Let* $\mathcal{B}_n$ *be the* $n \times n$ *Brualdi-Li matrix. Then* $\rho(\mathcal{B}_n) \geq \rho(T)$ *for every* $n \times n$ *tournament* $T$*; in case of equality,* $T$ *is permutationally similar to* $\mathcal{B}_n$.

**4. The Perron vector of the Brualdi-Li matrix.** The main focus of this section is to investigate relationships that exist among the entries of the Perron vector of the Brualdi-Li matrix and its Perron value. Our motivation is partly provided by the following general comparison result, which connects the ordering of the Perron values of tournaments to the (right and left) Perron vectors.

We recall and emphasize to the reader that, by definition, Perron vectors are nonnegative and normalized to have sum of entries equal to one.

THEOREM 4.1. *Let* $T$ *and* $\widehat{T}$ *be two* $n \times n$ *tournament matrices with Perron vectors* $x$, $\widehat{x}$, *respectively. Let also* $y$, $\widehat{y}$ *be left Perron vectors of* $T$, $\widehat{T}$, *respectively. Then the following are equivalent:*

**(a)** $\rho(T) \geq \rho(\widehat{T})$, **(b)** $\|x\|_2 \leq \|\widehat{x}\|_2$, **(c)** $\|y\|_2 \leq \|\widehat{y}\|_2$.

*Furthermore, either in all of the above statements the inequalities are strict, or they all hold as equalities.*

*Proof.* Given a tournament matrix $T$ with Perron vector $x$ and left Perron vector $y$, we have $\mathbf{1}^t x = \mathbf{1}^t y = 1$ and

$$x^t(T + T^t)x = 2\rho(T)x^t x, \tag{4.1}$$

as well as

$$x^t(T + T^t)x = x^t(J - I)x = x^t\mathbf{1}\mathbf{1}^t x - x^t x = 1 - x^t x. \tag{4.2}$$

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By (4.1) and (4.2), it follows that

$$\rho(T) = \frac{1 - x^t x}{2x^t x} = \frac{1}{2\|x\|_2^2} - \frac{1}{2}. \tag{4.3}$$

The result now follows by noticing that the Perron value in (4.3) is a decreasing function of the norm of the Perron vector and applying this fact to $T$, $\widehat{T}$ and their transposes. □

In light of Theorems 3.5 and 4.1, the following holds for the 2-norms of Perron vectors.

COROLLARY 4.2. *Let $n = 2m$ $(m \geq 2)$ and let $x$ and $y$ denote the right and left Perron vectors of $\mathcal{B}_n$, respectively. Let also $T$ be any $n \times n$ tournament matrix with right and left Perron vectors $\widehat{x}$ and $\widehat{y}$, respectively. Then $\|x\|_2 \leq \|\widehat{x}\|_2$ and $\|y\|_2 \leq \|\widehat{y}\|_2$. Equalities hold if and only if $T$ is permutationally similar to $\mathcal{B}_n$.*

The following theorem establishes relationships among the eigenvalues and eigenvector entries of a certain type of tournament matrices.

THEOREM 4.3. *Let $\mathcal{A} = \left[\begin{array}{c|c} T & T^t \\ \hline T^t + I_m & T \end{array}\right]$, where $T$ is a $m \times m$ $(m \geq 2)$ tournament matrix. Let also $\lambda \in \mathbb{C}$ and $x \in \mathbb{C}^{2m}$ be an eigenpair of $\mathcal{A}$ and consider $v, w \in \mathbb{C}^m$ so that $x = \left[\begin{array}{c} v \\ \hline w \end{array}\right]$. Then for each $k = 1, 2, \ldots, m$,*

$$\lambda = \frac{\mathbf{1}_{2m}^t\, x - w_k}{w_k + v_k} \quad \text{and} \quad \lambda = m - \frac{\mathbf{1}_m^t w}{\mathbf{1}_{2m}^t x}.$$

*Proof.* First notice that

$$\left[\begin{array}{c} \lambda v \\ \hline \lambda w \end{array}\right] = \lambda x = \mathcal{A}x = \left[\begin{array}{c} Tv + T^t w \\ \hline T^t v + v + Tw \end{array}\right] = \left[\begin{array}{c} (J_m - I_m - T^t)v + T^t w \\ \hline T^t v + v + Tw \end{array}\right].$$

It follows that

$$\begin{aligned} J_m v + T^t w + Tw &= \lambda(w + v) \\ \Longrightarrow J_m v + (J_m - I_m)w &= \lambda(w + v) \\ \Longrightarrow \mathbf{1}_m \left(\mathbf{1}_m^t(v + w)\right) - w &= \lambda(w + v) \\ \Longrightarrow \mathbf{1}_m^t \mathbf{1}_m \left(\mathbf{1}_m^t(w + v)\right) - \mathbf{1}_m^t w &= \lambda\left(\mathbf{1}_m^t(w + v)\right) \\ \Longrightarrow m - \frac{\mathbf{1}_m^t w}{\mathbf{1}_m^t(w + v)} &= \lambda. \end{aligned}$$

Furthermore, the equation $\mathbf{1}_m \left(\mathbf{1}_m^t(v + w)\right) - w = \lambda(w + v)$ found above yields

$$\lambda(w_k + v_k) = \mathbf{1}_m \left(\mathbf{1}_m^t(v + w)\right) - w_k \Longrightarrow \lambda = \frac{\mathbf{1}_m \left(\mathbf{1}_m^t(v + w)\right) - w_k}{(w_k + v_k)}. \quad \square$$

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The following result follows readily from Theorem 4.3 applied to the Brualdi-Li matrix.

