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# POSTLIE ALGEBRA STRUCTURES ON THE LIE ALGEBRA SL(2, ℂ)*

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**Abstract.** The PostLie algebra is an enriched structure of the Lie algebra that has recently arisen from operadic study. It is closely related to pre-Lie algebra, Rota-Baxter algebra, dendriform trialgebra, modified classical Yang-Baxter equations and integrable systems. This paper gives a complete classification of PostLie algebra structures on the Lie algebra sl(2, ℂ) up to isomorphism. The classification problem is first reduced to solving an equation of $3 \times 3$ matrices. Then the latter problem is solved by making use of the classification of complex symmetric matrices up to the congruent action of orthogonal groups.

**Key words.** Lie algebra, PostLie algebra, Symmetric matrices, Classification.

**AMS subject classifications.** 17A30, 17A42, 17B60, 18D50.

**1. Introduction.** We begin with recalling background on PostLie algebras [7].

DEFINITION 1.1.

1. A *(left) PostLie* ℂ*-algebra* is a ℂ-vector space $L$ with two binary operations $\circ$ and $[\, ,\, ]$ which satisfy the relations:

$$[x, y] = -[y, x], \tag{1.1}$$

$$[[x, y], z] + [[z, x], y] + [[y, z], x] = 0, \tag{1.2}$$

$$z \circ (y \circ x) - y \circ (z \circ x) + (y \circ z) \circ x - (z \circ y) \circ x + [y, z] \circ x = 0, \tag{1.3}$$

$$z \circ [x, y] - [z \circ x, y] - [x, z \circ y] = 0, \tag{1.4}$$

for all $x, y \in L$.

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2. Let $(\mathfrak{G}(L), [\,,\,])$ denote the Lie algebra defined by Eq. (1.1) and Eq. (1.2). Call $(L, [\,,\,], \circ)$ a *PostLie algebra on* $(\mathfrak{G}(L), [\,,\,])$.
3. Let $(\mathfrak{g}, [\,,\,])$ be a Lie algebra. Two PostLie algebras $(\mathfrak{g}, [\,,\,], \circ)$ and $\mathfrak{g}, [\,,\,], \star)$ on the Lie algebra $\mathfrak{g}$ are called *isomorphic on the Lie algebra* $(\mathfrak{g}, [\,,\,])$ if there is an automorphism $f$ of the Lie algebra $(\mathfrak{g}, [\,,\,])$ such that

$$f(x \circ y) = f(x) \star f(y), \quad \forall x, y \in \mathfrak{g}.$$

The concept of a PostLie algebra was recently introduced by Vallette from an operadic study [7]. It is closely related to pre-Lie algebra, Rota-Baxter algebra, dendriform trialgebra and modified classical Yang-Baxter equation, and has found applications to integrable systems [2]. For example, any PostLie algebra gives a solution of the modified classical Yang-Baxter equation introduced by Semenov-Tian-Shansky in [6] and a natural triple Lie algebra constructing a self-dual nonabelian generalized Lax pair. Further the corresponding operad plays the role of "splitting" a binary quadratic operad into three pieces in terms of Manin black products [1, 8].

It is important to give examples before attempting to achieve certain classification in an algebraic structure. Considering the great challenge in giving a complete classification for the well-known algebraic structures such as Lie algebras and associative algebras, it is reasonable to begin with studying the classification of PostLie algebras on some well-behaved Lie algebras, such as complex semisimple Lie algebras. As applications, the resulted PostLie algebras with explicit structure would be applied directly to the above mentioned fields. Thus, as a first step and as a guide for further investigations, we determine all isomorphic classes of PostLie algebra structures on the Lie algebra $(\mathrm{sl}(2,\mathbb{C}), [\,,\,])$.

Let

$$e_1 := \frac{1}{2}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad e_2 := \frac{1}{2\sqrt{-1}}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad e_3 := \frac{1}{2\sqrt{-1}}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

They form a $\mathbb{C}$-linear basis of $\mathrm{sl}(2,\mathbb{C})$ and determine the Lie algebra $(\mathrm{sl}(2,\mathbb{C}), [\,,\,])$ through the relations

$$[e_2, e_3] = e_1, \quad [e_3, e_1] = e_2, \quad [e_1, e_2] = e_3.$$

Our main result of this paper is the following classification theorem.

THEOREM 1.2. *The following is a complete set of representatives for the isomorphic classes of PostLie algebras* $(\mathrm{sl}(2,\mathbb{C}), [\,,\,], \circ)$ *on the Lie algebra* $(\mathrm{sl}(2,\mathbb{C}), [\,,\,])$.

1. $e_i \circ e_j = 0, \quad i, j = 1, 2, 3;$
2. $e_i \circ e_j = [-e_i, e_j], \quad i, j = 1, 2, 3;$

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*3.* $e_1 \circ e_i = [-e_1, e_i]$, $e_2 \circ e_i = [-\frac{1+\sqrt{-1}}{2}e_2 + \frac{\sqrt{-1}-1}{2}e_3, e_i]$,
$e_3 \circ e_i = [-\frac{1+\sqrt{-1}}{2}e_2 + \frac{\sqrt{-1}-1}{2}e_3, e_i]$, $i = 1, 2, 3$;

*4.* $e_1 \circ e_i = [(\sqrt{-1} - \frac{1}{2})e_1 + (1 - \frac{\sqrt{-1}}{2})e_2, e_i]$,
$e_2 \circ e_i = [(1 + \frac{\sqrt{-1}}{2})e_1 - (\sqrt{-1} + \frac{1}{2})e_2, e_i]$, $e_3 \circ e_i = 0$, $i = 1, 2, 3$;

*5.* $e_1 \circ e_i = [ke_1, e_i]$, $e_2 \circ e_i = [-\frac{1}{2}e_2 + \frac{\sqrt{-1}}{2}e_3, e_i]$, $e_3 \circ e_i = [-\frac{\sqrt{-1}}{2}e_2 - \frac{1}{2}e_3, e_i]$,
$i = 1, 2, 3$, $k \in \mathbb{C}$.

As pointed out in [2], such a classification problem is related to the classification of the modified classical Yang-Baxter equation [6]. While the classification in [6] works for the so-called graded r-matrices in a finite-dimensional semisimple Lie algebra, in terms of extensions of linear operators associated to the parabolic subalgebras, our classification in the case that we are considering is given without any constraint graded conditions and is more precise in the sense that the structural constants are spelled out explicitly.

