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ON COPOSITIVE AND COMPLETELY POSITIVE CONES, AND Z-TRANSFORMATIONS*

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Abstract. A well-known result of Lyapunov on continuous linear systems asserts that a real square matrix A is positive stable if and only if for some symmetric positive definite matrix X, $AX + XA^T$ is also positive definite. A recent result of Moldovan-Gowda says that a $\mathbf{Z}$-matrix A is positive stable if and only if for some symmetric strictly copositive matrix X, $AX + XA^T$ is also strictly copositive. In this paper, these results are unified/extended by replacing $\mathbb{R}^n$ and $\mathbb{R}^n_+$ by a closed convex cone $\mathcal{C}$ satisfying $\mathcal{C}-\mathcal{C} = \mathbb{R}^n$. This is achieved by relating the $\mathbf{Z}$-property of a matrix on this cone with the $\mathbf{Z}$-property of the corresponding Lyapunov transformation $L_A(X) := AX + XA^T$ on the completely positive cone of $\mathcal{C}$ and the $\mathbf{Z}$-property of L_{A^T} on the copositive cone of $\mathcal{C}$ in $\mathcal{S}^n$ (the space of all real $n \times n$ symmetric matrices). A similar analysis is carried out for the Stein transformation $S_A(X) = X - AXA^T$.

Key words. Copositive matrix, Copositive and completely positive cones, $\mathbf{Z}$-transformation, Lyapunov and Stein transformations.

AMS subject classifications. 15A48, 34A30, 34D23, 17B45.

1. Introduction. Given a closed convex cone K in a real finite dimensional Hilbert space $(H, \langle\cdot,\cdot\rangle)$, and a linear transformation L on H, we say that L has the $\mathbf{Z}$-property on K (or that it is a $\mathbf{Z}$-transformation on K) and write $L \in \mathbf{Z}(K)$ if

$$[x \in K,\ y \in K^*,\ \text{and } \langle x, y\rangle = 0] \Rightarrow \langle L(x), y\rangle \leq 0, \tag{1.1}$$

where K^* denotes the dual cone of K. As a generalization of a $\mathbf{Z}$-matrix (which is a real square matrix with nonpositive off-diagonal entries), such transformations were introduced in [19] in the form of cross-positive matrices. $\mathbf{Z}$-matrices/transformations have numerous properties and appear in many areas, e.g., see [3], [13]. Our motivation for this article comes from dynamical systems. Consider $\mathcal{S}^n$, the space of all $n \times n$ real symmetric matrices, with the inner product $\langle X, Y\rangle = trace(XY)$ and the cone $\mathcal{S}^n_+$ of all positive semidefinite matrices in $\mathcal{S}^n$. Then for any matrix $A \in \mathbb{R}^{n\times n}$, the *Lyapunov transformation* L_A and *Stein transformation* S_A, respectively defined on $\mathcal{S}^n$ by

$$L_A(X) := AX + XA^T \quad \text{and} \quad S_A(X) := X - AXA^T,$$

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are **Z**-transformations on $\mathcal{S}^n_+$ [13]. These transformations have been well studied in dynamical systems theory, starting from Lyapunov's paper [16] on continuous dynamical systems and Stein's paper [22] on discrete dynamical systems. The celebrated result of Lyapunov deals with the stability of the linear system $\dot{x} + Ax = 0$, and, in particular with the equivalence of the following statements [7]:

(i) The system $\dot{x} + Ax = 0$ is asymptotically stable in $\mathbb{R}^n$ (which means that the trajectory of the system from any starting point in $\mathbb{R}^n$ converges to the origin as $t \to \infty$).
(ii) A is positive stable (that is, all eigenvalues of A lie in the open right-half plane).
(iii) There exists $X \in \mathcal{S}^n$ such that X and $L_A(X)$ are positive definite.
(iv) For every positive definite $Y \in \mathcal{S}^n$, the equation $L_A(X) = Y$ has a (unique) positive definite solution X in $\mathcal{S}^n$.

For a discrete system of the form $x(k+1) = Ax(k)$, $k = 1, 2, \ldots$, similar equivalent statements can be made by replacing the positive stability of A with Schur stability of A (which means that all eigenvalues of A lie in the open unit disk) and L_A by S_A.

Now consider a linear system $\dot{x} + Ax = 0$ whose trajectories are constrained to lie in the nonnegative orthant $\mathbb{R}^n_+$. It is well known that this can happen if and only if A is a **Z**-matrix. Analogous to Lyapunov's result, we have the equivalence of the following when A is a **Z**-matrix:

(i) The system $\dot{x} + Ax = 0$ is asymptotically stable in $\mathbb{R}^n_+$.
(ii) A is positive stable.
(iii) There exists $X \in \mathcal{S}^n$ such that X and $L_A(X)$ are strictly copositive on $\mathbb{R}^n_+$.
(iv) For every $Y \in \mathcal{S}^n$ that is strictly copositive on $\mathbb{R}^n_+$, the equation $L_A(X) = Y$ has a (unique) strictly copositive solution X in $\mathcal{S}^n$.
(v) There exists a vector $d > 0$ (i.e., d belongs to the interior of $\mathbb{R}^n_+$) such that $Ad > 0$.