COROLLARY 4.4. *Let* $v = [v_j] \in \mathbb{R}^m$ *and* $w = [w_j] \in \mathbb{R}^m$ $(m \geq 2)$ *so that* $x = \left[\begin{array}{c} v \\ \hline w \end{array}\right] \in \mathbb{R}^{2m}$*, is the Perron vector of* $\mathcal{B}_{2m}$ *and* $\rho = \rho(\mathcal{B}_{2m})$*. Then*

$$\rho \;=\; m - \mathbf{1}^t w \;=\; (m-1) + \mathbf{1}^t v,$$

*and for each* $k = 1, 2, \ldots, m$*,*

$$w_k = \frac{1-\rho v_k}{\rho+1}, \quad \textit{and} \quad v_k - w_k = v_k - \frac{1-\rho v_k}{\rho+1} = \frac{v_k(2\rho+1)-1}{\rho+1}.$$

In the next theorem, we refine the ordering of the entries of the Perron vector of the Brualdi-Li matrix observed in Proposition 3.3.

THEOREM 4.5. *Let* $v = [v_j] \in \mathbb{R}^m$ *and* $w = [w_j] \in \mathbb{R}^m$ $(m \geq 2)$ *so that* $x = \left[\begin{array}{c} v \\ \hline w \end{array}\right] \in \mathbb{R}^{2m}$*, is the Perron vector of* $\mathcal{B}_{2m}$*. Then for all* $k = 1, 2, \ldots, m-1$*,*

$$v_{k+1} < v_k \quad \textit{and} \quad w_k < w_{k+1}.$$

*Proof.* Since $\mathcal{B}_{2m} x = \rho x$, we have that for $k = 1, 2, \ldots, m-1$,

$$\rho v_k = \sum_{j=1}^{k-1} v_j + \sum_{j=k+1}^{m} w_j$$

from which it follows that for all $k = 1, 2, \ldots, m-1$,

$$\rho v_k - \rho v_{k+1} = \left( \sum_{j=1}^{k-1} v_j + \sum_{j=k+1}^{m} w_j \right) - \left( \sum_{j=1}^{k} v_j + \sum_{j=k+2}^{m} w_j \right) = w_{k+1} - v_k.$$

In light of Proposition 3.3, we have that for $k = 1, 2, \ldots, m$,

$$v_{k+1} \;<\; v_k.$$

By Corollary 4.4, $w_k = \dfrac{1-\rho v_k}{\rho+1}$ for each $k = 1, \ldots, m$. Since $0 \leq v_{k+1} < v_k$ for $k = 1, 2, \ldots, m-1$, we have $w_k < w_{k+1}$ for $k = 1, 2, \ldots, m-1$. □

By Theorem 4.5 and Proposition 3.3 we now have the following complete ordering of the entries of the Perron vector, which is further discussed in Section 5.

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COROLLARY 4.6. *Let* $v = [v_j] \in \mathbb{R}^m$ *and* $w = [w_j] \in \mathbb{R}^m$ $(m \geq 2)$ *so that* $x = \begin{bmatrix} v \\ \hline w \end{bmatrix} \in \mathbb{R}^{2m}$, *is the Perron vector of* $\mathcal{B}_{2m}$. *Then*

$$v_m < v_{m-1} < \cdots < v_1 < w_1 < w_2 < \cdots < w_m.$$

THEOREM 4.7. *Let* $v = [v_j] \in \mathbb{R}^m$ *and* $w = [w_j] \in \mathbb{R}^m$ $(m \geq 2)$ *so that* $x = \begin{bmatrix} v \\ \hline w \end{bmatrix} \in \mathbb{R}^{2m}$ *is an eigenvector for* $\mathcal{B}_{2m}$ *corresponding to* $\lambda$. *Then*

$$\frac{w_{k+\ell} - v_{k+\ell}}{w_{k+j} - v_{k+j}} = \left(\frac{\lambda+1}{\lambda}\right)^{2(\ell-j)},$$

*where* $j, \ell = 0, 1, 2, \ldots, m$ *and* $k = 1, 2, \ldots, m$ *such that* $1 \leq k + \ell \leq m$, *and* $1 \leq k + j \leq m$.

*Proof.* By the definition of the Brualdi-Li matrix, it follows that for $k = 1, 2, \ldots, m-1$,

$$\lambda v_k = \sum_{j=1}^{k-1} v_j + \sum_{j=k+1}^{m} w_j \quad \text{and} \quad \lambda w_k = \sum_{j=k}^{m} v_j + \sum_{j=1}^{k-1} w_j.$$

Therefore, for all $\ell = 0, 1, 2, \ldots, m$ such that $k + \ell \leq m$,

$$\lambda(w_{k+\ell} - w_k) = \sum_{j=k+\ell}^{m} v_j + \sum_{j=1}^{k+\ell-1} w_j - \sum_{j=k}^{m} v_j - \sum_{j=1}^{k-1} w_j = \sum_{j=k}^{k+\ell-1} w_j - \sum_{j=k}^{k+\ell-1} v_j \tag{4.4}$$

and

$$\lambda(v_k - v_{k+\ell}) = \sum_{j=1}^{k-1} v_j + \sum_{j=k+1}^{m} w_j - \sum_{j=1}^{k+\ell-1} v_j - \sum_{j=k+\ell+1}^{m} w_j = \sum_{j=k+1}^{k+\ell} w_j - \sum_{j=k}^{k+\ell-1} v_j. \tag{4.5}$$

Subtracting (4.5) from (4.4) yields

$$\begin{aligned} \lambda(w_{k+\ell} - w_k + v_{k+\ell} - v_k) &= \sum_{j=k}^{k+\ell-1} w_j - \sum_{j=k+1}^{k+\ell} w_j = w_k - w_{k+\ell} \\ \implies \lambda(v_{k+\ell} - v_k) &= (\lambda+1)(w_k - w_{k+\ell}) \\ \implies \frac{w_k - w_{k+\ell}}{v_{k+\ell} - v_k} &= \frac{\lambda}{\lambda+1}\,; \end{aligned} \tag{4.6}$$

whereas, the sum of (4.4) and (4.5), yields

$$\lambda(w_{k+\ell} - w_k + v_k - v_{k+\ell}) = \sum_{j=k}^{k+\ell-1} w_j + \sum_{j=k+1}^{k+\ell} w_j - 2\sum_{j=k}^{k+\ell-1} v_j$$