Our proof of the theorem consists of two steps:

*Step 1.* Give a one-one correspondence from the isomorphic classes of PostLie algebra structures on $(\mathrm{sl}(2, \mathbb{C}), [\,,\,])$ to the congruent classes of solutions of the matrix equation Eq. (2.3). This will be carried out in Section 2.

*Step 2.* Classify the congruent classes of solutions of the matrix equation Eq. (2.3). This will be carried out in Section 3.
For this purpose, we make use of a result on the canonical forms for complex symmetric matrices under the congruent action of $\mathrm{SO}(3, \mathbb{C})$ (Proposition 3.2). When a solution $A$ of Eq. (2.3) has full rank, it can be shown that $A$ is symmetric. Thus, we only need to check against Eq. (2.3) the complex symmetric matrices with full rank which are in the above canonical forms. When a solution $A$ does not have full rank, $A$ is no longer symmetric. Then we try to relate Eq. (2.3) of $A$ to equations of various symmetrizations of $A$, such as $A'A$ and $A' + A$ so that we can still apply Proposition 3.2. This strategy turns out to work quite nicely.

## 2. A matrix equation from PostLie algebras.

In this section, we carry out the first step in establishing Theorem 1.2 by proving Theorem 2.4. We begin with recalling two results on PostLie algebras.

Lemma 2.1. ([7]) *Let $(L, [\,,\,], \circ)$ be a PostLie algebra. Then the binary operation given by*

$$\{x, y\} := x \circ y - y \circ x + [x, y], \;\; \forall x, y \in L,$$

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*defines a Lie algebra.*

Proposition 2.2. ([2]) *Let* $(\mathfrak{g},[\,,\,])$ *be a semisimple Lie algebra. Then any PostLie algebra structure* $(\mathfrak{g},[\,,\,],\circ)$ *(on* $(\mathfrak{g},[\,,\,])$*) is given by*

$$x\circ y=[f(x),y],\ \ \forall x,y\in\mathfrak{g},$$

*where* $f:\mathfrak{g}\to\mathfrak{g}$ *is a linear map satisfying*

$$[f(x),f(y)]=f([f(x),y]+[x,f(y)]+[x,y]),\ \ \forall x,y\in\mathfrak{g}. \tag{2.1}$$

Remark 2.3.

1. A linear map $f$ satisfying Eq. (2.1) is called a *Rota-Baxter operator of weight* 1 [4].
2. We also note that Eq. (2.1) is equivalent to the condition that $[f(x),f(y)]=f(\{x,y\})$, for any $x,y\in\mathfrak{g}$, that is, $f$ is a homomorphism between the two Lie algebras $(\mathfrak{g},[\,,\,])$ and $(\mathfrak{g},\{,\})$ from Lemma 2.1.

Theorem 2.4. *Let* $\circ$ *be a binary operation on* $\mathrm{sl}(2,\mathbb{C})$*. The following statements are equivalent.*

1. *The triple* $(\mathrm{sl}(2,\mathbb{C}),[\,,\,],\circ)$ *is a PostLie algebra on* $(\mathrm{sl}(2,\mathbb{C}),[\,,\,])$*;*
2. *The operation* $\circ$ *is given by*
$$x\circ y=[f(x),y],\quad \forall x,y\in\mathrm{sl}(2,\mathbb{C}), \tag{2.2}$$
*where* $f:\mathrm{sl}(2,\mathbb{C})\to\mathrm{sl}(2,\mathbb{C})$ *is a linear map satisfying Eq. (2.1);*
3. *The operation* $\circ$ *is given by*
$$x\circ y=[f(x),y],\quad \forall x,y\in\mathrm{sl}(2,\mathbb{C}),$$
*where* $f:\mathrm{sl}(2,\mathbb{C})\to\mathrm{sl}(2,\mathbb{C})$ *is a linear map whose matrix* $A$ *with respect to the basis* $\{e_1,e_2,e_3\}$ *satisfies*
$$A'((\mathrm{tr}(A)+1)I_3-A)=A^*. \tag{2.3}$$
*Here* $A'$ *is the transpose matrix of* $A$ *and* $A^*$ *is the adjugate matrix of* $A$.

*Furthermore, the linear map* $f$ *(and hence, its matrix* $A$*) in Item (3) is unique for a given* $\circ$.

Because of their uniqueness, the linear map $f$ (resp., its matrix $A$) in the theorem is called the *linear map* (resp., the *matrix) of the PostLie algebra* $(\mathrm{sl}(2,\mathbb{C}),[\,,\,],\circ)$ and is denoted $f_\circ$ (resp., $A_\circ$).

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*Proof.* The part (1) $\Longrightarrow$ (2) is a special case of Proposition 2.2.

(2) $\Longrightarrow$ (1): Applying Eq. (2.1) to $z \in \mathrm{sl}(2,\mathbb{C})$ gives Eq. (1.3). Since the left multiplication $y \mapsto [f(x), y], y \in \mathrm{sl}(2,\mathbb{C})$ is a derivation, from Eq. (2.2) we obtain Eq. (1.4).

(2) $\Longrightarrow$ (3): Set

$$f(e_i) = \sum_{j=1}^{3} a_{ij} e_j, \quad \text{where } A = (a_{ij}), \quad a_{ij} \in \mathbb{C}, i, j = 1, 2, 3.$$

Substituting the above equation into Eq. (2.1), we obtain

$$f((a_{22}+a_{33}+1)e_1 - a_{21}e_2 - a_{31}e_3) = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} e_1 + \begin{vmatrix} a_{23} & a_{21} \\ a_{33} & a_{31} \end{vmatrix} e_2 + \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} e_3,$$

$$f(-a_{12}e_1 + (a_{11}+a_{33}+1)e_2 - a_{32}e_3) = \begin{vmatrix} a_{32} & a_{33} \\ a_{12} & a_{13} \end{vmatrix} e_1 + \begin{vmatrix} a_{33} & a_{31} \\ a_{13} & a_{11} \end{vmatrix} e_2 + \begin{vmatrix} a_{31} & a_{32} \\ a_{11} & a_{12} \end{vmatrix} e_3,$$

$$f(-a_{13}e_1 - a_{23}e_2 + (a_{11}+a_{22}+1)e_3) = \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} e_1 + \begin{vmatrix} a_{13} & a_{11} \\ a_{23} & a_{21} \end{vmatrix} e_2 + \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} e_3.$$

By the linearity of $f$, we can express these equations in the matrix equation

$$\begin{bmatrix} a_{22}+a_{33}+1 & -a_{21} & -a_{31} \\ -a_{12} & a_{11}+a_{33}+1 & -a_{32} \\ -a_{13} & -a_{23} & a_{11}+a_{22}+1 \end{bmatrix} \begin{bmatrix} f(e_1) \\ f(e_2) \\ f(e_3) \end{bmatrix} = (A^*)' \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}.$$

That is,

$$((\mathrm{tr}(A)+1)I_3 - A')A \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = (A^*)' \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}.$$

Since $\{e_1, e_2, e_3\}$ is a basis of $\mathrm{sl}(2,\mathbb{C})$, we obtain

$$((\mathrm{tr}(A)+1)I_3 - A')A = (A^*)'.$$

This is Eq. (2.3).