Here, the strict copositivity (copositivity) of X on $\mathbb{R}^n_+$ is defined by: $x^TXx > 0$ (≥ 0), for all $0 \neq x \in \mathbb{R}^n_+$. The equivalence of Items (i), (ii), and (v) is well known in the literature, e.g., see [17]. The new items (iii) and (iv) were proved by Moldovan and Gowda [18] by relying on the equivalence of the following statements for any $A \in \mathbb{R}^{n\times n}$:

(a) A is a **Z**-matrix.
(b) L_A has the **Z**-property on the cone of completely positive matrices in $\mathcal{S}^n$.
(c) L_{A^T} has the **Z**-property on the cone of copositive matrices in $\mathcal{S}^n$.

In a recent article [5], Bundfuss and Dür raise the question of studying the dynamics of $\dot{x}+Ax = 0$ which is constrained to a (polyhedral) cone K by asking for the existence

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of a symmetric matrix X that is strictly copositive on K for which $AX + XA^T$ is also strictly copositive on K. Motivated by the similarities in the above results of Lyapunov and Moldovan-Gowda, and the question of Bundfuss and Dür, in this paper, we present a unifying result (Theorem 3.8) by relating the $\mathbf{Z}$-property of a matrix A on a closed convex cone in $\mathbb{R}^n$ with the $\mathbf{Z}$-property of L_A (L_{A^T}) on the corresponding completely positive cone (respectively, copositive cone) in $\mathcal{S}^n$.

Consider $\mathbb{R}^n$ with the usual inner product. Given a closed convex cone $\mathcal{C}$ in $\mathbb{R}^n$ with dual $\mathcal{C}^*$, we consider two related cones in $\mathcal{S}^n$: The *copositive cone* of $\mathcal{C}$ defined by

$$\mathcal{E} = \mathit{copos}(\mathcal{C}) := \{A \in \mathcal{S}^n : A \text{ copositive on } \mathcal{C}\} \tag{1.2}$$

and the *completely positive cone of* $\mathcal{C}$ defined by

$$\mathcal{K} = \mathit{compos}(\mathcal{C}) := \{BB^T : \text{columns of } B \text{ belong to } \mathcal{C}\}. \tag{1.3}$$

When $\mathcal{C} = \mathbb{R}^n$, these two cones reduce to $\mathcal{S}^n_+$ which is the underlying cone in semidefinite programming and semidefinite linear complementarity problems [1], [10], [11]. In the case of $\mathcal{C} = \mathbb{R}^n_+$, these cones reduce, respectively, to the cones of copositive matrices and completely positive matrices which have appeared prominently in statistical and graph theoretic literature [4] and (recently) in the study of (combinatorial) optimization problems [6], [8].

With the notation $L \in \mathbf{Z}(K)$ to mean that the transformation L has the $\mathbf{Z}$-property on K and $L \in \Pi(K)$ to mean that $L(K) \subseteq K$, we show in this article (see Theorems 3.3 and 5.1) that

$$\begin{aligned} &A \in \mathbf{Z}(\mathcal{C}) \Rightarrow L_A \in \mathbf{Z}(\mathcal{K}) \Leftrightarrow L_{A^T} \in \mathbf{Z}(\mathcal{E}) \quad \text{and} \\ &A \in \Pi(\mathcal{C}) \Rightarrow S_A \in \mathbf{Z}(\mathcal{K}) \Leftrightarrow S_{A^T} \in \mathbf{Z}(\mathcal{E}). \end{aligned}$$

These results, along with the properties of $\mathbf{Z}$-transformations, will allow us to extend the results of Lyapunov and Moldovan-Gowda, and (partially) answer the question of Bundfuss and Dür.

Here is an outline of the paper. Section 2 deals with the preliminaries. The $\mathbf{Z}$-property of L_A is covered in Section 3 and that of S_A is covered in Section 5. In Section 4, we study Lyapunov-like transformations. Finally, Section 6 deals with some results relating $\mathbf{Z}$-property, cone spectrum, and copositivity.

2. Preliminaries. Throughout this paper, H denotes a finite dimensional real Hilbert space with inner product given by $\langle x, y\rangle$. For a set K in H, K° and $K^\perp$ denote, respectively, the interior and orthogonal complement of K. For a closed convex cone K in H the dual is given by

$$K^* := \{y \in H : \langle y, x\rangle \geq 0 \ \forall x \in K\}.$$

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We use the notation

$$K \ni x \perp y \in K^* \text{ to mean that } x \in K,\ y \in K^*, \text{ and } \langle x, y \rangle = 0.$$

Recall [3] that a closed convex cone K in H is a *proper cone* if K is *reproducing* (that is, $K - K = H$) and *pointed* (that is, $K \cap -K = \{0\}$) (or equivalently, K and K^* have nonempty interiors [3]).

For a linear transformation L on H, L^* denotes its adjoint. It said to be

- *copositive on* K (*strictly copositive on* K) if $\langle L(x), x \rangle \geq 0$ (> 0) for all $0 \neq x \in K$;
- *monotone* if $\langle L(x), x \rangle \geq 0$ for all $x \in H$;
- *positive stable* (*Schur stable*) if all the eigenvalues of L lie in the open right-half plane (respectively, in the open unit disk).

In the space $H = \mathbb{R}^n$, vectors are written as column vectors and the usual inner product is written as $\langle x, y \rangle$ or as $x^T y$. Following standard terminology,

- *Copositive matrices* (*positive semidefinite* matrices) are those which are copositive on $\mathbb{R}^n_+$ (respectively, on $\mathbb{R}^n$);
- *Completely positive matrices* are of the form BB^T with columns of B coming from $\mathbb{R}^n_+$.

