

A GENERALIZATION FOR THE CLIQUE AND INDEPENDENCE NUMBERS*

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Abstract. In this paper, lower and upper bounds for the clique and independence numbers are established in terms of the eigenvalues of the signless Laplacian matrix of a given graph G .

Key words. Clique and independence numbers, Signless Laplacian eigenvalues.

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1. Introduction and preliminaries. Let $G = (V, E)$ be a simple connected graph with n vertices and m edges such that, for $i, j = 1, 2, \dots, n$ and $i \neq j$, $v_i \in V$ and $\{v_i, v_j\} \in E$. Let $A(G)$ (or, shortly, A) be the adjacency matrix of G and $D = \text{diag}(d(v_1), d(v_2), \dots, d(v_n))$ be the diagonal matrix of vertex degrees of G . The matrix $L = D - A$ is known as the *Laplacian matrix* ([3]) of G and has been studied extensively in the literature. Recently, there has also been interest in the matrix $Q = D + A$ known as the *signless Laplacian matrix* ([6, 8, 9]) of G . It is known that all eigenvalues of Q are non-negative since it is a positive semidefinite matrix. In the present paper, the eigenvalues of Q will be ordered by $q_1 \geq q_2 \geq \dots \geq q_n \geq 0$, and the corresponding normalized eigenvectors will be represented by $u_1 > 0, u_2, \dots, u_n$. In the meantime, for an arbitrary matrix M , the i -th eigenvalue of M will be denoted by $\lambda_i(M)$.

Let us consider the graph G again. The *clique number* $\omega(G)$ (or ω) and the *independence number* $\alpha(G)$ (or α) of G are defined to be the numbers of vertices of the largest clique and the largest independent sets in G , respectively. It is always true that $\omega(G) = \alpha(G^c)$, where G^c is the complement of G . We note that computing the clique number is generally called an *NP-hard problem* ([1, 7]). The recent survey paper [1] provides an account of the fairly rich literature including applications, formulations, exact algorithms, heuristics, bounds and estimates. Meanwhile, the clique number and a maximum clique can be computed in polynomial time for certain classes of graphs, such as perfect graphs and complements of t -perfect graphs (see [4]). We note

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that, in the literature, several studies have been presented about the connections between the clique number of G and the spectral properties of matrices A , L or Q of G (see, for example, [2, 3, 8, 11]). In particular, Budinich ([2]) proposed a lower bound on the clique number of a given G by combining the quadratic programming formulation of the clique number due to Motzkin and Straus's result ([10]) and due to the spectral decomposition of the matrix A . In fact, Budinich established that this lower bound improves the known spectral lower bounds on the Motzkin-Straus formulation.

In this paper, we mainly strengthen and improve the bounds, obtained in [8] previously, dependent on the eigenvalues of the signless Laplacian matrix Q .

2. Some known results for lower bounds over ω and α . For a given simple connected graph $G = (V, E)$ with n vertices, let S be the $(n - 1)$ -dimensional unit simplex in $\mathbb{R}^n$. In other words, $S = \{x \in \mathbb{R}^n : e^T x = 1, x \geq 0\}$ where e denotes the vector of all ones in $\mathbb{R}^n$. For ω and A as described in the previous section, Motzkin and Straus ([10]) showed that

$$(2.1) \quad 1 - \frac{1}{\omega} = \max_{x \in S} \langle x, Ax \rangle,$$

which is a continuous formulation of a combinatorial optimization problem. In addition to paving the way for the use of continuous optimization methods to solve the maximum clique number, this formulation plays a central role in the derivation of lower bounds on the clique number of G . Note that the dual of (2.1) also holds by the equality

$$(2.2) \quad \frac{1}{\alpha} = \min_{x \in S} \langle x, (I + A)x \rangle.$$

Hence, by using equality (2.2), we can get a great opportunity for the investigation of the independence number in the meaning of spectra.

Considering (2.1) and using the fact $\tilde{x} = (1/n)e \in S$, we clearly have

$$(2.3) \quad \omega \geq \frac{n^2}{n^2 - 2m} \geq 1.$$

This bound on ω matches the *clique number for complete graphs* K_n and their complements. Most of the other lower bounds in the literature are obtained by deriving (2.1) in spectral graph theory. From this point on, we assume that G is a connected graph since the maximum clique problem can be decomposed into smaller problems on each connected component of G . Therefore, we can collect some results about the spectra of such graphs. The reader is referred to [7] for further details.

For a given a connected graph G with the adjacency matrix A , let $\lambda_1 > 0$ and $x_1 \in \mathbb{R}^n$ denote the *Perron root* and the *positive Perron eigenvector* of A , respectively.