(3) $\Longrightarrow$ (2): Reversing the above calculation, from Eq. (2.3), we have

$$\begin{bmatrix} f(\{e_2,e_3\}) \\ f(\{e_3,e_1\}) \\ f(\{e_1,e_2\}) \end{bmatrix} = ((\mathrm{tr}(A)+1)I_3 - A')A \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = (A^*)' \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} [f(e_2), f(e_3)] \\ [f(e_3), f(e_1)] \\ [f(e_1), f(e_2)] \end{bmatrix}.$$

So by Remark 2.3.(2), Eq. (2.1) holds.

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Finally, suppose there are linear maps $f$ and $g$ on $\mathrm{sl}(2,\mathbb{C})$ such that $x\circ y = [f(x),y]=[g(x),y], x,y\in \mathrm{sl}(2,\mathbb{C})$. Since the center of $\mathrm{sl}(2,\mathbb{C})$ is zero, we have $f(x)=g(x), x\in \mathrm{sl}(2,\mathbb{C})$. This proves the uniqueness of $f$. ☐

We cite the following result [5] for later reference.

LEMMA 2.5. *A linear map on* $\mathrm{sl}(2,\mathbb{C})$ *is a Lie algebra automorphism if and only if its matrix with respect to the basis* $\{e_1,e_2,e_3\}$ *is in* $\mathrm{SO}(3,\mathbb{C})$.

THEOREM 2.6.

1. *Two PostLie algebras* $(\mathrm{sl}(2,\mathbb{C}),[\,,\,],\circ)$ *and* $(\mathrm{sl}(2,\mathbb{C}),[\,,\,],\star)$ *on the Lie algebra* $(\mathrm{sl}(2,\mathbb{C}),[\,,\,])$ *are isomorphic if and only if their matrices* $A_\circ$ *and* $A_\star$ *are congruent under* $\mathrm{SO}(3,\mathbb{C})$*, that is, there is* $T\in\mathrm{SO}(3,\mathbb{C})$ *such that* $A_\star = T'A_\circ T$.
2. *If* $A$ *is the matrix of a PostLie algebra, then all the matrices in its congruent class under* $\mathrm{SO}(3,\mathbb{C})$ *are matrices of PostLie algebras. Thus,* $\mathrm{SO}(3,\mathbb{C})$ *acts on the matrices of PostLie algebras on* $(\mathrm{sl}(2,\mathbb{C}),[\,,\,])$.
3. *The map that sends a solution of Eq. (2.3)to its corresponding PostLie algebra in Theorem 2.4 induces a bijection between congruent classes* (*under* $\mathrm{SO}(3,\mathbb{C})$) *of solutions of Eq. (2.3) and isomorphic classes of PostLie algebra structures on* $(\mathrm{sl}(2,\mathbb{C}),[\,,\,])$.

*Proof.* (1). ($\Longrightarrow$) By the assumption, there is a linear isomorphism $\varphi:\mathrm{sl}(2,\mathbb{C})\to\mathrm{sl}(2,\mathbb{C})$ such that

$$\varphi([x,y]) = [\varphi(x),\varphi(y)], \tag{2.4}$$
$$\varphi(x\circ y) = \varphi(x)\star\varphi(y), \quad \forall x,y\in\mathrm{sl}(2,\mathbb{C}). \tag{2.5}$$

Let $f_\circ$ and $f_\star$ be the linear maps on the PostLie algebras $(\mathrm{sl}(2,\mathbb{C}),[\,,\,],\circ)$ and $(\mathrm{sl}(2,\mathbb{C}),[\,,\,],\star)$ respectively from Proposition 2.2. By Eqs. (2.4) and (2.5), we have

$$[\varphi(f_\circ(x)),\varphi(y)] = \varphi([f_\circ(x),y]) = [f_\star(\varphi(x)),\varphi(y)], \quad \forall x,y\in\mathrm{sl}(2,\mathbb{C}).$$

Since the center of $\mathrm{sl}(2,\mathbb{C})$ is zero, we have

$$\varphi(f_\circ(x)) = f_\star(\varphi(x)), \quad \forall x\in\mathrm{sl}(2,\mathbb{C}).$$

Thus, $\varphi f_\circ = f_\star\varphi$, that is, $A_\circ T = TA_\star$ for the matrix $T$ of $\varphi$ with respect to the basis $\{e_1,e_2,e_3\}$ of $\mathrm{sl}(2,\mathbb{C})$. Since $T$ is in $\mathrm{SO}(3,\mathbb{C})$ by Lemma 2.5, we have $A_\star = T'A_\circ T$.

($\Longleftarrow$) Suppose there is $T\in\mathrm{SO}(3,\mathbb{C})$ such that $A_\star = T'A_\circ T$. Let $\psi$ be the linear operator on $\mathrm{sl}(2,\mathbb{C})$ whose matrix with respect to the basis $\{e_1,e_2,e_3\}$ is $T$. By Lemma 2.5, $\psi$ is an automorphism of the Lie algebra $\mathrm{sl}(2,\mathbb{C})$. Thus, Eq. (2.4) holds.

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Furthermore, from $A_\star = T'A_\circ T$ we obtain $\psi f_\circ = f_\star \psi$. Thus we have

$$\begin{aligned}\psi(x\circ y) &= \psi([f_\circ(x), y]) = [\psi(f_\circ(x)), \psi(y)] = [(\psi f_\circ)(x), \psi(y)]\\ &= [(f_\star\psi)(x), \psi(y)] = [f_\star(\psi(x)), \psi(y)] = \psi(x)\star\psi(y),\end{aligned}$$

proving Eq. (2.5).

