Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 23, pp. 171-179, February 2012

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BOUNDS OF SPECTRAL RADII OF $K_{2,3}$-MINOR FREE GRAPHS*

GUANGLONG YU†, JINLONG SHU‡, AND YUAN HONG‡

Abstract. Let $A(G)$ be the adjacency matrix of a graph G. The largest eigenvalue of $A(G)$ is called spectral radius of G. In this paper, an upper bound of spectral radii of $K_{2,3}$-minor free graphs with order n is shown to be $\frac{3}{2}+\sqrt{n-\frac{7}{4}}$. In order to prove this upper bound, a structural characterization of $K_{2,3}$-minor free graphs is presented in this paper.

Key words. Bound, Spectral radius, Minor free.

AMS subject classifications. 05C50.

1. Introduction. All graphs considered in this paper are undirected and simple (i.e., loops and multiple edges are not allowed). Let $G = G[V(G), E(G)]$ be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$, where $|V(G)| = n$ is the order and $|E(G)| = m$ is the size of G. Let $N_G(v)$ denote the neighbor set of vertex v in a graph G. The degree of v in G, denoted by $deg(v)$, is equal to $|N_G(v)|$. We denote by δ or $\delta(G)$ for the minimal vertex degree of G, and denote by Δ or $\Delta(G)$ the maximal vertex degree of G. In a connected graph G, the length of a shortest path from v_i to v_j is called the distance between v_i and v_j, denoted by $d(v_i, v_j)$. We denote by C_k a cycle of length k, denote by P_n a path of order n and by K_n the complete graph of order n. For $S \subseteq V(G)$, let $G[S]$ denote the subgraph induced by S. For a vertex set $\{v_1, v_2, \ldots, v_k\}$, we sometimes abbreviate $G[\{v_1, v_2, \ldots, v_k\}]$ by $G[v_1, v_2, \ldots, v_k]$. $G[S]$ is called a clique if it is a complete subgraph of G; $G[S]$ is called a k-clique if $|S| = k$. For a connected graph G which is not complete, the vertex connectivity, commonly referred to simply as connectivity, denoted by $\kappa(G)$, is the minimum number of vertices whose deletion yields the resulting graph disconnected. We define $\kappa(K_n)$ to be $n-1$.

There are several definitions of k-sum of two graphs (see [1], for example). Here,

*Received by the editors on January 31, 2011. Accepted for publication on January 15, 2012. Handling Editor: Bryan L. Shader.

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we cite the definition of k-sum given in [3].

DEFINITION 1.1. [3] Given two disjoint graphs G and H each of order at least $k+1$, a graph J is a k-sum of G and H if it can be obtained from G and H by identifying the vertices of a k-clique in G with the vertices of a k-clique in H and possibly deleting some of the edges of the now common k-clique.

In particular, in the k-sum of G and H, if no edge of the new common k-clique is deleted, J is a k-complete-sum of G and H. We denote by $J = G \oplus_k H$ the k-sum of graphs G and H and denote by $J = G \oplus_k^c H$ the k-complete-sum of G and H. We abbreviate the k-complete-sum of G and H by the k-sum of G and H and abbreviate $J = G \oplus_k^c H$ by $J = G \oplus_k H$ in this paper. A graph J is called separable if $J = J_1 \oplus_{k_1} J_2 \oplus_{k_2} \cdots \oplus_{k_{t-1}} J_t$ $(t \geq 2)$; the J_i $(i = 1, 2, \ldots, t)$ are called the summing factors of J. Complete separable and complete summing factor are defined similarly. In this paper, all *k-sums* of any two graphs we considered are *k-complete-sums*, and a k-complete-summing factor of J is also called a k-summing factor.

Let $A(G)$ denote the adjacency matrix of a graph G and $S_v(A^l)$ denote the row sum corresponding to v in A^l. In algebraic graph theory, it is well known that $S_v(A^l)$ is equal the number of the walks which have length l and start from vertex v in graph G. It is easy to see that for a graph G, $S_v(A) = deg(v)$ and $S_v(A^2) = \sum_{u \in N_G(v)} deg(u)$. The characteristic polynomial (or A-polynomial) of G, denoted by $P(G)$ or $P(G, \lambda)$, is defined as $\det(\lambda I - A(G))$, where I is the identity matrix. The largest eigenvalue of $P(G)$ is called the *spectral radius* of G, denoted by $\rho(G)$. We call the eigenvector corresponding to $\rho(G)$ the *Perron eigenvector* of graph G. By the Perron-Frobenius theorem [10], we know that the Perron eigenvector is a positive vector for a connected graph.