Throughout this paper, K denotes a closed convex cone in H and $\mathcal{C}$ denotes a closed convex cone in $\mathbb{R}^n$. Corresponding to $\mathcal{C}$, the copositive cone $\mathcal{E}$ and the completely positive cone $\mathcal{K}$ in $\mathcal{S}^n$ are defined, respectively, by (1.2) and (1.3).

PROPOSITION 2.1. *Let L be a self-adjoint linear transformation on H that is copositive on K. Then*

$$x \in K, \langle L(x), x \rangle = 0 \Rightarrow L(x) \in K^*.$$

Proof. Suppose $x \in K$ and $\langle L(x), x \rangle = 0$. Then for any $y \in K$,

$$0 \leq \lim_{t \downarrow 0} \frac{1}{t} \langle L(x + ty), x + ty \rangle = 2\langle L(x), y \rangle.$$

This shows that $L(x) \in K^*$. ∎

PROPOSITION 2.2. *The following statements hold:*

(*i*) *$\mathcal{E}$ is a closed convex cone in $\mathcal{S}^n$ and $\mathcal{K} \subseteq \mathcal{S}^n_+ \subseteq \mathcal{E}$.*
(*ii*) *$\mathcal{K}$ is a closed convex cone; moreover, $\mathcal{K}$ is the dual of $\mathcal{E}$.*
(*iii*) *$\mathcal{E}$ (likewise, $\mathcal{K}$) is proper if and only if $\mathcal{C} - \mathcal{C} = \mathbb{R}^n$.*

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Proof. Item (i) is obvious and Item (ii) is well known, see [24]. We prove Item (iii). As $\mathcal{S}^n_+ \subseteq \mathcal{E}$ and $\mathcal{S}^n_+ - \mathcal{S}^n_+ = \mathcal{S}^n$, we have $\mathcal{E} - \mathcal{E} = \mathcal{S}^n$. So, to see (iii), it is enough to show that $\mathcal{E} \cap -\mathcal{E} = \{0\}$ if and only if $\mathcal{C} - \mathcal{C} = \mathbb{R}^n$.

Now, let $A \in \mathcal{S}^n$. By an application of Proposition 2.1 with $L = A$ and $K = \mathcal{C}$, we have

$$\begin{aligned} A \in \mathcal{E} \cap -\mathcal{E} \Leftrightarrow \quad & x^T A x = 0 \ \forall x \in \mathcal{C} \\ \Leftrightarrow \quad & -Ax, Ax \in \mathcal{C}^* \ \forall x \in \mathcal{C} \\ \Leftrightarrow \quad & A(\mathcal{C}) \subseteq \mathcal{C}^* \cap -\mathcal{C}^* = \mathcal{C}^\perp = (\mathcal{C} - \mathcal{C})^\perp \\ \Leftrightarrow \quad & A(\mathcal{C} - \mathcal{C}) \subseteq (\mathcal{C} - \mathcal{C})^\perp. \end{aligned}$$

Hence, when $\mathcal{C} - \mathcal{C} = \mathbb{R}^n$, we have $A = 0$ for any $A \in \mathcal{E} \cap -\mathcal{E}$. On the other hand, when $\mathcal{C} - \mathcal{C} \neq \mathbb{R}^n$, (as $\mathcal{C} - \mathcal{C}$ is a subspace) there exists $0 \neq v \in (\mathcal{C} - \mathcal{C})^\perp$. Then $x^T(vv^T)x = 0$ for all $x \in \mathcal{C}$ and so $0 \neq A = vv^T \in \mathcal{E} \cap -\mathcal{E}$.

Finally, $\mathcal{E}$ is proper if and only if its dual $\mathcal{K}$ is proper. Thus, $\mathcal{K}$ is proper if and only if $\mathcal{C} - \mathcal{C} = \mathbb{R}^n$. ∎

3. The Z-property and Lyapunov transformations. The **Z**-property of a matrix or a linear transformation with respect to a cone is defined by (1.1). The following result shows the importance of studying this property in dynamical systems.

Proposition 3.1. ([9], [19]) *Suppose L is a linear transformation on H and K be a proper cone in H. Then the following are equivalent:*

(a) $L \in \mathbf{Z}(K)$.
(b) $e^{-tL}(K) \subseteq K$ *for all* $t \geq 0$ *in* $\mathbb{R}$.
(c) The trajectory of the dynamical system $\dot{x} + L(x) = 0$ *with any initial point in* K *stays in* K.

As noted in the Introduction, when $\mathcal{C} = \mathbb{R}^n_+$, $A \in \mathbf{Z}(\mathcal{C})$ if and only if all the off-diagonal entries of A are nonpositive. Here is a non-trivial example.

Example 3.2. Consider $\mathbb{R}^n$ with $n > 1$ and write any element in the form $x = \left[t, u^T\right]^T$, where $t \in \mathbb{R}$ and $u \in \mathbb{R}^{n-1}$. Let

$$\mathcal{C} = \mathcal{L}^n_+ := \left\{ x = \begin{bmatrix} t \\ u \end{bmatrix} : t \geq ||u|| \right\}.$$

This is a symmetric cone (that is, a self-dual, homogeneous, closed convex cone) in the Jordan spin algebra $\mathcal{L}^n$, called the Lorentz cone (or the second order cone or the ice-cream cone). For this (proper) cone, the copositive cone $\mathcal{E}$, the completely positive cone $\mathcal{K}$, and $\mathbf{Z}(\mathcal{L}^n_+)$ are described below.