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$$\Longrightarrow \frac{w_{k+\ell}-v_{k+\ell}}{w_k-v_k} = \frac{1}{\lambda(w_k-v_k)}\left(\sum_{j=k}^{k+\ell-1} w_j + \sum_{j=k+1}^{k+\ell} w_j - 2\sum_{j=k}^{k+\ell-1} v_j\right) + 1. \tag{4.7}$$

By the above relations, we obtain

$$\begin{aligned}
\frac{w_{k+1}-v_{k+1}}{w_k-v_k} &= \frac{1}{\lambda}+1+\frac{1}{\lambda}\left(\frac{w_{k+1}-v_k}{w_k-v_k}\right) \quad \text{(by (4.7) for } \ell=1)\\
&= \frac{1}{\lambda}+1+\frac{1}{\lambda}\left(\frac{w_{k+1}-v_k}{\lambda(w_{k+1}-w_k)}\right) \quad \text{(by (4.4) for } \ell=1)\\
&= \frac{1}{\lambda}+1+\frac{1}{\lambda^2}\left(\frac{w_{k+1}-v_{k+1}}{w_{k+1}-w_k} - \frac{v_k-v_{k+1}}{w_{k+1}-w_k}\right)\\
&= \frac{1}{\lambda}+1+\frac{1}{\lambda^2}\left(\frac{w_{k+1}-v_{k+1}}{w_{k+1}-w_k} - \frac{\lambda+1}{\lambda}\right) \quad \text{(by (4.6) for } \ell=1)\\
&= \frac{(\lambda+1)(\lambda^2-1)}{\lambda^3} + \frac{1}{\lambda^2}\left(\frac{w_{k+1}-v_{k+1}}{w_{k+1}-w_k}\right)\\
&= \frac{(\lambda+1)(\lambda^2-1)}{\lambda^3} + \frac{1}{\lambda}\left(\frac{w_{k+1}-v_{k+1}}{w_k-v_k}\right)
\end{aligned}$$

$$\Longrightarrow \left(1-\frac{1}{\lambda}\right)\frac{w_{k+1}-v_{k+1}}{w_k-v_k} = \frac{(\lambda+1)(\lambda^2-1)}{\lambda^3}$$

$$\Longrightarrow \frac{w_{k+1}-v_{k+1}}{w_k-v_k} = \left(\frac{\lambda+1}{\lambda}\right)^2.$$

It now follows that

$$\begin{aligned}
\frac{w_{k+\ell}-v_{k+\ell}}{w_k-v_k} &= \left(\frac{w_{k+1}-v_{k+1}}{w_k-v_k}\right)\left(\frac{w_{k+2}-v_{k+2}}{w_{k+1}-v_{k+1}}\right)\cdots\left(\frac{w_{k+\ell}-v_{k+\ell}}{w_{k+\ell-1}-v_{k+\ell+1}}\right)\\
&= \left(\frac{\lambda+1}{\lambda}\right)^2\left(\frac{\lambda+1}{\lambda}\right)^2\cdots\left(\frac{\lambda+1}{\lambda}\right)^2\\
&= \left(\frac{\lambda+1}{\lambda}\right)^{2\ell}.
\end{aligned}$$

This result can be further generalized for each $j = 0, 1, \ldots, m$ with $k+j \le m$ through the following process:

$$\begin{aligned}
\frac{w_{k+\ell}-v_{k+\ell}}{w_{k+j}-v_{k+j}} &= \left(\frac{w_{k+\ell}-v_{k+\ell}}{w_k-v_k}\right)\left(\frac{w_k-v_k}{w_{k+1}-v_{k+1}}\right)\cdots\\
&\qquad \cdots\left(\frac{w_{k+(j-2)}-v_{k+(j-2)}}{w_{k+(j-1)}-v_{k+(j-1)}}\right)\left(\frac{w_{k+(j-1)}-v_{k+(j-1)}}{w_{k+j}-v_{k+j}}\right)\\
&= \left(\frac{w_{k+\ell}-v_{k+\ell}}{w_k-v_k}\right)\left(\frac{w_k-v_k}{w_{k+1}-v_{k+1}}\right)\cdots\\
&\qquad \cdots\left(\frac{w_{(k+j-2)}-v_{(k+j-2)}}{w_{(k+j-2)+1}-v_{(k+j-2)+1}}\right)\left(\frac{w_{(k+j-1)}-v_{(k+j-1)}}{w_{(k+j-1)+1}-v_{(k+j+1)-1}}\right)
\end{aligned}$$

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$$= \left(\frac{\lambda+1}{\lambda}\right)^{2\ell}\left[\left(\frac{\lambda}{\lambda+1}\right)^{2}\cdots\left(\frac{\lambda}{\lambda+1}\right)^{2}\left(\frac{\lambda}{\lambda+1}\right)^{2}\right]$$
$$= \left(\frac{\lambda+1}{\lambda}\right)^{2\ell}\left(\frac{\lambda}{\lambda+1}\right)^{2j}$$
$$= \left(\frac{\lambda+1}{\lambda}\right)^{2(\ell-j)}. \qquad \square$$