(2). If $A$ is the matrix of a PostLie algebra on the Lie algebra $(\mathrm{sl}(2,\mathbb{C}), [\,,\,])$, then A is a complex matrix satisfying Eq. (2.3). Thus, for any $T \in \mathrm{SO}(3,\mathbb{C})$, we have

$$T'A'TT'((\mathrm{tr}(A)+1)I_3 - A)T = T'A^*T.$$

Since $T' = T^*$, this gives

$$(T'AT)'((\mathrm{tr}(A)+1)I_3 - T'AT) = T^*A^*(T^*)' = T^*A^*(T')^* = (T'AT)^*,$$

showing that $B = T'AT$ also satisfies Eq. (2.3). So $B$ is also the matrix of a PostLie algebra on $\mathrm{sl}(2,\mathbb{C})$.

(3). By Theorem 2.4, we have a bijective map from the set of solutions of Eq. (2.3) to the set of PostLie algebras on $(\mathrm{sl}(2,\mathbb{C}), [\,,\,])$. By Item (1), this bijective map induces a bijective map from the set of congruent classes (under the action of $\mathrm{SO}(3,\mathbb{C})$) of the solutions of Eq. (2.3) to the set of isomorphic classes of PostLie algebras on $(\mathrm{sl}(2,\mathbb{C}), [\,,\,])$. ◻

**3. Classification of the matrix solutions.** According to Theorem 2.6.(3), in order to prove our main Theorem 1.2, we only need to prove the following theorem on congruent classes of solutions of Eq. (2.3).

Theorem 3.1. *A complete list of representatives A of congruent classes for the solutions of Eq. (2.3) is given as follows:*

$$\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix};\quad \begin{bmatrix}-1&0&0\\0&-1&0\\0&0&-1\end{bmatrix};\quad \begin{bmatrix}-1&0&0\\0&-\frac{1+\sqrt{-1}}{2}&\frac{\sqrt{-1}-1}{2}\\0&-\frac{1+\sqrt{-1}}{2}&\frac{\sqrt{-1}-1}{2}\end{bmatrix};$$
$$\begin{bmatrix}k&0&0\\0&-\frac{1}{2}&\frac{\sqrt{-1}}{2}\\0&-\frac{\sqrt{-1}}{2}&-\frac{1}{2}\end{bmatrix}, k\in\mathbb{C};\quad \begin{bmatrix}-\frac{1}{2}+\sqrt{-1}&1-\frac{\sqrt{-1}}{2}&0\\1+\frac{\sqrt{-1}}{2}&-\frac{1}{2}-\sqrt{-1}&0\\0&0&0\end{bmatrix}.$$

The proof of this theorem will be presented in this section. After discussing a preparatory result on congruent classes of complex symmetric matrices, we will divide our proof into the three cases when the rank of $A$ is three, two or one. Since the case when the rank of $A$ is zero gives us the trivial solution, we will not discuss it further.

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The next result is essentially due to [3, Chapter XI, Corollary 2]. We modify it in its general form and spell out the details in the dimension three case to fit the application in this paper.

Proposition 3.2. *Consider the $k \times k$ complex matrix*

$$D_k := \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 1 & 0 & 1 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & 1 & 0 \end{bmatrix} + \sqrt{-1} \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & \cdots & 1 & 0 & -1 \\ 0 & 0 & \cdots & 0 & -1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & \cdots & 0 & 0 & 0 \end{bmatrix}.$$

1. *For a complex symmetric $n \times n$ matrix $A$ with elementary factors $(\lambda - \lambda_1)^{k_1}, \dots, (\lambda - \lambda_t)^{k_t}$, $k_1 + \cdots + k_t = n$, there exists $T \in O(n, \mathbb{C})$ such that*

$$TAT^{-1} = P := \operatorname{diag}\left(\lambda_1 I_{k_1} + D_{k_1}, \lambda_2 I_{k_2} + D_{k_2}, \dots, \lambda_t I_{k_t} + D_{k_t}\right).$$

2. *When $n$ is odd, the above matrix $T$ can be chosen to be in $SO(n, \mathbb{C})$.*
3. *For each $3 \times 3$ complex symmetric matrix $A$, there exists $T \in SO(3, \mathbb{C})$ and a unique $P$ in the following list such that $P = TAT^{-1}$.*
   *(a) When* $\mathrm{r}(A) = 3$,

$$\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}; \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 + \sqrt{-1} & 1 \\ 0 & 1 & \lambda_2 - \sqrt{-1} \end{bmatrix};$$

$$\begin{bmatrix} \lambda & 1 + \sqrt{-1} & 0 \\ 1 + \sqrt{-1} & \lambda & 1 - \sqrt{-1} \\ 0 & 1 - \sqrt{-1} & \lambda \end{bmatrix}. \tag{3.1}$$

   *(b) When* $\mathrm{r}(A) = 2$,

$$\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}; \begin{bmatrix} \lambda + \sqrt{-1} & 1 & 0 \\ 1 & \lambda - \sqrt{-1} & 0 \\ 0 & 0 & 0 \end{bmatrix};$$

$$\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \sqrt{-1} & 1 \\ 0 & 1 & -\sqrt{-1} \end{bmatrix}; \begin{bmatrix} 0 & 1 + \sqrt{-1} & 0 \\ 1 + \sqrt{-1} & 0 & 1 - \sqrt{-1} \\ 0 & 1 - \sqrt{-1} & 0 \end{bmatrix}. \tag{3.2}$$

   *(c) When* $\mathrm{r}(A) = 1$,

$$\begin{bmatrix} \lambda & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}; \quad \begin{bmatrix} \sqrt{-1} & 1 & 0 \\ 1 & -\sqrt{-1} & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{3.3}$$

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*(d) When* $\mathrm{r}(A) = 0$,

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{3.4}$$

*Here all constants are non-zero.*

*Proof.* (1). By [3, Chapter XI, Corollary 2], for a complex symmetric matrix $A$ with the elementary factors as in the proposition, there exists $T \in O(n, \mathbb{C})$ such that

$$TAT^{-1} = P := \mathrm{diag}\left(\lambda_1 I_{k_1} + \frac{1}{2}D_{k_1}, \lambda_2 I_{k_2} + \frac{1}{2}D_{k_2}, \dots, \lambda_t I_{k_t} + \frac{1}{2}D_{k_t}\right).$$

Note that $\frac{1}{2}A$ is a complex symmetric matrix whose elementary factors are $(\lambda - \frac{1}{2}\lambda_1)^{k_1}$, $\dots, (\lambda - \frac{1}{2}\lambda_t)^{k_t}$. Applying the above result in [3] to $\frac{1}{2}A$, there is $T \in O(n, \mathbb{C})$ such that

$$T\frac{1}{2}AT^{-1} = \mathrm{diag}\left(\frac{1}{2}\lambda_1 I_{k_1} + \frac{1}{2}D_{k_1}, \dots, \frac{1}{2}\lambda_t I_{k_t} + \frac{1}{2}D_{k_t}\right).$$

Hence, we have

$$TAT^{-1} = \mathrm{diag}\left(\lambda_1 I_{k_1} + D_{k_1}, \dots, \lambda_t I_{k_t} + D_{k_t}\right).$$

(2). When $n$ is odd, we can get the matrix $T \in O(n, \mathbb{C})$ to be a matrix $S \in SO(n, \mathbb{C})$ by keeping $T$ if $\det\ T = 1$ and replacing $T$ by $-T$ if $\det\ T = -1$.