Let $J := \text{diag}(1, -1, -1, \ldots, -1) \in \mathbb{R}^{n \times n}$ and $A \in \mathbb{R}^{n \times n}$. Then

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(*i*) $A \in \mathcal{E}$ if and only if $A - \mu J$ is positive semidefinite for some $\mu \geq 0$, see [15], Lemma 2.2;

(*ii*) $A \in \mathcal{K}$ if and only if A is a (finite) sum of matrices of the form $\begin{bmatrix} t^2 & tu^T \\ tu & uu^T \end{bmatrix}$, where $t \in \mathbb{R}$, $u \in \mathbb{R}^{n-1}$ with $t \geq ||u||$;

(*iii*) $A \in \mathbf{Z}(\mathcal{L}^n_+)$ if and only if $\alpha J - (JA + A^T J)$ is positive semidefinite for some $\alpha \in \mathbb{R}$, see Example 4 in [13];

(*iv*) $A, -A \in \mathbf{Z}(\mathcal{L}^n_+)$ if and only if $JA + A^T J = \alpha J$ for some $\alpha \in \mathbb{R}$.

Note: Item (*ii*) follows from the definition and Item (*iv*) is a simple consequence of (*iii*).

We now come to one of the main results of the paper. Before stating this, we observe that for any $A \in \mathbb{R}^{n\times n}$,

$$L_A \in \mathbf{Z}(\mathcal{K}) \Leftrightarrow L_{A^T} \in \mathbf{Z}(\mathcal{E}).$$

This follows easily as $(L_A)^* = L_{A^T}$ and $\mathcal{E}^* = \mathcal{K}$ in $\mathcal{S}^n$.

THEOREM 3.3. *For any closed convex cone $\mathcal{C}$ in $\mathbb{R}^n$,*

$$A \in \mathbf{Z}(\mathcal{C}) \Rightarrow L_A \in \mathbf{Z}(\mathcal{K}).$$

The reverse implication holds under the following condition on $\mathcal{C}$:

$$\mathcal{C} \ni u \perp v \in \mathcal{C}^*,\ u \neq 0 \ \Rightarrow\ \exists\ Y \in \mathcal{E} \text{ such that } Yu = v. \tag{3.1}$$

Proof. Let $A \in \mathbf{Z}(\mathcal{C})$ and $\mathcal{K} \ni X \perp Y \in \mathcal{K}^* = \mathcal{E}$. Writing $X = \sum_1^N u_i u_i^T$, with $u_i \in \mathcal{C}$ for all i, we have

$$0 = \langle X, Y \rangle = trace(XY) = \sum_1^N u_i^T Y u_i.$$

This implies, as Y is copositive on $\mathcal{C}$, $u_i^T Y u_i = 0$ for all i. From Proposition 2.1, $v_i := Y u_i \in \mathcal{C}^*$. So, for all i, $\mathcal{C} \ni u_i \perp v_i \in \mathcal{C}^*$. As $A \in \mathbf{Z}(\mathcal{C})$, $v_i^T A u_i = \langle A u_i, v_i \rangle \leq 0$ for all i. Now,

$$\langle L_A(X), Y \rangle = 2\ trace(AXY) = 2\sum_1^N trace(A u_i u_i^T Y) = 2\sum_1^N v_i^T A u_i \leq 0.$$

Thus, $L_A \in \mathbf{Z}(\mathcal{K})$.

Now to see the reverse implication, assume that $\mathcal{C}$ satisfies (3.1), $L_A \in \mathbf{Z}(\mathcal{K})$, and let $u \in \mathcal{C}$, $v \in \mathcal{C}^*$ and $\langle u, v \rangle = 0$. We have to show that $\langle Au, v \rangle \leq 0$. We may assume

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without loss of generality, that u is nonzero. Then there exists a $Y \in \mathcal{E}$ such that $Yu = v$. We have

$$X = uu^T \in \mathcal{K},\, Y \in \mathcal{K}^* = \mathcal{E}, \quad \text{and} \quad \langle X, Y\rangle = u^T Y u = u^T v = 0.$$

Hence $trace(L_A(X)Y) \leq 0$. This leads to $trace(AXY) \leq 0$ and $\langle Au, v\rangle = v^T Au = trace(AXY) \leq 0$. Thus, $A \in \mathbf{Z}(\mathcal{C})$. ◻

Example 3.4. When $\mathcal{C} = \mathbb{R}^n$, we have $\mathcal{C}^* = \{0\}$ and $\mathcal{K} = \mathcal{S}^n_+$. In this case, every matrix $A \in \mathbb{R}^{n\times n}$ belongs to $\mathbf{Z}(\mathcal{C})$ and consequently, for any $A \in \mathbb{R}^{n\times n}$, both L_A and $-L_A = L_{-A}$ belong to $\mathbf{Z}(\mathcal{S}^n_+)$. Hence,

$$\mathcal{S}^n_+ \ni X \perp Y \in \mathcal{S}^n_+ \Rightarrow \langle L_A(X), Y\rangle = 0.$$

(This motivates the definition of Lyapunov-like transformations, see Section 4.)

The following example shows that the reverse implication in Theorem 3.3 may not always hold.