THEOREM 4.8. *Let $v = [v_j] \in \mathbb{R}^m$ and $w = [w_j] \in \mathbb{R}^m$ $(m \geq 2)$ so that $x = \begin{bmatrix} v \\ \hline w \end{bmatrix} \in \mathbb{R}^{2m}$ is the Perron vector of $\mathcal{B}_{2m}$ and let $\rho = \rho(\mathcal{B}_{2m})$. Then for $j, k = 0, 1, 2, \ldots, m$ with $1 \leq k + j \leq m$,*

$$w_{k+j} - v_{k+j} = \frac{1 - v_{k+j}(2\rho+1)}{\rho+1}$$

*and*

$$v_{j+k} = \frac{1}{2\rho+1} - \left(\frac{2\left(\rho + \frac{1-m}{2}\right)^2 + (1-m)\left(\frac{1+m}{2}\right)}{\rho(2\rho+1)}\right)\left(\frac{\rho+1}{\rho}\right)^{2(j+k)-1}.$$

*Proof.* For the sake of brevity denote $\Lambda = \frac{\rho+1}{\rho}$ and $c = w_{k+j} - v_{k+j}$. The sum of the two identities $\rho = m - \mathbf{1}^t w$ and $\rho = (m-1) + \mathbf{1}^t v$, derived in Corollary 4.4, leads to

$$\begin{aligned}
& 2\rho = (m-1) + \mathbf{1}^t v + m - \mathbf{1}^t w \\
\implies\ & 2m - 2\rho - 1 = \mathbf{1}^t w - \mathbf{1}^t v \\
\implies\ & \frac{2m-2\rho-1}{w_{k+j} - v_{k+j}} = \sum_{i=1}^{m} \frac{w_i - v_i}{w_{k+j} - v_{k+j}} = \sum_{\ell=1-k}^{m-k} \frac{w_{k+\ell} - v_{k+\ell}}{w_{k+j} - v_{k+j}} \\
& = \sum_{\ell=1-k}^{m-k} \Lambda^{2(\ell-j)} = \Lambda^{-2j} \sum_{\ell=1-k}^{m-k} \Lambda^{2\ell} \\
& = \frac{\Lambda^{2(1-k-j)}\left(\Lambda^{2m} - 1\right)}{\Lambda^2 - 1} \\
& = \frac{\rho^2}{2p+1}\left(\frac{1+\rho}{\rho}\right)^{2(1-k-j)}\left(\left(\frac{1+\rho}{\rho}\right)^{2m} - 1\right).
\end{aligned}$$

By Proposition 3.4, it follows that

$$\left(\frac{2m-2\rho-1}{w_{k+j} - v_{k+j}}\right)\left(\frac{2p+1}{\rho^2}\right)\left(\frac{1+\rho}{\rho}\right)^{2(k+j-1)} = \left(\frac{1+\rho}{\rho}\right)^{2m} - 1$$

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$$
\begin{aligned}
&\Longrightarrow \left(\frac{2m-2\rho-1}{w_{k+j}-v_{k+j}}\right)\left(\frac{2p+1}{\rho^2}\right)\left(\frac{1+\rho}{\rho}\right)^{2(k+j-1)}+2=\left(\frac{1+\rho}{\rho}\right)^{2m}+1\\
&\quad\Longrightarrow \frac{c\rho^2}{2c\rho^2-(2\rho+1)\left(\frac{\rho+1}{\rho}\right)^{2(j+k-1)}(2\rho-2m+1)}=\left[\left(\frac{1+\rho}{\rho}\right)^{2m}+1\right]^{-1}\\
&\quad\Longrightarrow \frac{c\rho^2}{2c\rho^2-(2\rho+1)\left(\frac{\rho+1}{\rho}\right)^{2(j+k-1)}(2\rho-2m+1)}=2\rho^2-2(m-1)\rho-(m-1).
\end{aligned}
$$

Further algebraic manipulations show that

$$
c\rho^2=\left(2c\rho^2-(2\rho+1)\left(\frac{\rho+1}{\rho}\right)^{2(j+k-1)}(2\rho-2m+1)\right)\big(2\rho^2-2(m-1)\rho-(m-1)\big)
$$

or

$$
\big((2p+1)(2m-2p-1)\big)\left(\left(\frac{\rho+1}{\rho}\right)^{2(j+k-1)}\big(p^2+(p+1)^2-m(1+2p)\big)-cp^2\right)=0.
$$

Since $\dfrac{2m-2}{2}\leq\rho\leq\dfrac{2m-1}{2}$ and $0<c$, it must be that the above equation holds provided that

$$
\begin{aligned}
0&=\left(\frac{\rho+1}{\rho}\right)^{2(j+k-1)}\big(p^2+(p+1)^2-m(1+2p)\big)-cp^2\\
cp^2&=\left(\frac{\rho+1}{\rho}\right)^{2(j+k-1)}\big(p^2+(p+1)^2-m(1+2p)\big).
\end{aligned}
$$

By Corollary 4.4, $c=\dfrac{1-v_{k+j}(2\rho+1)}{\rho+1}$, and thus,

$$
\begin{aligned}
c\rho^2&=\left(\frac{\rho+1}{\rho}\right)^{2(j+k-1)}\big(\rho^2+(\rho+1)^2-m(1+2\rho)\big)\\
\Longrightarrow \left(\frac{1-v_{k+j}(2\rho+1)}{\rho+1}\right)\rho^2&=\left(\frac{\rho+1}{\rho}\right)^{2(j+k-1)}\big(\rho^2+(\rho+1)^2-m(1+2\rho)\big)\\
\Longrightarrow v_{k+j}&=\frac{1}{2p+1}-\left(\frac{(p+1)^2+(p-m)^2-m(m+1)}{p(2p+1)}\right)\\
&\qquad\times\left(\frac{\rho+1}{\rho}\right)^{2(j+k)-1}\\
&=\frac{1}{2p+1}-\left(\frac{2\left(\rho+\frac{1-m}{2}\right)^2+(1-m)\left(\frac{1+m}{2}\right)}{p(2p+1)}\right)\\
&\qquad\times\left(\frac{\rho+1}{\rho}\right)^{2(j+k)-1}. \qquad\square
\end{aligned}
$$

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As a consequence of Theorem 4.8, we can express $v_1$ and $w_1$ in terms of $\rho$.