(3). This part is a detailed enumeration of Item (2) except the uniqueness of $P$ which follows since different matrices in the list of Eqs. (3.1) – (3.4) have different Jordan canonical forms. ∎

**3.1. Case 1 of Theorem 3.1: The rank of $A$ is three.** In this case, $A$ is invertible. Then by Eq. (2.3), we obtain

$$(\mathrm{tr}(A) + 1)I_3 - A = (A')^{-1}A^* = (A^{-1})'A^{-1}\det A.$$

Then

$$A = (\mathrm{tr}(A) + 1)I_3 - (A^{-1})'A^{-1}\det A.$$

So A is a symmetric matrix. By Proposition 3.2, we can assume that $A$ is one of the three matrices in Eq. (3.1).

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**3.1.1. Case 1.1.** $A = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}, \lambda_1\lambda_2\lambda_3 \neq 0.$

Applying Eq. (2.3) to $A$ and comparing the entries on the main diagonal of the matrices on the two sides, we have

$$\lambda_1(\lambda_2+\lambda_3+1)=\lambda_2\lambda_3, \quad \lambda_2(\lambda_3+\lambda_1+1)=\lambda_3\lambda_1, \quad \lambda_3(\lambda_1+\lambda_2+1)=\lambda_1\lambda_2.$$

Adding two of the equations at a time and then dividing by $\lambda_1\lambda_2\lambda_3$, we obtain

$$\frac{1}{\lambda_1}+\frac{1}{\lambda_2}+2=0, \quad \frac{1}{\lambda_2}+\frac{1}{\lambda_3}+2=0, \quad \frac{1}{\lambda_3}+\frac{1}{\lambda_1}+2=0,$$

yielding the unique solution $\lambda_1=\lambda_2=\lambda_3=-1$. Therefore,

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

This is indeed a solution of Eq. (2.3).

**3.1.2. Case 1.2.** $A = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2+\sqrt{-1} & 1 \\ 0 & 1 & \lambda_2-\sqrt{-1} \end{bmatrix}, \lambda_1\lambda_2 \neq 0.$

Applying Eq. (2.3) to $A$ and comparing entries, we have

$$2\lambda_1\lambda_2+\lambda_1=\lambda_2^2, \quad (2\lambda_1+1)\sqrt{-1}+\lambda_2^2+\lambda_2=0, \quad 2\lambda_1+1=0.$$

From the second and third equations, we find that $\lambda_2$ is 0 or $-1$, both contradicting the first equation. Thus, the equation set does not have any solution and this case does not give any solution of Eq. (2.3).

**3.1.3. Case 1.3.** $A = \begin{bmatrix} \lambda & 1+\sqrt{-1} & 0 \\ 1+\sqrt{-1} & \lambda & 1-\sqrt{-1} \\ 0 & 1-\sqrt{-1} & \lambda \end{bmatrix}, \lambda \neq 0.$

Applying Eq. (2.3) to $A$ and comparing the $(1,3)$-entries of the two sides yield a contradiction $-2=2$. Therefore, this case does not give any solution of Eq. (2.3).

**3.2. Case 2 of Theorem 3.1: The rank of $A$ is two.** In this case, $A$ is not necessarily symmetric. But $A'A$ is still a symmetric matrix. We will use this observation to relate Eq. (2.3) for $A$ to an equation for $A'A$.

Let $A$ be a solution of Eq. (2.3). Multiplying $A$ from the right to the two sides of Eq. (2.3), we have $A'((\mathrm{tr}A+1)I_3-A)A=\det AI_3=0$. Thus,

$$(\mathrm{tr}A+1)A'A=A'AA. \tag{3.5}$$

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Therefore, for a solution $A$ of Eq. (2.3), $A'A$ is symmetric and satisfies Eq. (3.5).

Furthermore, since $\mathrm{r}(A) = 2$ and $\mathrm{r}(A'A) \le \mathrm{r}(A)$, $\mathrm{r}(A'A)$ is 0, or 1 or 2. However, $\mathrm{r}(A'A) \neq 0$. Otherwise, $\mathrm{r}(A') = 2$ would be the dimension of the solution space of $XA = 0$, which is $3 - \mathrm{r}(A) = 1$, a contradiction. Thus, $\mathrm{r}(A'A) = 1$ or 2. Then by Proposition 3.2, there is $T \in \mathrm{SO}(3,\mathbb{C})$ such that $P := TA'AT'$ is one of the six matrices in Eqs. (3.2) and (3.3).

By Theorem 2.6, $B := TAT'$ is also a solution of Eq. (2.3) that is congruent to $A$. Further, multiplying $T$ (resp., $T'$) to the left (resp., right) hand side of Eq. (3.5), we find that $B'B = TA'AT' = P$ satisfies Eq. (3.5) as well and is congruent to $A'A$.

To summarize, in order to find solutions of Eq. (2.3) of rank 2 up to congruent by $\mathrm{SO}(3,\mathbb{C})$, we only need to consider every solution $A$ of Eq. (2.3) such that $A'A$ is one of the six matrices in Eqs. (3.2) and (3.3), and satisfies Eq. (3.5). We now consider the corresponding six cases separately.

**3.2.1. Case 2.1.** $A'A = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \lambda_1\lambda_2 \neq 0.$

Substituting $A'A$ into Eq. (3.5) and comparing entries in the first two rows, we have

$$a_{12} = a_{13} = a_{21} = a_{23} = 0, \quad a_{11} = a_{22} = \mathrm{tr}(A) + 1. \tag{3.6}$$

If $\mathrm{tr}(A) + 1 = 0$, then the first two rows of $A$ are zero. Then $\mathrm{r}(A) \leqslant 1$, which is a contradiction to our assumption. So $a_{11} = a_{22} = \mathrm{tr}(A) + 1 \neq 0$. Then, from $\mathrm{r}(A) = 2$, we get $a_{33} = 0$. Substituting it into Eq. (2.3) and comparing the $(3,3)$-entries of the two sides, we get $0 - 0 = a_{11}a_{22} \neq 0$, which is a contradiction. Therefore, this case does not give any solution of Eq. (2.3).