Example 3.5. In $\mathbb{R}^2$, let $\mathcal{C}$ be the closed upper half-plane. In this case, $\mathcal{C}^*$ is the nonnegative y-axis and $\mathcal{E} = \mathcal{S}^2_+$; hence $\mathcal{K} = \mathcal{S}^2_+$. Now consider a matrix $A \in \mathbb{R}^{2\times 2}$ whose $(2, 1)$ entry is one. Then for the standard coordinate vectors e_1 and e_2, we have $\mathcal{C} \ni e_1 \perp e_2 \in \mathcal{C}^*$. However, $\langle Ae_1, e_2\rangle = 1$. Therefore, $A \notin \mathbf{Z}(\mathcal{C})$ while $L_A \in \mathbf{Z}(\mathcal{K})$.

Corollary 3.6. *Suppose $\mathcal{C}$ is a closed convex pointed cone in $\mathbb{R}^n$. Then*

$$A \in \mathbf{Z}(\mathcal{C}) \Leftrightarrow L_A \in \mathbf{Z}(\mathcal{K}).$$

Proof. We show that the given $\mathcal{C}$ satisfies condition (3.1) and quote the previous theorem. To this end, let $\mathcal{C} \ni u \perp v \in \mathcal{C}^*$, $u \neq 0$. Since $\mathcal{C}$ is pointed, $\mathcal{C}^*$ has nonempty interior. Let $w \in (\mathcal{C}^*)^\circ$ such that $w^T u = 1$. Define $Y := vw^T + wv^T$. Clearly $Y \in \mathcal{S}^n$ and for all $x \in \mathcal{C}$,

$$x^T Y x = x^T v\ w^T x + x^T w\ v^T x \geq 0;$$

thus, $Y \in \mathcal{E}$. Also, $Yu = v\, w^T u + w\, v^T u = v$. ◻

Our next objective is to present a result that extends the results of Lyapunov and Moldovan-Gowda. First, we recall a basic result on $\mathbf{Z}$-transformations.

Proposition 3.7. ([3], [13]) *Suppose L is a linear transformation on H, K a proper cone in H, and $L \in \mathbf{Z}(K)$. Then the following are equivalent:*

(1) *There exists $d \in K^\circ$ such that $L(d) \in K^\circ$.*
(2) *L is invertible with $L^{-1}(K^\circ) \subseteq K^\circ$.*

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(3) *L is positive stable.*
(4) *$L + tI$ is invertible for all $t \in [0, \infty)$.*
(5) *All real eigenvalues of L are positive.*
(6) *There exists $e \in (K^*)^\circ$ such that $L^*(e) \in (K^*)^\circ$.*

Moreover, when $H = \mathbb{R}^n$, $K = \mathbb{R}^n_+$, and $L = A$, the above properties (for a $\mathbf{Z}$-matrix A) are further equivalent to

(7) *A is a $\mathbf{P}$-matrix, that is, all principal minors of A are positive.*
(8) *There exists a positive definite diagonal matrix D in $\mathcal{S}^n$ such that $AD+DA^T$ is positive definite.*

As a consequence, we have the following.

Theorem 3.8. *Suppose $\mathcal{C}$ is a closed convex cone in $\mathbb{R}^n$ such that $\mathcal{C} - \mathcal{C} = \mathbb{R}^n$. Then the following are equivalent:*

(a) *A is positive stable.*
(b) *The system $\dot{x}+Ax = 0$ is asymptotically stable in $\mathcal{C}$ (that is, from any starting point in $\mathcal{C}$, its trajectory converges to the origin).*

When $A \in \mathbf{Z}(\mathcal{C})$, these are further equivalent to:

(c) *There exists $D \in \mathcal{K}^\circ$ such that $AD + DA^T \in \mathcal{K}^\circ$.*
(d) *There exists $D \in \mathcal{E}^\circ$ such that $A^TD + DA \in \mathcal{E}^o$.*

If, in addition, $\mathcal{C}$ is also proper, then the above conditions are equivalent to

(e) *There exists $d \in \mathcal{C}^\circ$ such that $Ad \in \mathcal{C}^\circ$.*

Proof. The proof of $(a) \Rightarrow (b)$ is standard, see the proof of Theorem 3.1 in [7]. The proof of $(b) \Rightarrow (a)$ is as in [7], except that the starting point should be allowed to vary in the interior of $\mathcal{C}$ (which is nonempty because $\mathcal{C} - \mathcal{C} = \mathbb{R}^n$).

Now assume that $A \in \mathbf{Z}(\mathcal{C})$. Since $\mathcal{C} - \mathcal{C} = \mathbb{R}^n$, by Proposition 2.2, both $\mathcal{E}$ and $\mathcal{K}$ are proper; hence they have nonempty interiors. Also, since $A \in \mathbf{Z}(\mathcal{C})$, by Theorem 3.3, $L_A \in \mathbf{Z}(\mathcal{K})$ and $L_{A^T} \in \mathbf{Z}(\mathcal{E})$. Since the eigenvalues of L_A on $\mathcal{S}^n$ are of the form $\lambda+\overline{\mu}$, where λ and μ are eigenvalues of A, see [26], it follows that A is positive stable if and only if L_A is positive stable. Now the equivalence of Items $(a), (c)$, and (d) follows from the previous result applied to L_A on $\mathcal{K}$.