COROLLARY 4.9. *Under the assumptions and in the notation of* Theorem 4.8*,*

$$v_1 = \frac{(m-1)(\rho+1)-\rho^2}{\rho^2} \quad \textit{and} \quad w_1 = \frac{\rho+1-m}{\rho}.$$

*Proof.* In Theorem 4.8, it was shown that $c = w_{k+j} - v_{k+j} = \frac{1-v_{k+j}(2\rho+1)}{\rho+1}$, which holds for $j, k = 0, 1, \ldots, m$ with $1 \leq k+j \leq m$. By setting $k = 1$ and $j = 0$, and using the identity derived in Theorem 4.8, we have

$$\begin{aligned} cp^2 &= p^2 + (p+1)^2 - m(1+2p) \\ \Longrightarrow \left(\frac{1-v_1(2\rho+1)}{\rho+1}\right) p^2 &= p^2 + (p+1)^2 - m(1+2p) \\ \Longrightarrow 0 &= \frac{(2\rho+1)\big(-(v_1+1)p^2 + (m-1)p + (m-1)\big)}{\rho+1}. \end{aligned}$$

Since $\frac{2m-2}{2} \leq \rho \leq \frac{2m-1}{2}$ and $0 < c$, it must be that $(v_1 1+1)p^2 - (m-1)p - (m-1) = 0$; i.e.,

$$\begin{aligned} \rho &= \frac{m + \sqrt{4v_1(m-1) + m(m+2) - 3} - 1}{2(v_1+1)} \\ &= \frac{m - 1 + \sqrt{(m-1)(m+4v_1+3)}}{2(v_1+1)}. \end{aligned}$$

Solving for $v_1$ yields $v_1 = \dfrac{(m-1)(\rho+1)-\rho^2}{\rho^2}$; furthermore, by applying the identity $w_k = \dfrac{1-\rho v_k}{\rho+1}$, we have $w_1 = \dfrac{\rho+1-m}{\rho}$. ∎

Although it is possible to use Theorem 4.8 in order to determine $v_m$, and in retrospect $w_m$, their forms would be cumbersome. An alternative approach is to employ the inverse of $\mathcal{B}_{2m}$ found in Theorem 3.1 and the fact that $\rho^{-1}x = \mathcal{B}_{2m}{}^{-1}x$, leading to the following result.

LEMMA 4.10. *Under the assumptions and in the notation of* Theorem 4.8*,*

$$v_m = \frac{\rho+1-m}{\rho+1} \quad \textit{and} \quad w_m = \frac{m\rho+1-\rho^2}{(\rho+1)^2}.$$

*Proof.*

$$\left(\frac{m-1}{\rho}\right) v_m = (1-m)v_m + (m-1)w_1$$

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$$
\begin{aligned}
&\Longrightarrow v_m = \rho(w_1 - v_m)\\
&\Longrightarrow v_m = \left(\frac{\rho}{\rho+1}\right) w_1\\
&\Longrightarrow = \frac{\rho+1-m}{\rho+1}.
\end{aligned}
$$

Furthermore, using the identity $w_k = \dfrac{1-\rho v_k}{\rho+1}$, it follows that $w_m = \dfrac{m\rho+1-\rho^2}{(\rho+1)^2}$. ☐

Using a similar approach, it is possible to derive not only $v_m$ in terms of $\rho$ but also the following recursive relationship among the entries of the Perron vector.

Theorem 4.11. *Let $v = [v_j] \in \mathbb{R}^m$ and $w = [w_j] \in \mathbb{R}^m$ $(m \geq 2)$ so that $x = \left[\begin{array}{c} v \\ \hline w \end{array}\right] \in \mathbb{R}^{2m}$ is the Perron vector of $\mathcal{B}_{2m}$ and let $\rho = \rho(\mathcal{B}_{2m})$. Then for $k = 1, \ldots, m-1$,*

$$
v_k = \frac{1+\rho^2 v_{k+1}}{(\rho+1)^2} \quad \textit{and} \quad w_{k+1} = \frac{w_k(\rho^2+2\rho+1)-1}{\rho^2}.
$$

*Proof.* Since $\mathcal{B}_{2m}x = \rho x$, $\rho > 0$, we have

$$
\frac{m-1}{\rho}\left[\begin{array}{c} v \\ \hline w \end{array}\right] = \left[\begin{array}{c|c} A & B \\ \hline J_m - (m-1)I_m & PA^tP \end{array}\right]\left[\begin{array}{c} v \\ \hline w \end{array}\right] = \left[\begin{array}{c} Av + Bw \\ \hline J_m v - (m-1)v + PA^tPw \end{array}\right].
$$

Considering a further partition $v = \left[\begin{array}{c} \nu \\ \hline v_m \end{array}\right]$ and $w = \left[\begin{array}{c} w_1 \\ \hline \omega \end{array}\right]$, where $\nu, \omega \in \mathbb{R}^{m-1}$, it follows that

$$
\begin{aligned}
\left(\frac{m-1}{\rho}\right)\nu &= \big(J_{m-1}\nu - (m-1)I_{m-1}\nu + v_m\big) + \big(J_{m-1}\omega - (m-1)I_{m-1}\omega\big)\\
&= J_{m-1}(\nu+\omega) - (m-1)(\nu+\omega) + v_m\mathbf{1}_{m-1}\\
&= \big(\mathbf{1}_{2m}^t x - v_m - w_1\big)\mathbf{1}_{m-1} - (m-1)(\nu+\omega) + v_m\mathbf{1}_{m-1}\\
&= \left(\mathbf{1}_{2m}^t x - \frac{(2\rho+1)(\rho+1-m)}{\rho(\rho+1)}\right)\mathbf{1}_{m-1} - (m-1)(\nu+\omega)\\
&\quad + \left(\frac{\rho+1-m}{\rho+1}\right)\mathbf{1}_{m-1}\\
&= \mathbf{1}_{2m}^t x + \frac{m-1-\rho}{\rho} - (m-1)(\nu+\omega)\\
\Longrightarrow \nu &= \frac{\lambda(\mathbf{1}_{2m}^t x - 1) + m - 1 + (p - mp)\omega}{-(\lambda+1)(m-1)}\\
&= \frac{\rho(1-\mathbf{1}_{2m}^t x)}{1-m} + \left(\frac{1-\rho}{\rho+1}\right)\omega.
\end{aligned}
$$