DEFINITION 1.2. [6] A graph H is called a minor or H-minor of G, or G is called a H-minor graph if H can be obtained from G by deleting edges, contracting edges, and deleting isolated (degree zero) vertices. Given a graph H, a graph G is H-minor free if H is not a minor of G.

The investigation of H-minor free graphs is of great significance; it is very useful for studying the structures and properties of graphs (see [2, 3, 14, 15]). In 1937, Wagner [14] had shown that a finite graph is planar if and only if it has no minor isomorphic to K_5 or to $K_{3,3}$. In 2003, Yuan Hong [6] established some sharp bounds for the maximal and minimal spectral radius of a K_5-minor free graph. It is well known that a simple graph G is outer-planar if and only if G is both K_4-minor free and $K_{2,3}$-minor free (see [1, 16], for example), so the study of K_4-minor free graphs and $K_{2,3}$-minor free graphs is very useful for the study of outer-planar graphs. In 2000, J.L. Shu and Y. Hong [12] determined that, for any connected simple maximal

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A publication of the International Linear Algebra Society
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outer-planar graph G of order n, $\rho(G) \leq \frac{3}{2} + \sqrt{n - \frac{7}{4}}$. In 2001, J. Shi and Y. Hong [11] studied K_4-minor free graphs. They obtained a sharp upper bound for their spectral radii and characterized the extremal graphs with the sharp upper bound. An interesting thing is that an upper bound for spectral radii of $K_{2,3}$-minor free graphs of order n is also shown to be $\frac{3}{2} + \sqrt{n - \frac{7}{4}}$ in this paper. To prove this upper bound, a structural characterization of $K_{2,3}$-minor free graphs is presented. This paper is organized as follows: Section 2 presents some working lemmas; Section 3 presents some upper bounds for the spectral radii of $K_{2,3}$-minor free graphs.

2. Preliminaries. The reader is referred to [1, 4, 5, 16] for the facts about outer-planar and maximal outer-planar graphs.

Definition 2.1. A simple graph is an outer-planar graph if it has an embedding in the plane so that every vertex lies on the unbounded (exterior) face.

Definition 2.2. A simple outer-planar graph is maximal if no edge can be added to the graph without violating outer-planarity.

Lemma 2.3. *A simple graph G is an outer-planar graph if and only if G is both K_4-minor free and $K_{2,3}$-minor free.*

Lemma 2.4. *If a simple graph G of order $n \geq 3$ is a maximal outer-planar graph, then G has a planar embedding whose outer face is a Hamilton cycle, all other faces being triangles.*

Lemma 2.5. [8, 9] *Let G_1 be a connected graph. If G_2 is a proper subgraph of G_1, then $\rho(G_2) < \rho(G_1)$ and for $\lambda \geq \rho(G_1)$, $P(G_2, \lambda) > P(G_1, \lambda)$.*

Lemma 2.6. [13] *Let u and v be two vertices of a connected graph G. Suppose $v_1, v_2, \ldots, v_s$ $(1 \leq s \leq d(v))$ are vertices of $N_G(v) \backslash N_G[u]$ $(N_G[u] = N_G(u) \bigcup \{u\})$ and $X = (x_1, x_2, \ldots, x_n)^T$ is the Perron eigenvector of G, where x_i corresponds to the vertex v_i $(1 \leq i \leq n)$. Let G^* be the graph obtained from G by deleting the edges (v, v_i) and adding the edges (u, v_i) $(1 \leq i \leq s)$. If $x_u \geq x_v$, then $\rho(G) < \rho(G^*)$.*

Lemma 2.7. [7] *Let G be a connected graph with n vertices, m edges and minimum degree δ. Then*

$$\rho(G) \leq \frac{\delta - 1 + \sqrt{(\delta+1)^2 + 4(2m - \delta n)}}{2},$$

where equality holds if and only if G is isomorphic to a regular graph or to a graph in which the degree of each vertex is either $n-1$ or δ.

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3. Upper bounds.

DEFINITION 3.1. The union of simple graphs H and G is the simple graph $G\bigcup H$ with vertex set $V(G)\bigcup V(H)$ and edge set $E(G)\bigcup E(H)$. The intersection $G\bigcap H$ of simple graphs H and G is defined analogously.