Using the feasible solution $\tilde{x} = (1/s_1)x_1 \in S$ of (2.1), where $s_1 = e^T x_1$, Wilf (see [11]) proved the lower bound

$$(2.4) \quad \omega \geq \frac{s_1^2}{s_1^2 - \lambda_1} = 1 + \frac{\lambda_1}{s_1^2 - \lambda_1}$$

with the truth that equality holds if G is complete. The lower bound in (2.4) is an actual improvement over the lower bound in (2.3). On the other hand, in [8], lower bounds for ω in terms of the eigenvalues of the signless Laplacian matrix of G were obtained. More recently, in [2], Budinich proposed a new lower bound that makes use of all the eigenvectors of A . Unless G is a complete multipartite graph, in the same paper, it was also showed that this lower bound strictly improves upon (2.4).

As main results of this paper (see Theorems 3.1, 3.2 below), we give generalizations for the bounds defined in [8] over ω and α . In order to do that we need to present the following material.

If $u^1 > 0, u^2, \dots, u^n$ denote the eigenvectors of Q of unit Euclidean norm corresponding to the eigenvalues $q_1 \geq q_2 \geq \dots \geq q_n \geq 0$, respectively, then one can construct a family of unit vectors $y_i(\beta) = \beta u^i + \sqrt{1 - \beta^2} u^1 \in \mathbb{R}^n$, where $i = 2, 3, \dots, n$ and $\beta \in (-1, 1)$. Then

$$(2.5) \quad z_i(\beta) = (1/e^T y_i(\beta))/y_i(\beta)$$

is a feasible solution of (2.1) for $\beta \in [l_i, u_i]$, where

$$l_i = \max_{j: u_j^i > 0} \frac{-u_j^1}{\sqrt{(u_j^1)^2 + (u_j^i)^2}} < 0 \quad \text{and} \quad u_i = \min_{j: u_j^i < 0} \frac{u_j^1}{\sqrt{(u_j^1)^2 + (u_j^i)^2}} > 0,$$

for the values $i = 2, 3, \dots, n$. We note that, in here, u_j is the normalized eigenvector corresponding to q_j and u_j^i is the i -th entry of u_j . For $i = 2, 3, \dots, n$, let

$$(2.6) \quad \begin{aligned} g_i &= \max_{\beta \in [l_i, u_i]} z_i(\beta)^T A z_i(\beta), \\ &= \max_{\beta \in [l_i, u_i]} \frac{\beta^2 \lambda_i + (1 - \beta^2) \lambda_1}{[\beta s_i + \sqrt{1 - \beta^2} s_1]^2}, \end{aligned}$$

where $s_i = e^T u^i$. Then, by assuming $g^* = \max_{i=2,3,\dots,n} g_i$, it follows from (2.1) that

$$(2.7) \quad \omega \geq \frac{1}{1 - g^*}.$$

3. Main results. The main results of this paper are the following.

THEOREM 3.1. *Let G be a graph with maximum degree Δ . Then*

$$\omega \geq \max_{i=2,3,\dots,n} \frac{n}{n - [\beta^2 q_i + (1 - \beta^2) q_1 - \Delta]},$$

where β as defined in (2.5).

Proof. Let $q_1 \geq q_2 \geq \dots \geq q_n \geq 0$ be the eigenvalues of Q , and let $u^1 > 0, u^2, \dots, u^n$ be the corresponding normalized eigenvectors, respectively. Then, by (2.6) and (2.7),

$$\begin{aligned} \langle z_i(\beta), Q z_i(\beta) \rangle &= \langle z_i(\beta), D z_i(\beta) \rangle + \langle z_i(\beta), A z_i(\beta) \rangle \\ &\leq \Delta \langle z_i(\beta), z_i(\beta) \rangle + g_i, \\ &\leq \frac{\Delta}{\left[\beta s_i + \sqrt{1 - \beta^2 s_1} \right]^2} + 1 - \frac{1}{\omega}. \end{aligned}$$

On the other hand,

$$\langle z_i(\beta), Q z_i(\beta) \rangle = \frac{\beta^2 q_i + (1 - \beta^2) q_1}{\left[\beta s_i + \sqrt{1 - \beta^2 s_1} \right]^2}.$$

Therefore,

$$\frac{\beta^2 q_i + (1 - \beta^2) q_1}{\left[\beta s_i + \sqrt{1 - \beta^2 s_1} \right]^2} \leq \frac{\Delta}{\left[\beta s_i + \sqrt{1 - \beta^2 s_1} \right]^2} + 1 - \frac{1}{\omega}.$$

It is easy to see that

$$\left[\beta s_i + \sqrt{1 - \beta^2 s_1} \right]^2 \leq n.$$

So we obtain

$$\frac{\beta^2 q_i + (1 - \beta^2) q_1 - \Delta}{n} \leq \frac{\beta^2 q_i + (1 - \beta^2) q_1 - \Delta}{\left[\beta s_i + \sqrt{1 - \beta^2 s_1} \right]^2} \leq 1 - \frac{1}{\omega}.$$

Hence, the result follows. $\square$

By applying similar procedure as in the proof of Theorem 3.1 and using the idea of the equality over α given in (2.2), we can also obtain the following theorem.