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The above relationship among the vectors $\nu$ and $\omega$ yields the identity $v_k = \left(\frac{1-\lambda w_{k+1}}{1+\rho}\right)$, where $k = 1, 2, \ldots, m-1$. From this and the fact that $w_k = \frac{1-\rho v_k}{\rho+1}$ for $k = 1, \ldots, m$, we have

$$\begin{aligned} v_k &= \left(\frac{1}{\rho+1}\right)(1-\rho w_{k+1}) \\ &= \left(\frac{1}{\rho+1}\right)\left(1 - \frac{1-\rho v_{k+1}}{\rho+1}\right) \\ &= \frac{1+\rho^2 v_{k+1}}{(\rho+1)^2}. \end{aligned}$$

Using the latter result, converting $v_\ell$ into its corresponding $w_\ell$ counterparts, and some algebraic manipulations, we obtain the following recursive relationship among the entries for the bottom block of the Perron vector:

$$w_{k+1} = \frac{w_k(\rho^2 + 2\rho + 1) - 1}{\rho^2}, \quad k = 1, 2, \ldots, m-1. \quad \square$$

**5. Ranking the Brualdi-Li tournament.** Referring to the Kendall-Wei ranking scheme [7, 15], that is, the ranking of tournament teams according to the magnitude of the entries of the Perron vector, it follows from Corollary 4.6 that in the round robin tournament represented by $\mathcal{B}_{2m}$ ($m \geq 2$), team $2m$ ranks the highest, followed in decreasing rank by teams $2m-1, 2m-2, \ldots, m+1, 1, 2, \ldots, m$.

An alternative ranking system is due to Ramanajucharyula [18]. It takes into account both the right and left Perron vectors $x$, $y$, respectively, by considering the ratios $x_j/y_j$. The larger this ratio is, the higher the $j$-th team ranks. This is because $x_j$ is viewed as a measure of relative strength (accounting for the teams defeated by team $j$) and $y_j$ as a measure of relative weakness of team $j$ (accounting for the teams that defeated team $j$). In the next result, we can indeed order these ratios for the Brualdi-Li matrix.

Theorem 5.1. *Let $v = [v_j] \in \mathbb{R}^m$ and $w = [w_j] \in \mathbb{R}^m$ ($m \geq 2$) so that $x = \begin{bmatrix} v \\ \hline w \end{bmatrix} \in \mathbb{R}^{2m}$ is the Perron vector of $\mathcal{B}_{2m}$ and let $\rho = \rho(\mathcal{B}_{2m})$; furthermore, let $y$ denote the left Perron vector of $\mathcal{B}_{2m}$. Then, we have the following interlacing relationships*

$$\frac{x_m}{y_m} < \frac{x_1}{y_1} < \frac{x_{m-1}}{y_{m-1}} < \frac{x_2}{y_2} < \frac{x_{m-2}}{y_{m-2}} < \cdots < \frac{x_{\lceil m/2 \rceil}}{y_{\lceil m/2 \rceil}} < 1,$$

$$1 < \frac{x_{2m-\lceil m/2 \rceil+1}}{y_{2m-\lceil m/2 \rceil+1}} < \cdots < \frac{x_{m+3}}{y_{m+3}} < \frac{x_{2m-1}}{y_{2m-1}} < \frac{x_{m+2}}{y_{m+2}} < \frac{x_{2m}}{y_{2m}} < \frac{x_{m+1}}{y_{m+1}}.$$

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*Proof.* For the sake of brevity, denote

$$\Lambda = \frac{\rho+1}{\rho}, \quad \alpha = \frac{1}{2\rho+1}, \quad F = \left( \frac{2\left(\rho + \frac{1-m}{2}\right)^2 + (1-m)\left(\frac{1+m}{2}\right)}{\rho(2\rho+1)} \right).$$

In Corollary 4.4, it is shown that $w_k = \dfrac{1-\rho v_k}{\rho+1}$ $(k = 1,2,\ldots,m)$, thereby giving the following relationships:

$$v_k = \alpha - F\Lambda^{2k-1} \quad \text{and} \quad w_k = \alpha + F\Lambda^{2k-2} \quad \text{for } \; k = 1,\ldots,m.$$

Define now for $k = 1,\ldots,m$ the function $f(k) = \dfrac{y_k}{x_k}$ so that

$$f(k) = \frac{y_k}{x_k} = \frac{w_{m-k+1}}{v_k} = \frac{\alpha + F\Lambda^{2m-2k}}{\alpha - F\Lambda^{2k-1}}.$$