When $\mathcal{C}$ is proper, the previous result can be applied to A and $\mathcal{C}$ (note that $\mathbf{Z}(\mathcal{C})$) to get the equivalence of (a) and (e). ☐

Remark 3.9. (i) In Item (d) of the above theorem, the matrix $D \in \mathcal{E}^o$ is necessarily strictly copositive on $\mathcal{C}$: For any nonzero $u \in \mathcal{C}$, $u^T(D-\varepsilon I)u \geq 0$ for small $\varepsilon > 0$.

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(ii) When $\mathcal{C} - \mathcal{C} = \mathbb{R}^n$ and $A \in \mathbf{Z}(\mathcal{C})$, the equation

$$A^T X + XA = Y, \quad Y \text{ strictly copositive on } \mathcal{C}$$

has a unique solution X which is also strictly copositive on $\mathcal{C}$ for some Y (equivalently for all Y) if and only if A is positive stable. This follows from Items (1) and (2) in Proposition 3.7 with $K = \mathcal{E}$ and $L = L_{A^T}$. When $\mathcal{C}$ is proper and A is positive stable, this unique solution is given by

$$X = \int_0^\infty e^{-tA^T} Y e^{-tA} dt.$$

(iii) The results of Lyapunov and Moldovan-Gowda (stated in the Introduction) follow by taking $\mathcal{C} = \mathbb{R}^n$ and $\mathcal{C} = \mathbb{R}^n_+$ respectively.

(iv) Suppose $\mathcal{C}$ is proper. When the conditions of the above result are in place, (any) trajectory of the system $\dot{x} + Ax = 0$ from any starting point in $\mathcal{C}$ stays in $\mathcal{C}$ and converges to the origin as $t \to \infty$. In this setting, $f(x) := d^T x$ (with d as in Item (e)) acts as a linear Lyapunov function and $g(x) := x^T D x$ (with D as in Item (d)) acts as a quadratic Lyapunov function.

(v) Instead of our condition $\mathcal{C} - \mathcal{C} = \mathbb{R}^n$ in Theorem 3.8, Stern [23] assumes that $\mathcal{C}$ in $\mathbb{R}^n$ satisfies $\mathcal{C} \cap -\mathcal{C} = \{0\}$. He proves that when $A \in \mathbf{Z}(\mathcal{C})$ and $\mathcal{C} \cap -\mathcal{C} = \{0\}$, the system $\dot{x} + Ax = 0$ is asymptotically stable if and only if the following implication holds:

$$[x \in \mathcal{C}, \ -Ax \in \mathcal{C}] \Rightarrow x = 0.$$

It may be noted that if, in addition, $\mathcal{C} - \mathcal{C} = \mathbb{R}^n$, that is, if $\mathcal{C}$ is proper, then the above condition is equivalent to Item (e) in Theorem 3.8.

The following result (partially) answers a question of Bundfuss and Dür [5]:

Corollary 3.10. *Suppose $\mathcal{C} = M(\mathbb{R}^m_+)$ is a polyhedral cone in $\mathbb{R}^n$, where M is an $n \times m$-matrix. Assume that M has rank n and $A \in \mathbf{Z}(\mathcal{C})$. Then there exists a symmetric matrix D such that D and $A^T D + DA$ are strictly copositive on $\mathcal{C}$ if and only if A is positive stable.*

4. Lyapunov-like transformations. Motivated by Example 3.4, a linear transformation L on H is said to be *Lyapunov-like* with respect to a closed convex cone K in H if both L and $-L$ have the $\mathbf{Z}$-property on K. This simply means that

$$K \ni x \perp y \in K^* \Rightarrow \langle L(x), y \rangle = 0.$$

For any matrix $A \in \mathbb{R}^{n \times n}$, the Lyapunov transformation L_A is Lyapunov-like with respect to $\mathcal{S}^n_+$ in $\mathcal{S}^n$ (see Example 3.4). In the setting of the cone $\mathbb{R}^n_+$ in $\mathbb{R}^n$, Lyapunov-

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like matrices are just diagonal matrices. Because of Proposition 3.1, Lyapunov-like transformations are intimately connected to automorphism groups and Lie algebras.

In the rest of this section, we assume that $\mathcal{C}$ is a *proper cone* in $\mathbb{R}^n$ and use the notation $\mathcal{B}(\mathcal{S}^n, \mathcal{S}^n)$ to denote the set of all (bounded) linear transformations on $\mathcal{S}^n$. We consider two automorphism groups:

- $Aut(\mathcal{C}) := \{A \in \mathbb{R}^{n\times n} : A(\mathcal{C}) = \mathcal{C}\}$.
- $Aut(\mathcal{K}) := \{L \in \mathcal{B}(\mathcal{S}^n, \mathcal{S}^n) : L(\mathcal{K}) = \mathcal{K}\}$.

(Note that elements of these groups are necessarily invertible, as $\mathcal{C}$ and $\mathcal{K}$ have nonempty interiors.) Since these groups can be regarded as matrix groups, the corresponding Lie algebras are given, see [2], by:

- $\mathrm{Lie}(Aut(\mathcal{C})) := \{A \in \mathbb{R}^{n\times n} : e^{tA} \in Aut(\mathcal{C}) \;\forall\, t \in \mathbb{R}\}$.
- $\mathrm{Lie}(Aut(\mathcal{K})) := \{L \in \mathcal{B}(\mathcal{S}^n, \mathcal{S}^n) : e^{tL} \in Aut(\mathcal{K}) \;\forall\, t \in \mathbb{R}\}$.