**3.2.2. Case 2.2.** $A'A = \begin{bmatrix} \lambda + \sqrt{-1} & 1 & 0 \\ 1 & \lambda - \sqrt{-1} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \lambda \neq 0.$

Substituting $A'A$ into Eq. (3.5) and comparing entries in the first two rows, we again have Eq. (3.6). Therefore, as in Case 2.1, the current case does not give any solution of Eq. (2.3).

**3.2.3. Case 2.3.** $A'A = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \sqrt{-1} & 1 \\ 0 & 1 & -\sqrt{-1} \end{bmatrix}, \lambda \neq 0.$

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Substituting the form of $A'A$ into Eq. (3.5), we obtain

$$(\mathrm{tr}(A)+1)\begin{bmatrix}\lambda & 0 & 0\\ 0 & \sqrt{-1} & 1\\ 0 & 1 & -\sqrt{-1}\end{bmatrix} = \begin{bmatrix}\lambda & 0 & 0\\ 0 & \sqrt{-1} & 1\\ 0 & 1 & -\sqrt{-1}\end{bmatrix} A. \quad (3.7)$$

Since $\lambda \neq 0$, we have

$$a_{11} = \mathrm{tr}(A)+1, \quad a_{12} = a_{13} = 0.$$

By the assumption of the form of $A'A$ in this case, we obtain

$$\begin{bmatrix} a_{22} & a_{32}\\ a_{23} & a_{33}\end{bmatrix}\begin{bmatrix} a_{22} & a_{23}\\ a_{32} & a_{33}\end{bmatrix} = \begin{bmatrix}\sqrt{-1} & 1\\ 1 & -\sqrt{-1}\end{bmatrix}.$$

Hence,

$$(a_{22})^2 + (a_{32})^2 = \sqrt{-1}, \quad (a_{23})^2 + (a_{33})^2 = -\sqrt{-1}, \quad \det\begin{bmatrix} a_{22} & a_{23}\\ a_{32} & a_{33}\end{bmatrix} = 0. \quad (3.8)$$

From the third equation in Eq. (3.8) and $a_{12} = a_{13} = 0$, we find that the first row of $A^*$ is 0. Thus, from Eq. (2.3), we obtain

$$(\mathrm{tr}(A)+1)\begin{bmatrix} a_{11} & a_{21} & a_{31}\\ 0 & a_{22} & a_{32}\\ 0 & a_{23} & a_{33}\end{bmatrix} - \begin{bmatrix}\lambda & 0 & 0\\ 0 & \sqrt{-1} & 1\\ 0 & 1 & -\sqrt{-1}\end{bmatrix} = \begin{bmatrix} 0 & 0 & 0\\ (A^*)_{21} & (A^*)_{22} & (A^*)_{23}\\ (A^*)_{31} & (A^*)_{32} & (A^*)_{33}\end{bmatrix}.$$

Comparing the (1, 1)-entries, we have $(\mathrm{tr}(A)+1)a_{11} = (\mathrm{tr}(A)+1)^2 = \lambda \neq 0$. Then $a_{21} = a_{31} = 0$. So from Eq. (3.7), we get

$$(\mathrm{tr}(A)+1)\sqrt{-1} = \sqrt{-1}a_{22} + a_{32}, \quad -(\mathrm{tr}(A)+1)\sqrt{-1} = a_{23} - \sqrt{-1}a_{33}. \quad (3.9)$$

It is easy to derive the solutions of Eq. (3.8) and Eq. (3.9):

$$a_{22} = \frac{1}{2}(\mathrm{tr}(A)+1+\frac{\sqrt{-1}}{\mathrm{tr}(A)+1}); \quad a_{32} = \frac{1}{2}((\mathrm{tr}(A)+1)\sqrt{-1}+\frac{1}{\mathrm{tr}(A)+1}).$$
$$a_{23} = \frac{1}{2}(\frac{1}{\mathrm{tr}(A)+1} - (\mathrm{tr}(A)+1)\sqrt{-1}); \quad a_{33} = \frac{1}{2}((\mathrm{tr}(A)+1) - \frac{\sqrt{-1}}{\mathrm{tr}(A)+1}).$$

Hence, $a_{22} + a_{33} = \mathrm{tr}(A)+1$. On the other hand, since $\mathrm{tr}(A)+1 = a_{11}$, we have $a_{22} + a_{33} = -1$. Therefore, $\mathrm{tr}(A)+1 = -1$. So

$$A = \begin{bmatrix} -1 & 0 & 0\\ 0 & -\frac{1+\sqrt{-1}}{2} & \frac{\sqrt{-1}-1}{2}\\ 0 & -\frac{1+\sqrt{-1}}{2} & \frac{\sqrt{-1}-1}{2}\end{bmatrix}.$$

It is straightforward to check that it satisfies Eq. (2.3).

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**3.2.4. Case 2.4.** $A'A = \begin{bmatrix} 0 & 1+\sqrt{-1} & 0 \\ 1+\sqrt{-1} & 0 & 1-\sqrt{-1} \\ 0 & 1-\sqrt{-1} & 0 \end{bmatrix}$.

Substituting $A'A$ into Eq. (3.5), we obtain

$$a_{21} = a_{23} = 0, \quad a_{22} = \mathrm{tr}(A) + 1, \quad a_{12}^2 + a_{32}^2 = 0.$$

On the other hand, comparing the $(2,2)$-entries on both sides of $A'A$ in its assumed form in this case, we obtain $a_{12}^2 + a_{22}^2 + a_{32}^2 = 0$. So $a_{22} = 0$. Then $\mathrm{tr}(A) + 1 = 0$. Thus, by Eq. (3.5) in this case again, we obtain

$$(1+\sqrt{-1}, 1-\sqrt{-1}) \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = 0.$$

Note that $(a_{21}, a_{22}, a_{23}) = 0$. So $\mathrm{r}(A) = 1$, which is a contradiction to our assumption. Therefore, this case does not give any solution of Eq. (2.3).