It follows from Corollary (4.6) that for $k = 1,\ldots,m,$ $f(k) > 1 > \dfrac{1}{f(2m+1-k)}$. Also, notice that the structures of $y$ and $x$ yield

$$f(k) = \frac{1}{f(2m+1-k)} \quad \text{for } k = 1,\ldots,m. \tag{5.1}$$

We shall first show that $f(m) > f(m-j)$ for $j = 1,\ldots,m-1$. Since $\Lambda > 1$, the following inequalities are equivalent:

$$\begin{aligned}
\Lambda^{2m} &> \Lambda^{2j} \\
\Lambda^{2m}\left(1 - \Lambda^{-2j}\right) &> \Lambda^{2j} - 1 \\
\Lambda^{2m} + 1 &> \Lambda^{2j} - \Lambda^{2m-2j} \qquad (5.2) \\
\frac{\alpha\Lambda - \alpha}{F} &> \Lambda^{2j} - \Lambda^{2m-2j} \\
\Lambda &> \frac{\alpha + F\Lambda^{2j}}{\alpha - F\Lambda^{2m-2j-1}} \\
\frac{\alpha + F}{\alpha - F\Lambda^{2m-1}} &> \frac{\alpha + F\Lambda^{2j}}{\alpha - F\Lambda^{2m-2j-1}} \\
f(m) &> f(m-j), \qquad (5.3)
\end{aligned}$$

and thus, (5.3) holds. Next we will show that

$$f(k) > f(m-k) \quad \text{for } \; k = 1,2,\ldots,\lceil m/2 \rceil. \tag{5.4}$$

By way of contradiction, assume that $f(k) \le f(m-k)$ for $1 \le k \le \lceil m/2 \rceil$. Notice that this inequality is strict when $k \ne \lceil m/2 \rceil$. The following equivalent inequalities

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ensue:

$$
\begin{aligned}
f(k) \;&\leq\; f(m-k) \\
\frac{\alpha + F\Lambda^{2m-2k}}{\alpha - F\Lambda^{2k-1}} \;&\leq\; \frac{\alpha + F\Lambda^{2m-2(m-k)}}{\alpha - F\Lambda^{2(m-k)-1}} = \frac{\alpha + F\Lambda^{2k}}{\alpha - F\Lambda^{2(m-k)-1}} \\
\left(\alpha + F\Lambda^{2m-2k}\right)\left(\alpha - F\Lambda^{2(m-k)-1}\right) \;&\leq\; \left(\alpha - F\Lambda^{2k-1}\right)\left(\alpha + F\Lambda^{2k}\right) \\
\alpha\left(\Lambda^{2m-2k} - \Lambda^{2m-2k-1}\right) - F\Lambda^{4m-4k-1} \;&\leq\; \alpha\left(\Lambda^{2k} - \Lambda^{2k-1}\right) - F\Lambda^{4k-1} \\
\alpha\left(1 - \Lambda^{-1}\right)\left(\Lambda^{2m-2k} - \Lambda^{2k}\right) \;&\leq\; F\Lambda^{-1}\left(\Lambda^{4m-4k} - \Lambda^{4k}\right) \\
\alpha\left(\frac{1}{\rho}\right)\left(\Lambda^{2m-2k} - \Lambda^{2k}\right) \;&\leq\; F\Lambda^{-1}\left(\Lambda^{2m-2k} - \Lambda^{2k}\right)\left(\Lambda^{2m-2k} + \Lambda^{2k}\right) \\
\frac{\Lambda\left(\Lambda^{2m-2k} - \Lambda^{2k}\right)}{2\rho^2 - 2\rho(m-1) - (m-1)} \;&\leq\; \left(\Lambda^{2m-2k} - \Lambda^{2k}\right)\left(\Lambda^{2m-2k} + \Lambda^{2k}\right) \\
\Lambda\left(\Lambda^{2m-2k} - \Lambda^{2k}\right)\left(\Lambda^{2m} + 1\right) \;&\leq\; \left(\Lambda^{2m-2k} - \Lambda^{2k}\right)\left(\Lambda^{2m-2k} + \Lambda^{2k}\right).
\end{aligned}
$$

Because $1 \leq k \leq \lceil m/2 \rceil$, we have $\Lambda^{2m-2k} \leq \Lambda^{2k}$, which in turn implies

$$\Lambda\left(\Lambda^{2m} + 1\right) \leq \Lambda^{2m-2k} + \Lambda^{2k}.$$

By (5.2), it follows that

$$\Lambda\left(\Lambda^{2m} + 1\right) \leq \Lambda^{2m-2k} + \Lambda^{2k} \leq \Lambda^{2m} + 1,$$

thereby yielding the contradiction $\Lambda \leq 1$, thus showing (5.4). Finally, we will show that

$$f(m-k) > f(k+1) \quad \text{for} \;\; k = 1, 2, \ldots, \lceil m/2 \rceil. \tag{5.5}$$

By way of contradiction, assume that $f(m-k) \leq f(k+1)$ for $1 \leq k \leq \lceil m/2 \rceil$. Notice that this inequality is strict when $k \neq \lceil m/2 \rceil$. The following equivalent inequalities ensue:

$$
\begin{aligned}
f(m-k) \;&\leq\; f(k+1) \\
\frac{\alpha + F\Lambda^{2k}}{\alpha - F\Lambda^{2m-2k-1}} \;&\leq\; \frac{\alpha + F\Lambda^{2m-2k-2}}{\alpha - F\Lambda^{2k+1}}
\end{aligned}
$$

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$$\left(\alpha + F\Lambda^{2k}\right)\left(\alpha - F\Lambda^{2k+1}\right) \;\leq\; \left(\alpha + F\Lambda^{2m-2k-2}\right)\left(\alpha - F\Lambda^{2m-2k-1}\right)$$

$$\alpha\left(\Lambda^{2k} - \Lambda^{2k+1}\right) - F\Lambda^{4k+1} \;\leq\; \alpha\left(\Lambda^{2m-2k-2} - \Lambda^{2m-2k-1}\right) - F\Lambda^{4m-4k-3}$$

$$\alpha\left(1-\Lambda\right)\left(\Lambda^{2k} - \Lambda^{2m-2k-2}\right) \;\leq\; F\Lambda\left(\Lambda^{4k} - \Lambda^{4m-4k-4}\right)$$

$$\frac{\rho(1-\Lambda)}{\Lambda}\left(\Lambda^{2m}+1\right)\left(\Lambda^{2k} - \Lambda^{2m-2k-2}\right) \;\leq\; \left(\Lambda^{4k} - \Lambda^{4m-4k-4}\right)$$

$$\frac{\rho}{\rho+1}\left(\Lambda^{2m}+1\right)\left(\Lambda^{2m-2k-2} - \Lambda^{2k}\right) \;\leq\; \left(\Lambda^{2k} - \Lambda^{2m-2k-2}\right)\left(\Lambda^{2k} + \Lambda^{2m-2k-2}\right).$$