LEMMA 3.2. *A graph G of order n is $K_{2,3}$-minor free if and only if each component of G is a 1-sum, namely, $G_1 \oplus_1 G_2 \oplus_1 \cdots \oplus_1 G_k$, in which, each G_i $(1 \leq i \leq k)$ is either a K_4 or an outer-planar graph.*

Proof. We prove the necessity by induction on n. For $n = 1, 2, 3, 4$, the lemma obviously holds. Suppose the $N \geq 5$ and that the lemma holds for $n < N$. Next we prove the lemma holds for $n = N$.

If G is disconnected, then the conclusion follows at once from the induction hypothesis. Next, we suppose G is connected.

Case 1: G is K_4-minor free. Then G is an outer-planar graph by Lemma 2.3. Hence, the lemma holds.

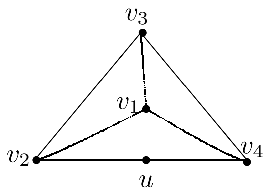

Fig. 3.1. H'.

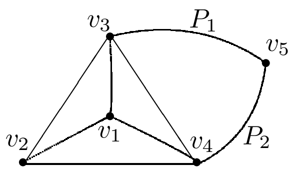

Fig. 3.2. F.

Case 2: K_4 is a minor of G. Because $K_{2,3}$ is a minor of any subdivision (obtained from K_4 by subdividing some of its edges) of K_4 (see Fig. 3.1, $H = H' - v_1v_3$ is a $K_{2,3}$), there exists a K_4 but no subdivision of K_4 as a subgraph in G.

We assert $\kappa(G) = 1$. Suppose otherwise that $\kappa(G) \geq 2$ and $G[v_1, v_2, v_3, v_4] = K_4$. Then there are two internal disjoint independent paths P_1, P_2 from vertex v_5 to $G[v_1, v_2, v_3, v_4]$, which causes a $K_{2,3}$-minor subgraph $F = G[v_1, v_2, v_3, v_4]\bigcup P_1 \bigcup P_2$ in G (see Fig. 3.2), which contradicts that G is $K_{2,3}$-minor free. So our assertion holds, and then $\kappa(G) = 1$. Suppose G is obtained from G_1 and G_2 by their 1-sum. Because G_1 and G_2 are both $K_{2,3}$-minor free, by induction, then G_1 and G_2 are both obtained from some $K_4's$ with some outer-planar graphs by their 1-sum. So G is also obtained from some $K_4's$ with some outer-planar graphs by their 1-sum.

By Case 1 and Case 2, the necessity is proved.

Conversely, suppose that G is obtained from some $K_4's$ with some outer-planar graphs by their 0-sum or 1-sum. Since the outer-planar graph and K_4 are both

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 23, pp. 171-179, February 2012

$K_{2,3}$-minor free, and $\kappa(K_{2,3}) = 2$, G is also $K_{2,3}$-minor free. □

LEMMA 3.3. *Let e be a cut-edge of graph G and $G = G_1eG_2$, where one end vertex of e is in G_1 and the other one is in G_2. If G_1 and G_2 are both $K_{2,3}$-minor free, then G is still $K_{2,3}$-minor free.*

Proof. Note that $\kappa(K_{2,3}) = 2$, so e does not belong to any $K_{2,3}$-minor subgraph in G. Thus, G is still $K_{2,3}$-minor free. □

LEMMA 3.4. *If a graph G^* on n vertices satisfies $\rho(G^*) = \max\{\rho(H) |\ H$ is $K_{2,3}$-minor free$\}$, then for some integer $t \geq 0$, G^* is obtained by attaching t $K_4^{'}s$ to a vertex of a maximal outer-planar graph of order $n - 3t$ (see Fig. 3.3).*

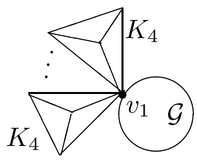

Fig. 3.3. G^*.

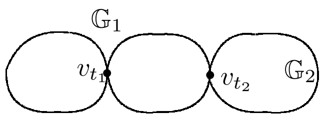

Fig. 3.4. $G = \mathbb{G}_1 \oplus \mathbb{G}_2$.

Proof. We claim that G^* is connected. To see this, assume to the contrary that G^* is disconnected and let $G_1, G_2, \ldots, G_k$ be the connected components of G^*. By Lemma 3.3, we can get a connected $K_{2,3}$-minor free graph $H_1 = G_1e_1G_2e_2 \cdots e_{k-1}G_k$ by adding edges $e_1, e_2, \ldots, e_{k-1}$. But $\rho(H_1) > \rho(G^*)$ by Lemma 2.5, which contradicts $\rho(G^*) = \max\{\rho(H) |\ H$ is $K_{2,3}$-minor free$\}$. So G^* is connected.