**3.2.5. Case 2.5.** $A'A = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \lambda \neq 0$.

Substituting the form of $A'A$ into Eq. (3.5), we obtain

$$a_{11} = \mathrm{tr}(A) + 1, \quad a_{12} = a_{13} = 0.$$

On the other hand, by the assumption of the form of $A'A$ in this case, we obtain

$$\begin{bmatrix} a_{22} & a_{32} \\ a_{23} & a_{33} \end{bmatrix} \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} = 0. \tag{3.10}$$

From Eq. (2.3), we have

$$(\mathrm{tr}(A)+1) \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ 0 & a_{22} & a_{32} \\ 0 & a_{23} & a_{33} \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ (A^*)_{21} & (A^*)_{22} & (A^*)_{23} \\ (A^*)_{31} & (A^*)_{32} & (A^*)_{33} \end{bmatrix}.$$

Comparing the $(1,1)$-entries of the two sides, we have $(\mathrm{tr}(A)+1)a_{11} = (\mathrm{tr}(A)+1)^2 = \lambda \neq 0$. So $a_{11} \neq 0$. Moreover, comparing the $(1,2)$ and (1,3)-entries of the two sides, we see that $a_{21} = a_{31} = 0$. Since $a_{11} \neq 0$, we obtain $a_{22} = a_{33}, a_{23} = -a_{32}$. Since $a_{11} = \mathrm{tr}(A) + 1$, we have $a_{22} + a_{33} = -1$. So $a_{22} = a_{33} = -\frac{1}{2}$. Substituting them into Eq. (3.10) gives

$$A = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{\sqrt{-1}}{2} \\ 0 & -\frac{\sqrt{-1}}{2} & -\frac{1}{2} \end{bmatrix} \text{ and } A = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & -\frac{1}{2} & -\frac{\sqrt{-1}}{2} \\ 0 & \frac{\sqrt{-1}}{2} & -\frac{1}{2} \end{bmatrix}.$$

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The above two matrices are orthogonal congruent by $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}$. So every matrix in this case is orthogonal congruent to one of

$$\begin{bmatrix} a_{11} & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{\sqrt{-1}}{2} \\ 0 & -\frac{\sqrt{-1}}{2} & -\frac{1}{2} \end{bmatrix}, \quad a_{11} \neq 0,$$

which satisfies Eq. (2.3) for any $a_{11} \neq 0$. Thus in this case, we obtain a parameterized family of solutions. Since the trace of a matrix is preserved by any orthogonal congruent operator, the matrices with different values of $a_{11}$ are not congruent. Therefore, different matrices in the family are in different congruency classes.

**3.2.6. Case 2.6.** $A'A = \begin{bmatrix} \sqrt{-1} & 1 & 0 \\ 1 & -\sqrt{-1} & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Substituting $A'A$ into Eq. (3.5), we obtain

$$(\mathrm{tr}(A)+1)\begin{bmatrix} \sqrt{-1} & 1 & 0 \\ 1 & -\sqrt{-1} & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \sqrt{-1} & 1 & 0 \\ 1 & -\sqrt{-1} & 0 \\ 0 & 0 & 0 \end{bmatrix} A. \tag{3.11}$$

If $(\mathrm{tr}(A)+1) = 0$, then

$$a_{11} = \sqrt{-1}\, a_{21}, \quad a_{12} = \sqrt{-1}\, a_{22}, \quad a_{13} = \sqrt{-1}\, a_{23}.$$

So $(A^*)_{11} = -\sqrt{-1}(A^*)_{12}$. However, from Eq. (2.3), we obtain $-A'A = A^*$. So $(A^*)_{11} = \sqrt{-1}(A^*)_{12}$ which is a contradiction. Therefore, this case does not give a solution of Eq. (2.3).

If $(\mathrm{tr}(A)+1) \neq 0$, then by Eq. (3.11) and the assumption of the form of $A'A$ in this case, we conclude

$$a_{13} = \sqrt{-1}\, a_{23}, \quad a_{33} = 0, \quad a_{21} = \sqrt{-1}\, a_{22}; \quad a_{11} = \sqrt{-1}\, a_{12}; \quad a_{31} = \sqrt{-1}\, a_{32}.$$

Thus, the last row of $A^*$ is 0. Furthermore, from the last row of Eq. (2.3), we obtain $a_{13} = a_{23} = a_{33} = 0$. Then $\mathrm{r}(A) = 1$, which is a contradiction to our assumption. Therefore, this case does not give any solution of Eq. (2.3).

**3.3. Case 3 of Theorem 3.1: The rank of $A$ is one.** In this case, $A^* = 0$. So Eq. (2.3) becomes

$$(\mathrm{tr}(A)+1)A' = A'A. \tag{3.12}$$

There are the following six subcases, including two subcases where $A$ is symmetric and four cases where $A$ is not symmetric.

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**3.3.1.** ***A*** **is symmetric.** Since $\mathrm{r}(A) = 1$, by Proposition 3.2, we have the following two subcases.

*Case 3.1:* $A = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \lambda \neq 0.$

Then from Eq. (3.12), we obtain $(\lambda+1)\lambda = (\lambda)^2$. So $\lambda = 0$ which is a contradiction to our assumption. Thus, this case does not give a solution of Eq. (2.3).

*Case 3.2:* $A = \begin{bmatrix} \sqrt{-1} & 1 & 0 \\ 1 & -\sqrt{-1} & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Note that $A'A = 0$. Combining it with Eq. (3.12), we have $(\mathrm{tr}(A) + 1)A' = 0$. Since $\mathrm{tr}(A) \neq -1$, we have $\mathrm{tr}(A) + 1 \neq 0$. Then $A' = 0$. Thus $A = 0$, which is a contradiction again. Therefore, this case does not give a solution of Eq. (2.3).

**3.3.2.** ***A*** **is not symmetric.** In this case, we apply a strategy similar to Case 2 by relating $A$ to its symmetrizer $\frac{1}{2}(A + A')$.

First note that if $\mathrm{tr}(A) + 1 \neq 0$, then $A$ is symmetric. So by our assumption, we obtain $\mathrm{tr}(A) + 1 = 0$. Then by Eq. (3.12), we also have $A'A = 0$.

Let $A$ be a solution of Eq. (2.3). Since $\frac{1}{2}(A + A')$ is symmetric, by Proposition 3.2.(3), there is $T \in \mathrm{SO}(3, \mathbb{C})$ such that $T\frac{1}{2}(A + A')T'$ is one of the matrices in Proposition 3.2.(3). By Theorem 2.6, $TAT'$ is a solution of Eq. (2.3) that is congruent to $A$ and its symmetrizer $\frac{1}{2}(TAT' + (TAT')') = T\frac{1}{2}(A + A')T'$ is one of the matrices in Proposition 3.2.(3).