Because $1 \leq k \leq \lceil m/2 \rceil$, we have $\Lambda^{2k} \leq \Lambda^{2m-2k-2}$, thereby yielding the contradiction

$$\frac{\rho}{\rho+1}\left(\Lambda^{2m}+1\right)\left(\Lambda^{2m-2k-2} - \Lambda^{2k}\right) \leq 0,$$

which proves (5.5). The first set of inequalities of the theorem now follow from (5.3), (5.4) and (5.5). The second set of inequalities follows by applying (5.1) to the first set of inequalities. □

It follows from Theorem 5.1 that in the round robin tournament represented by $\mathcal{B}_{2m}$ ($m \geq 2$), teams $m+1, m+2, \ldots, 2m$ have strength to weakness ratios greater than 1, and thus, rank higher than each of the teams $1, 2, \ldots, m$ whose ratios are all less than one. This is in agreement with the Kendall-Wei ranking. However, Theorem 5.1 points us to a different than Kendall-Wei (namely, interlacing) ranking within each group of teams, illustrated in the next example.

Example 5.2. To illustrate Theorem 5.1 and the resulting ranking, we compute the right and left Perron vectors $x, y$, respectively, of $\mathcal{B}_{12}$ and observe the interlacing relationships

$$\frac{x_6}{y_6} = 0.8454 < \frac{x_1}{y_1} = 0.8738 < \frac{x_5}{y_5} = 0.8761 > \frac{x_2}{y_2} = 0.8910 < \frac{x_4}{y_4} = 0.8927 < \frac{x_3}{y_3} = 0.8973.$$

$$\frac{x_{10}}{y_{10}} = 1.1144 < \frac{x_9}{y_9} = 1.1202 < \frac{x_{11}}{y_{11}} = 1.1 > \frac{x_8}{y_8} = 1.1414 < \frac{x_{12}}{y_{12}} = 1.1444 < \frac{x_7}{y_7} = 1.1829.$$

According to the strength to weakness ratios, the teams rank in decreasing order as follows:

7, 12, 8, 11, 9, 10, 3, 4, 2, 5, 1, 6,

whereas the Kendall-Wei ranking is

12, 11, 10, 9, 8, 7, 1, 2, 3, 4, 5, 6.

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REFERENCES

[1] A. Brauer and I.C. Gentry. On the characteristic roots of tournament matrices. *Bull. Amer. Math. Soc.*, 74:1133–1135, 1968.
[2] R.A. Brualdi and Q. Li. Problem 31. *Discrete Math.*, 43:329–330, 1983.
[3] S.W. Drury. Solution of the Conjecture of Brualdi and Li. *Linear Algebra Appl.*, to appear, doi:10.1016/j.laa.2011.11.031.
[4] C. Eschenbach, F. Hall, R. Hemasinha, S.J. Kirkland, Z. Li, B.L. Shader, J.L. Stuart, and J.R. Weaver. On almost regular tournament matrices. *Linear Algebra Appl.*, 306:103–121, 2000.
[5] F. Harary and L. Moser. The theory of round robin tournaments. *Amer. Math. Monthly*, 73:231–246, 1966.
[6] R.A. Horn and C.R. Johnson. *Matrix Analysis.* Cambridge University Press, Cambridge 1990.
[7] M.G. Kendall. Further contributions to the theory of paired comparisons. *Biometrics*, 11:43–62, 1955.
[8] S. Kirkland. Hypertournament matrices, score vectors and eigenvalues. *Linear Multilinear Algebra*, 30:261–274, 1991.
[9] S. Kirkland. A note on the sequence of Brualdi-Li matrices. *Linear Algebra Appl.*, 248:233–240, 1996.
[10] S. Kirkland. Perron vector bounds for a tournament matrix with applications to a conjecture of Brualdi and Li. *Linear Algebra Appl.*, 262:209–227, 1997.
[11] S. Kirkland. A note on Perron vectors for almost regular tournament matrices. *Linear Algebra Appl.*, 266:43–47, 1997.
[12] S. Kirkland, P. Psarrakos, and M. Tsatsomeros. On the location of the spectrum of hypertournament matrices. *Linear Algebra Appl.*, 323:37–49, 2001.
[13] J.W. Moon. *Topics on Tournaments.* Holt, Rhinehart and Winston, New York, 1968.
[14] J. McDonald, P. Psarrakos, and M. Tsatsomeros. Almost skew-symmetric matrices. *Rocky Mountain J. Math.*, 34:269–288, 2004.
[15] T.H. Wei. *The Algebraic foundations of Ranking Theory.* Ph.D. Thesis, Cambridge University, 1952.
[16] P. Psarrakos and M. Tsatsomeros. The Perron eigenspace of nonnegative almost skew-symmetric and Levinger's transformation. *Linear Algebra Appl.*, 360:43–57, 2003.
[17] P. Psarrakos and M. Tsatsomeros. Bounds for Levinger's function of nonnegative almost skew-symmetric matrices. *Linear Algebra Appl.*, 416:759–772, 2006.
[18] C. Ramanajucharyula. Analysis of potential experiments. *Psycometrica*, 29:257–261, 1964.