We assert that there is at most one cut-vertex in G^*. Otherwise, suppose that $v_{t_1}, v_{t_2}, \ldots, v_{t_l}$ $(l \geq 2)$ are the cut-vertices of G^*. Let $X = (x_1, x_2, \ldots, x_n)^T$ denote the Perron eigenvector of G^*, where x_i corresponds to vertex v_i $(1 \leq i \leq n)$. Suppose that $x_{t_1} = \max\limits_{1 \leq i \leq l}\{x_{t_1}, x_{t_2}, \ldots, x_{t_l}\}$ in the Perron eigenvector X of G^*, and suppose $G^* = \mathbb{G}_1 \oplus_1 \mathbb{G}_2$ where $V(\mathbb{G}_1) \bigcap V(\mathbb{G}_2) = \{v_{t_2}\}$, $v_{t_1} \in V(\mathbb{G}_1)$ and v_{t_2} is not a cut-vertex of $\mathbb{G}_1$ (see Fig. 3.4). Let

$$H_2 \;=\; G^* - \sum_{u \in N_{\mathbb{G}_2}(v_{t_2})} v_{t_2}u \;+\; \sum_{u \in N_{\mathbb{G}_2}(v_{t_2})} v_{t_1}u.$$

Now, the number of the cut-vertices in H_2 is less than the number of the cut-vertices in G^*, and H_2 is also a $K_{2,3}$-minor free graph. But $\rho(H_2) > \rho(G^*)$ by Lemma 2.6, which contradicts $\rho(G^*) = \max\{\rho(H) |\ H$ is $K_{2,3}$-minor free$\}$. So our assertion holds.

If G^* has no cut-vertex, then G^* must be either a K_4 or a maximal outer-planar graph. The lemma holds.

If G^* has only one cut-vertex, noting that a graph obtained by a 1-sum of some outer-planar graphs is also an outer-planar graph, and that any outer-planar graph

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ELA

can be expanded into a maximal outer-planar graph by adding edges, so for some integer $t \geq 0$, G^* is obtained by attaching t $K_4^{'}s$ to a vertex of a maximal outer-planar graph of order $n - 3t$. This completes the proof. ◻

LEMMA 3.5. *Let A be an irreducible nonnegative square real matrix of order n and spectral radius ρ. If there exists a nonnegative vector $y \neq 0$ and a polynomial function f whose coefficients are all real numbers such that $f(A)y \leq ry$ $(r \in \mathbb{R})$, then $f(\rho) \leq r$.*

Proof. Since $\rho(A^T) = \rho(A) = \rho$ and A^T is also irreducible and nonnegative, let x denote the Perron eigenvector of A^T. If $f(A)y \leq ry$ $(r \in \mathbb{R})$, then $f(\rho)x^T y = (f(A^T)x)^T y = x^T f(A)y \leq rx^T y$ and $f(\rho) \leq r$. ◻

LEMMA 3.6. *Let G^* be as in Lemma 3.4 and assume the order of its maximal outer-planar 1-summing factor is $h \geq 3$. Then $\rho(G^*) \leq \dfrac{3}{2} + \sqrt{n - \dfrac{7}{4}}$.*

Proof. Denote by v_1, $\mathcal{G}$ the only cut-vertex and the maximal outer-planar 1-summing factor in G^*, respectively. Let $C_1 = (v_1, v_2, v_3, \ldots, v_{p+1}, v_1)$ $(p \geq 2)$ denote the Hamilton cycle of $\mathcal{G}$. For any $v \in V(G^*)$, there are the following 4 cases.

Case 1: $v = v_1$. Along the clockwise direction, suppose the neighbors on C_1 of v_1 are v_{i_1} $(v_{i_1} = v_2)$, $v_{i_2}, \ldots, v_{i_k}, v_{i_{k+1}}$ $(v_{i_{k+1}} = v_{p+1})$. Suppose there are l_j $(1 \leq j \leq k)$ vertices between v_{i_j} and $v_{i_{j+1}}$ on C_1, and let $S_j = \{u_1, u_2, \ldots, u_{l_j}\}$ denote the set of such vertices between v_{i_j} and $v_{i_{j+1}}$ (see Fig 3.5 and Fig 3.6).

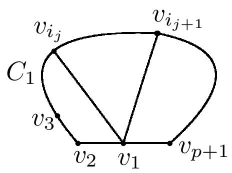

Fig. 3.5. F.