Therefore, to find solutions $A$ of Eq. (2.3) with $\mathrm{r}(A) = 1$ that is not symmetric, we only need to find from those $A$ such that $\frac{1}{2}(A + A')$ is from the matrices in Proposition 3.2.(3).

Furthermore, since $\mathrm{r}(A) = 1$, we can suppose $A = (\alpha_1, \alpha_2, \alpha_3)' \cdot (\beta_1, \beta_2, \beta_3)$, where not all $\alpha_i \in \mathbb{C}$ are zero and not all $\beta_j \in \mathbb{C}$ are zero. Then

$$\frac{1}{2}(A + A') = \begin{bmatrix} \alpha_1\beta_1 & \frac{1}{2}(\alpha_1\beta_2 + \alpha_2\beta_1) & \frac{1}{2}(\alpha_1\beta_3 + \alpha_3\beta_1) \\ \frac{1}{2}(\alpha_1\beta_2 + \alpha_2\beta_1) & \alpha_2\beta_2 & \frac{1}{2}(\alpha_2\beta_3 + \alpha_3\beta_2) \\ \frac{1}{2}(\alpha_1\beta_3 + \alpha_3\beta_1) & \frac{1}{2}(\alpha_2\beta_3 + \alpha_3\beta_2) & \alpha_3\beta_3 \end{bmatrix}.$$

If $\mathrm{r}(\frac{1}{2}(A + A')) < 2$, then all the $2 \times 2$ subdeterminants are 0. Thus, we have

$$\alpha_1\beta_2 = \alpha_2\beta_1, \quad \alpha_2\beta_3 = \alpha_3\beta_2, \quad \alpha_3\beta_1 = \alpha_1\beta_3.$$

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Therefore, $A$ is symmetric, which is a contradiction to our assumption. Thus, $\mathrm{r}(\frac{1}{2}(A+A')) \geqslant 2$. On the other hand, by basic linear algebra, we have

$$\mathrm{r}(\frac{1}{2}(A+A')) = \mathrm{r}(A+A') \leq \mathrm{r}(A) + \mathrm{r}(A').$$

Hence, $\mathrm{r}(\frac{1}{2}(A+A')) = 2$. Thus, by Proposition 3.2.(3), we only need to consider the following four cases:

*Case 3.3:* $\frac{1}{2}(A+A') = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \lambda_1\lambda_2 \neq 0.$

In this case,

$$a_{11} = \lambda_1 \neq 0, \quad a_{12}+a_{21} = 0, \quad a_{13}+a_{31} = 0, \quad a_{33} = 0.$$

Suppose $a_{21} = ka_{11}$. Then from $A'A = 0$, we obtain

$$A = a_{11}\begin{bmatrix} 1 & -k & \mp\sqrt{-(k^2+1)} \\ k & -k^2 & \mp k\sqrt{-(k^2+1)} \\ \pm\sqrt{-(k^2+1)} & \mp k\sqrt{-(k^2+1)} & k^2+1 \end{bmatrix}. \tag{3.13}$$

Since $a_{33} = 0$, we have $k = \pm\sqrt{-1}$ and since $\mathrm{tr}(A) = -1$, $a_{11} = -\frac{1}{2}$. Therefore,

$$A = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{-1}}{2} & 0 \\ -\frac{\sqrt{-1}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ or } A = \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{-1}}{2} & 0 \\ \frac{\sqrt{-1}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The above two matrices are congruent by $\begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

So up to orthogonal congruences, we have

$$A = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{-1}}{2} & 0 \\ -\frac{\sqrt{-1}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

It is straightforward to check that it gives a solution of Eq. (2.3).

*Case 3.4:* $\frac{1}{2}(A+A') = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \sqrt{-1} & 1 \\ 0 & 1 & -\sqrt{-1} \end{bmatrix}, \lambda \neq 0.$

In this case, we have $a_{11} = \lambda \neq 0$ and $a_{22}+a_{33} = \sqrt{-1}-\sqrt{-1} = 0$. By a similar argument as in Case 3.3, we see that Eq. (3.13) holds. Thus $a_{22}+a_{33} =$

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$(-k^2+k^2+1)a_{11}$, implying $a_{11}=0$. This is a contradiction. So this case does not give any solution of Eq. (2.3).

*Case 3.5:* $\frac{1}{2}(A+A') = \begin{bmatrix} \lambda+\sqrt{-1} & 1 & 0 \\ 1 & \lambda-\sqrt{-1} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \lambda \neq 0.$

Since $\mathrm{tr}(\frac{1}{2}(A+A')) = -1$, we have $\lambda = -\frac{1}{2}$. So $a_{11} = -\frac{1}{2}+\sqrt{-1} \neq 0$. Assume $a_{21} = ka_{11}$. Since $a_{13} = -a_{31}$ and $A'A = 0$, we obtain

$$A = \begin{bmatrix} a_{11} & a_{12} & \mp\sqrt{-(k^2+1)}a_{11} \\ ka_{11} & ka_{12} & \mp k\sqrt{-(k^2+1)}a_{11} \\ \pm\sqrt{-(k^2+1)}a_{11} & \pm\sqrt{-(k^2+1)}a_{12} & (k^2+1)a_{11} \end{bmatrix}.$$

Moreover, by assumption, we have

$$a_{33} = 0, \quad a_{11} = -\frac{1}{2}+\sqrt{-1}, \quad a_{22} = -\frac{1}{2}-\sqrt{-1}, \quad a_{12}+a_{21} = 2.$$

Therefore,

$$A = \begin{bmatrix} -\frac{1}{2}+\sqrt{-1} & 1-\frac{\sqrt{-1}}{2} & 0 \\ 1+\frac{\sqrt{-1}}{2} & -\frac{1}{2}-\sqrt{-1} & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

It is straightforward to check that it gives a solution of Eq. (2.3).

*Case 3.6:* $\frac{1}{2}(A+A') = \begin{bmatrix} 0 & 1+\sqrt{-1} & 0 \\ 1+\sqrt{-1} & 0 & 1-\sqrt{-1} \\ 0 & 1-\sqrt{-1} & 0 \end{bmatrix}.$

In this case, $\mathrm{tr}(\frac{1}{2}(A+A')) = 0$ which is a contradiction to our assumption that $\mathrm{tr}(\frac{1}{2}(A+A')) = \mathrm{tr}(A) = -1$. So this case does not give any solution of Eq. (2.3).

We have now completed the proof of Theorem 3.1.

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