Note that in these Lie algebras, the Lie bracket is the one induced by the (associative) product of matrices/transformations: $[A, B] = AB - BA$, etc.
In view of Proposition 3.1, we have

$$A, -A \in \mathbf{Z}(\mathcal{C}) \Leftrightarrow A \in Lie(Aut(\mathcal{C})) \quad \text{and} \quad L, -L \in \mathbf{Z}(\mathcal{K})) \Leftrightarrow L \in Lie(Aut(\mathcal{K})).$$

THEOREM 4.1. *For any proper cone $\mathcal{C}$ in $\mathbb{R}^n$, the mapping $A \mapsto L_A$ is an injective Lie algebra homomorphism from Lie($Aut(\mathcal{C})$) to Lie($Aut(\mathcal{K})$).*

Proof. For $A \in Lie(Aut(\mathcal{C}))$, we have $A, -A \in \mathbf{Z}(\mathcal{C})$. By Theorem 3.3, $L_A, -L_A \in \mathbf{Z}(\mathcal{K})$, that is, $L_A \in Lie(Aut(\mathcal{K}))$. Clearly, the mapping $A \mapsto L_A$ is linear. That it is a Lie algebra homomorphism follows from the identity $L_{[A,B]} = [L_A, L_B]$. To show that this is injective, suppose $L_A = 0$, that is, $AX + XA^T = 0$ for all $X \in \mathcal{S}^n$. By taking $X = I$ (Identity), we see that $A + A^T = 0$, that is, A is skew-symmetric. By taking X to be a diagonal matrix with distinct elements, we see that $A = 0$. ∎

5. The Z-property of Stein transformations. Recall that for a matrix $A \in \mathbb{R}^{n\times n}$, the corresponding Stein transformation S_A is defined on $\mathcal{S}^n$ by $S_A(X) := X - AXA^T$. We also recall that $\Pi(\mathcal{C}) := \{A \in \mathbb{R}^{n\times n} : A(\mathcal{C}) \subseteq \mathcal{C}\}$. As in the case of Lyapunov transformations, we have $S_A \in \mathbf{Z}(\mathcal{K}) \Leftrightarrow S_{A^T} \in \mathbf{Z}(\mathcal{E})$.

THEOREM 5.1. *Let $\mathcal{C}$ be any closed convex cone in $\mathbb{R}^n$. Then*

$$\pm A \in \Pi(\mathcal{C}) \Rightarrow S_A \in \mathbf{Z}(\mathcal{K}).$$

Proof. Without loss of generality, let $A \in \Pi(\mathcal{C})$. Let $X = \sum_1^N u_i u_i^T \in \mathcal{K}$, $Y \in \mathcal{K}^* = \mathcal{E}$, and $\langle X, Y\rangle = 0$, where $u_i \in \mathcal{C}$ for all i. Then $w_i := Au_i \in \mathcal{C}$ for all i.

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Now, as Y is copositive on $\mathcal{C}$,

$$trace(AXA^TY) = \sum_1^N w_i^T Y w_i \geq 0.$$

Hence

$$\langle S_A(X), Y\rangle = \langle X, Y\rangle - \langle AXA^T, Y\rangle = -trace(AXA^TY) \leq 0.$$

This proves that $S_A \in \mathbf{Z}(\mathcal{K})$. ☐

Example 5.2. By taking $\mathcal{C} = \mathbb{R}^n$ in the above theorem, we see that for any matrix $A \in \mathbb{R}^{n\times n}$, $S_A \in \mathbf{Z}(\mathcal{S}^n_+)$. Now, let $\mathcal{C}$ be the closed upper half-plane in $\mathbb{R}^2$ so that $\mathcal{K} = \mathcal{E} = \mathcal{S}^2_+$. Then for any 2×2 real matrix A, $S_A \in \mathbf{Z}(\mathcal{S}^2_+)$, while it is easy to construct a 2×2 real matrix which is not in $\Pi(\mathcal{C})$. Thus, the converse in the above theorem does not hold.

Analogous to Theorem 3.8, we have

Theorem 5.3. *Suppose $\mathcal{C}$ is a closed convex cone in $\mathbb{R}^n$ such that $\mathcal{C} - \mathcal{C} = \mathbb{R}^n$. Then the following are equivalent:*

- (a) *A is Schur stable.*
- (b) *The system $x(k+1) = Ax(k)$, $k = 0, 1, 2, \ldots$ is asymptotically stable in $\mathcal{C}$ (that is, from any starting point in $\mathcal{C}$, its trajectory converges to the origin).*

When $\pm A \in \Pi(\mathcal{C})$, these are further equivalent to:

- (c) *There exists $D \in \mathcal{K}^\circ$ such that $S_A(D) \in \mathcal{K}^\circ$.*
- (d) *There exists $D \in \mathcal{E}^\circ$ such that $S_{A^T}(D) \in \mathcal{E}^o$.*

Note: S_A is positive stable if and only if A is Schur stable, see [10].

6. Cone spectrum, copositivity, and Z-transformations. In this section, we relate the $\mathbf{Z}$-property, copositivity, and cone spectrum. Let L be a linear transformation on H and K be a *nonzero* closed convex cone in H. Then the *cone spectrum* [21] of L with respect to K is the set of all real λ for which there is an x such that

$$0 \neq x \in K,\ L(x) - \lambda x \in K^* \quad \text{and} \quad \langle x, L(x) - \lambda x\rangle = 0.$$

We denote this set by $\sigma(L, K)$.