THEOREM 3.2. *Let G be a graph with minimum degree δ . Then*

$$\alpha \geq \max_{i=2,3,\dots,n} \frac{U^2}{\beta^2 q_i + (1 - \beta^2) q_1 - \delta + U^2},$$

where $U = \beta s_i + \sqrt{1 - \beta^2 s_1}$ and $s_i = e^T u^i$.

REMARK 3.3. As a special case, if we take $i = 1$ in Theorems 3.1 and 3.2, then the lower bounds $\omega \geq \frac{n}{n - q_1 + \Delta}$ and $\alpha \geq \frac{s_1^2}{q_1^2 - \delta + 1}$, given in [8], can be clearly obtained.

In the rest of this section (see Theorem 3.5, Remark 3.6 below), we will present a lower and an upper bound for α (and for ω). They involve the signless Laplacian spectrum and extend the results given for regular graphs in [8].

Consider two sequences of real numbers $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ and $\mu_1 \geq \mu_2 \geq \dots \geq \mu_m$ with $m < n$. The second sequence is said to interlace the first one whenever $\lambda_i \geq \mu_i \geq \lambda_{n-m+i}$ for $i = 1, 2, \dots, m$. The interlacing is called tight if there exists an integer $k \in [1, m]$ such that $\lambda_i = \mu_i$ hold for $1 \leq i \leq k$ and $\lambda_{n-m+i} = \mu_i$ hold for $k+1 \leq i \leq m$.

For a given matrix

$$M_{n \times n} = \begin{bmatrix} M_{11} & \dots & M_{1m} \\ \vdots & \ddots & \vdots \\ M_{m1} & \dots & M_{mm} \end{bmatrix},$$

let us suppose that the rows and columns of are partitioned according to a partitioning $(X_1, X_2, \dots, X_m)$ of $\{1, 2, \dots, n\}$. The quotient matrix is Q_M whose entries are the average row sums of the blocks of $M_{n \times n}$. The partition is called *regular* if each block M_{ij} of M has constant row (and column) sum.

LEMMA 3.4. ([5]) *For a symmetric partitioned matrix M , the eigenvalues of the quotient matrix Q_M interlace the eigenvalues of M .*

Liu et al. ([8]) applied this theory to the signless Laplacian matrix of a graph and obtained a lower and upper bound on the independence and clique numbers for regular graphs. In here, by using again the signless Laplacian matrix, we showed that this result can actually be extended to all graphs.

Let G be a graph with order n and a partition $V(G) = V_1 \cup V_2$. Also, for $i = 1, 2$, let us assume that G_i is the subgraph of G induced by V_i with $n_i < n$ vertices, where $n_1 + n_2 = n$, and average degree r_i . Finally, again for $i = 1, 2$, let

$$\hat{d}_i = \sum_{v \in V_i} \frac{d_G(v)}{n_i}.$$

We then have the following theorem.

THEOREM 3.5. *Suppose V_1 is a largest independent set of a graph G with minimum degree δ and maximum degree Δ . Then*

$$\frac{n(2\delta - q_2)(\delta - q_2)}{q_2^2 + 4\Delta(\Delta - q_2)} \leq \alpha \leq \frac{n(2\Delta - q_n)(\Delta - q_n)}{q_n^2 + 4\delta(\delta - q_n)}.$$

Proof. We first note that

$$Q(G) = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} = \begin{bmatrix} D_{11} + A(G_1) & M_{12} \\ M_{21} & D_{22} + A(G_2) \end{bmatrix},$$

where

$$D_{11} = \text{diag}(d_G(v_1), d_G(v_2), \dots, d_G(v_{n_1})), \\ D_{22} = \text{diag}(d_G(v_{n_1+1}), d_G(v_{n_1+2}), \dots, d_G(v_n))$$

and $M_{21} = M_{12}^T$. Moreover the set V_1 gives rise to a partition of $Q(G)$ with the quotient matrix

$$Q_{Q(G)} = \begin{bmatrix} \hat{d}_1 & \hat{d}_1 \\ \frac{\alpha \hat{d}_1}{n-\alpha} & 2\hat{d}_2 - \frac{\alpha \hat{d}_1}{n-\alpha} \end{bmatrix}.$$

Therefore, by Lemma 3.4, we have $q_n \leq \lambda_2(Q_{Q(G)}) \leq q_2$ which gives the inequality

$$\frac{n(2\hat{d}_2 - q_2)(\hat{d}_1 - q_2)}{q_2^2 + 2(\hat{d}_1 + \hat{d}_2)(\hat{d}_1 - q_2)} \leq \alpha \leq \frac{n(2\hat{d}_2 - q_n)(\hat{d}_1 - q_n)}{q_n^2 + 2(\hat{d}_1 + \hat{d}_2)(\hat{d}_1 - q_n)}.$$

As the final stage, for $i = 1, 2$, we clearly have $\delta \leq \hat{d}_i \leq \Delta$. $\square$

REMARK 3.6. By applying a similar procedure as in the proof of Theorem 3.5, one can obtain a lower and an upper bound of the clique number ω for any graph G .

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