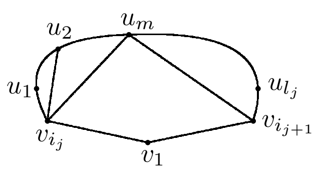

Fig. 3.6. H_1.

We claim that v_{i_j} is adjacent to $v_{i_{j+1}}$ in G^*. Let $H_1 = G^*[\{v_1, v_{i_j}, v_{i_{j+1}}\} \bigcup S_j]$. Then H_1 is also an maximal outer-planar graph by Definitions 2.1, 2.2 and Lemmas 2.3, 2.4. If v_{i_j} is not adjacent to $v_{i_{j+1}}$ in G^*, then since v_1 must be in a triangle of H_1, there exists at least one different vertex from v_{i_j} and $v_{i_{j+1}}$ adjacent to v_1 in H_1, which contradicts $N_{H_1}(v_1) = \{v_{i_j}, v_{i_{j+1}}\}$. So, our claim holds.

Note that any two edges in $\mathcal{G}$ either do not intersect or intersect only at their common end vertex, so there is at most one in S_j which is adjacent to both v_{i_j} and $v_{i_{j+1}}$. Hence, the contribution of S_j to $deg(v_{i_j}) + deg(v_{i_{j+1}})$ is at most $l_j + 1$ and the contribution of $\{v_{i_j}, v_{i_{j+1}}\} \bigcup S_j$ to $deg(v_{i_j}) + deg(v_{i_{j+1}})$ is at most $l_j + 3$.

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ELA

Let A denote the adjacency matrix of G^*. Note that $n = 3t+l_1+l_2+\cdots+l_k+k+2$ and $S_{v_1}(A) = k+1+3t$, so

$$\begin{aligned} S_{v_1}(A^2) &= 9t + \sum_{1}^{k+1} deg(v_{i_j}) \leq 9t + k + 1 + (l_1+3) + (l_2+3) + \cdots + (l_k+3) \\ &= 9t + k + 1 + 3k + (l_1 + l_2 + \cdots + l_k) = n + 3k - 1 + 6t, \end{aligned}$$

and so $S_{v_1}(A^2) - 3(k+1+3t) = S_{v_1}(A^2) - 3S_{v_1}(A) \leq n - 4 - 3t$.

Case 2: $v \neq v_1$ is a vertex of some K_4. Then $S_v(A) = 3$. Note that $deg(v_1) \leq n-1$. As a consequence, $S_v(A^2) = \sum_{vv_i \in E(G^*)} deg(v_i) = deg(v_1) + 6 \leq n - 1 + 6$, and so $S_v(A^2) - 3S_v(A) \leq n - 4$.

Case 3: $v \neq v_1$ is a vertex of $\mathcal{G}$ and $vv_1 \in E(G^*)$. Along the clockwise direction, suppose the neighbors of v on C_1 are $v_{i_1}, v_{i_2}, \ldots, v_{i_q}, v_{i_{q+1}}$ and l_j $(1 \leq j \leq q)$ denotes the number of vertices between v_{i_j} and $v_{i_{j+1}}$. Note that

$$n = 3t + l_1 + l_2 + \cdots + l_q + q + 2$$

and $S_v(A) = q + 1$; similar to Case 1, we get

$$\begin{aligned} S_v(A^2) &= \sum_{1}^{q+1} deg(v_{i_j}) \leq 3t + q + 1 + (l_1+3) + (l_2+3) + \cdots + (l_q+3) \\ &= 3t + q + 1 + 3q + (l_1 + l_2 + \cdots + l_q) = n + 3q - 1. \end{aligned}$$

So, we have

$$S_v(A^2) - 3(q+1) = S_v(A^2) - 3S_v(A) \leq n - 4.$$

Case 4: $v \neq v_1$ is a vertex of $\mathcal{G}$ and v is not adjacent to v_1. Suppose the neighbors of v along the clockwise direction on C_1 are v_{i_1}, v_{i_2}, $\ldots$, v_{i_s}, $v_{i_{s+1}}$ and l_j $(1 \leq j \leq s)$ denotes the number of vertices between v_{i_j} and $v_{i_{j+1}}$. Note that

$$n = 3t + l_1 + l_2 + \cdots + l_s + s + 2$$

and $S_v(A) = s + 1$; similar to Case 1, we get

$$\begin{aligned} S_v(A^2) &= \sum_{1}^{s+1} deg(v_{i_j}) \leq s + 1 + (l_1+3) + (l_2+3) + \cdots + (l_s+3) \\ &= s + 1 + 3s + (l_1 + l_2 + \cdots + l_s) = n + 3s - 1 - 3t. \end{aligned}$$