The following result gives the nonemptyness of the cone spectrum.

Proposition 6.1. *Let K be a nonzero closed convex cone in H and L be linear on H.*

- (i) *If K is proper, then $\sigma(L, K) \neq \emptyset$.*

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(*ii*) *If L is self-adjoint, then $\sigma(L,K) \neq \emptyset$; In fact,*

$$\lambda^* := \min\{\langle L(x), x\rangle : x \in K, \, ||x|| = 1\} \in \sigma(L,K).$$

Proof. The proof of (*i*) is given in [20], Corollary 2.1. While a proof of (*ii*) is given in [14], Corollary 2.4 and [21], Example 1, we offer a direct and simple proof. Let $\lambda^* = \langle L(x^*), x^*\rangle$, where $x^* \in K$ with $||x^*|| = 1$. Define $S := L - \lambda^* I$. Then for all $0 \neq x \in K$, we have

$$\langle S(x), x\rangle = ||x||^2 \left\{ \langle L(\frac{x}{||x||}), \frac{x}{||x||}\rangle - \lambda^* \right\} \geq 0.$$

This means that the self-adjoint transformation S is copositive on K. Since $\langle S(x^*), x^*\rangle = 0$, we have, from Proposition 2.1, $y^* = S(x^*) \in K^*$. Thus, we have

$$x^* \in K, \; y^* := L(x^*) - \lambda^* x^* \in K^* \quad \text{and} \quad \langle x^*, y^*\rangle = 0.$$

Hence, $\lambda^* \in \sigma(L,K)$. □

The above result, together with the observation that every $\lambda \in \sigma(L,K)$ is of the form $\lambda = \frac{\langle L(x),x\rangle}{||x||^2}$ for some nonzero $x \in K$, gives the following:

Corollary 6.2. *Suppose $\sigma(L,K)$ is nonempty. If L is copositive on K, then $\lambda \geq 0$ for all $\lambda \in \sigma(L,K)$. The converse holds when L is self-adjoint.*

In what follows, we write $\sigma(L)$ for the spectrum of L.

Theorem 6.3. *Suppose K is proper and $L \in \mathbf{Z}(K)$. Then*

$$\min \, Re\,\sigma(L) \in \sigma(L,K) \subseteq \sigma(L).$$

Proof. Let $\mu^* := \min\{Re\,\lambda : \lambda \in \sigma(L)\}$. Since K is proper and $L \in \mathbf{Z}(K)$, by Theorem 6 in [19], there exists a nonzero $u \in K$ such that $L(u) = \mu^* u$. Clearly, $\mu^* \in \sigma(L,K)$. This proves the first part of the inclusion. The second part is proved in Theorem 9, [27]; Here is its short proof: Let $\mu \in \sigma(L,K)$ so that for some nonzero $x \in K$, $y = L(x) - \mu x \in K^*$ and $\langle x, y\rangle = 0$. As $L - \mu I \in \mathbf{Z}(K)$, $\langle (L-\mu I)x, y\rangle \leq 0$. This leads to $y = 0$, that is, $L(x) = \mu x$ proving $\mu \in \sigma(L)$. □

Remark 6.4. In Theorem 3.8, the equivalence of (*a*) and (*d*) was proved under the assumptions that $\mathcal{C} - \mathcal{C} = \mathbb{R}^n$ and $A \in \mathbf{Z}(\mathcal{C})$. When $\mathcal{C}$ is proper and $A \in \mathbf{Z}(\mathcal{C})$, the following simple proof (an adaptation of the standard argument) can be given. We only prove the implication $(d) \Rightarrow (a)$. Assume that for some (symmetric) D that is strictly copositive on $\mathcal{C}$, $A^TD + DA = Y$ is also strictly copositive on $\mathcal{C}$. Let $\mu^* = \min \, Re\,\sigma(A)$ so that by Theorem 6 in [19], there is a nonzero $u \in \mathcal{C}$ such that $Au = \mu^* u$. Then $0 < u^TYu = u^T(A^TD + DA)u = 2\mu^* u^TDu$. Since u^TDu is also positive, we see that $\mu^* > 0$. This means that A is positive stable.

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The following result extends a result of J. Tao [25] proved in the setting of symmetric cones.

COROLLARY 6.5. *Let K be proper, $L \in \mathbf{Z}(K)$ and copositive on K. Then L is semi-positive stable (that is, all eigenvalues of L lie in the closed right-half plane). If, in addition, L is self-adjoint or K is self-dual, then L is monotone.*

Proof. That L is semi-positive stable follows from Corollary 6.2 and Theorem 6.3. If L is self-adjoint, then all eigenvalues of L are nonnegative, and hence L is monotone. When K is self dual, $L \in \mathbf{Z}(K) \Leftrightarrow L^* \in \mathbf{Z}(K^*) \Leftrightarrow L^* \in \mathbf{Z}(K)$. In this case, $L + L^*$ is self-adjoint, copositive on K, and belongs to $\mathbf{Z}(K)$. By the previous case, $L + L^*$ (and hence L) is monotone. □

A concluding remark. In a follow up paper [12], it is shown that the mapping $A \mapsto L_A$ in Theorem 4.1 is actually a bijection.

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