Hence, we have

$$S_v(A^2) - 3(s+1) = S_v(A^2) - 3S_v(A) \leq n - 4 - 3t \leq n - 4.$$

Electronic Journal of Linear Algebra ISSN 1081-3810
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ELA

By Cases 1–4, we know that $S_v(A^2) - 3S_v(A) \leq n-4$ for any vertex $v \in V(G^*)$. So, $\rho^2(G^*) - 3\rho(G^*) \leq n-4$ and $\rho(G^*) \leq \frac{3}{2} + \sqrt{n - \frac{7}{4}}$ by Lemma 3.5. ◻

Lemma 3.7. *Let G^* be as in Lemma 3.4 with no maximal outer-planar 1-summing factor; so, G^* is a 1-sum of t $K_4^{'}s$ for some positive integer t. Then $\rho(G^*) = 1 + \sqrt{n}$.*

Proof. Note that G^* is a bidegree graph with $\delta = 3$, $\Delta = n-1$ and $m = 2n-2$ now. Then the result follows immediately from Lemma 2.7. ◻

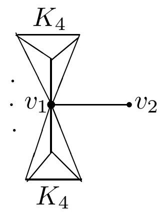

Fig. 3.7. G^*.

Lemma 3.8. *Let G^* be as in Lemma 3.4 with its maximal outer-planar 1-summing factor equal to K_2. Then $\rho(G^*) \leq 1 + \sqrt{n}$.*

Proof. Let $V(G^*) = \{v_1, v_2, \ldots, v_n\}$ and let A denote the adjacency matrix of G^*. We denote by v_1 the only cut-vertex in G^* and by v_2 the other vertex of the maximal outer-planar 1-summing factor (see Fig. 3.7). Then

$$S_{v_1}(A) = 3t+1, S_{v_1}(A^2) = \sum_{v_i \in N_{G^*}(v_1)} deg(v_i) = 3 \cdot 3t + 1 = 9t + 1 = n + 6t - 1,$$

$$S_{v_2}(A) = 1, S_{v_2}(A^2) = n - 1.$$

For any vertex v_j $(j \geq 3)$, we have $S_{v_j}(A) = 3, S_{v_j}(A^2) = \sum_{v_i \in N_{G^*}(v_j)} deg(v_i) = 6 + n - 1$. Therefore, for any vertex $v_j \in V(G^*)$, we have $S_{v_j}(A^2) - 2S_{v_j}(A) \leq n-1$. By Lemma 3.5, we get $\rho^2(G^*) - 2\rho(G^*) \leq n-1$. Hence, $\rho(G^*) \leq 1 + \sqrt{n}$. ◻

Theorem 3.9. *Let G be a $K_{2,3}$-minor free graph of order $n \geq 2$. Then*

$$\rho(G) \leq \frac{3}{2} + \sqrt{n - \frac{7}{4}}.$$

Proof. When $n = 1$, then $G \cong K_1$; when $n = 2$, then $G \cong 2K_1$ or $G \cong P_2$; when $n = 3$, then $G \cong 3K_1$, or $G \cong (P_2 \bigcup K_1)$, or $G \cong P_3$ or $G \cong K_3$. Note that $\rho(K_1) = 0$,

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 23, pp. 171-179, February 2012

$\rho(P_2) = 1$, $\rho(P_3) = \sqrt{2}$, $\rho(K_3) = 2$, hence the theorem holds for $n \leq 3$. When $n \geq 4$, the theorem follows from the Lemmas 3.4 and 3.6–3.8 since $1 + \sqrt{n} \leq \dfrac{3}{2} + \sqrt{n - \dfrac{7}{4}}$. $\square$

REMARK 3.10. Let $\rho^* = \max \{\rho(G) |\ G \text{ be a } K_{2,3}\text{-minor free graph}\}$. Note that $\rho(G^*) = 1 + \sqrt{n}$ in Lemma 3.7 and note that $\lim\limits_{n\to\infty} \dfrac{\frac{3}{2} + \sqrt{n - \frac{7}{4}}}{1 + \sqrt{n}} = 1$, so we say that the upper bound in Theorem 3.9 is good and that while $n \to \infty$, ρ^* is tight up to $O(1 + \sqrt{n})$.

Acknowledgment. We offer many thanks to the referees for their kind reviews and helpful suggestions.

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