

ADDITIVITY OF MAPS ON $\mathbb{N} \times \mathbb{N}$ MATRICES*

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Abstract. It was recently proved that if a map defined on the ring of all $n \times n$ matrices satisfies the condition $f(x + y) = f(x) + f(y)$ for all invertible x and y , then f is additive. In the present paper, it is proved that the same result holds for the ring of $\mathbb{N} \times \mathbb{N}$ matrices with a finite number of nonzero entries in each column defined over a field of at least three elements.

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1. Introduction. In [8], it was shown that a map $f: M_n(F) \rightarrow M_n(F)$ is additive if and only if it satisfies the additivity condition, i.e.,

$$(1.1) \quad f(x + y) = f(x) + f(y),$$

for all invertible matrices x, y . Soon after that two of the authors of [8] generalized this result to the case of matrices of a fixed rank. Namely, if $s \geq \frac{n}{2}$ and f satisfies (1.1) for all rank- s matrices, then f is additive. Analogous result holds for the tensor product of linear spaces [3]. However, the same claim does not hold if $s < \frac{n}{2}$. In particular, we can reshow the following example¹.

EXAMPLE 1.1. Let $n > 1$ and let $f: M_n(\mathbb{Q}) \rightarrow M_n(\mathbb{Q})$ be the map such that $f(A) = \det A \cdot A$. Then, for all rank-1 matrices, we have $f(A + B) = f(A) + f(B) = 0$. Yet, f is not additive.

It is a natural question to ask whether the same conclusions hold in subrings of $M_n(F)$. From [5], we know that we have an analogous theorem for the rank- s matrices for $\frac{n}{2} \leq s < n$ defined over an arbitrary field².

In this work, we would like to consider similar problems, but for the $\mathbb{N} \times \mathbb{N}$ matrices. These objects have quite an interesting history (see [1]). In order to introduce well-defined multiplication of such matrices, one usually limit themselves to the so-called column-finite matrices (for more details see for instance [2]). Let $\mathcal{M}_{cf}(F)$ denote the (latter) ring of all $\mathbb{N} \times \mathbb{N}$ matrices defined over a field F that in each column possess a finite number of nonzero entries and let $\mathcal{A}$ denote any ring. We will prove the following theorem.

THEOREM 1.2. Let F be a field of at least three elements. If a map $\phi: \mathcal{M}_{cf}(F) \rightarrow \mathcal{A}$ satisfies

$$(1.2) \quad \phi(A + B) = \phi(A) + \phi(B) \quad \text{for all invertible } A, B \in \mathcal{M}_{cf}(F),$$

then ϕ is additive.

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¹For more facts on this topic see [6].

²Apparently in [4], it was proven that if (1.1) is satisfied for any two invertible upper triangular matrices, then f is additive on the ring of $n \times n$ upper triangular matrices $T_n(F)$ for any field F of at least 3 elements, but unfortunately we do not have access to this paper.

Similarly to the $n \times n$ case, it seems interesting whether a theorem analogous to the above holds for the subrings of $\mathcal{M}_{Cf}(F)$. Probably, its most known subring is $\mathcal{T}_\infty(F)$ consisting of all $\mathbb{N} \times \mathbb{N}$ (upper) triangular matrices. We will show that, indeed, under some further constraints analogous theorem is true.

THEOREM 1.3. *Let F be a field of at least four elements. If a map $\phi: \mathcal{T}_\infty(F) \rightarrow \mathcal{A}$ satisfies*

$$(1.3) \quad \phi(A + B) = \phi(A) + \phi(B) \quad \text{for all invertible } A, B \in \mathcal{T}_\infty(F),$$

then ϕ is additive.

2. Proofs of results. Let us start by introducing the notation. The symbol I_∞ will stand for the $\mathbb{N} \times \mathbb{N}$ identity matrix. In our proofs, we are going to come across some subalgebras of $\mathcal{M}_{Cf}(F)$. As we have already declared, one of them is going to be $\mathcal{T}_\infty(F)$. Another subalgebra of $\mathcal{M}_{Cf}(F)$ that is going to be used in our investigations is the algebra consisting of block-diagonal matrices of the same fixed size n ; we will denote it by $\mathcal{M}_n^{\text{di}}(F)$. We also introduce the subalgebras $\mathcal{M}_n^{\text{di}+u}(F)$, $\mathcal{M}_n^{\text{di}+\ell}(F)$ whose elements have nonzero entries in the $n \times n$ diagonal blocks and above, respectively, below, main diagonal.

Similarly as in [7, 5], we can observe that the following holds.

LEMMA 2.1. *Suppose F is a field, and $\phi: \mathcal{M} \rightarrow \mathcal{A}$ is a map satisfying (1.1), where $\mathcal{M}$ is either $\mathcal{M}_{Cf}(F)$ or $\mathcal{T}_\infty(F)$. Then,*

1. $f(-A) = -f(A)$ for all invertible A ,
2. $f(A - B) = f(A) - f(B)$ for all invertible A, B .

Proof. To prove the first claim, observe that in the case $\text{char}(F) \neq 2$, then

$$f(A) = f(A + A - A) = f(A) + f(A) + f(-A),$$

so $0 = f(A) + f(-A)$.

In the case when $\text{char}(F) = 2$, we have $-A = A$, so $f(-A) = f(A) = -f(A)$.

Now we can use the first assertion to prove the second one. Namely

$$f(A) - f(B) = f(A) + f(-B) = f(A - B).$$

By the same arguments, we derive

COROLLARY 2.2. *Suppose F is a field, and $\phi: \mathcal{M} \rightarrow \mathcal{A}$ is a map satisfying (1.1) for all $x, y \in \mathcal{S}$, where $\mathcal{S}$ is a fixed set such that $\emptyset \neq \mathcal{S} \subset \mathcal{M}$, whereas $\mathcal{M}$ is either $\mathcal{M}_{Cf}(F)$ or $\mathcal{T}_\infty(F)$. Then,*

1. $f(-A) = -f(A)$ for all $A \in \mathcal{S}$,
2. $f(A - B) = f(A) - f(B)$ for all $A, B \in \mathcal{S}$.

In the next sections, we are going to prove our main results.

2.1. Proof of Theorem 1.2. We start by observing facts concerning some decompositions.

LEMMA 2.3. *Let F be a field of at least three elements and let $n > 1$. Suppose $U_1 \in \mathcal{M}_n^{\text{di}+u}(F)$ and $U_2 \in \mathcal{M}_n^{\text{di}+u}(F)$ is invertible. Then, there exists invertible $C \in \mathcal{M}_n^{\text{di}+u}(F)$ such that both matrices $U_1 + C$, $U_2 - C$ are invertible.*

Proof. Since a block diagonal matrix is invertible if and only if all its diagonal blocks are invertible, it suffices to prove the claim for the blocks. Thus, the claim follows from the result for $n \times n$ matrices, which was proved in [8, Lemma 4.1]. $\square$

By a similar, though not identical demonstration, we also obtain

COROLLARY 2.4. *Let F be a field of at least three elements. Suppose $U \in \mathcal{M}_2^{\text{di}+u}(F)$ and $L \in \mathcal{M}_2^{\text{di}+\ell}(F)$. Then, there exists an invertible $D \in \mathcal{M}_2^{\text{di}}(F)$ such that both matrices $U + D$, $L - D$ are invertible.*

Proof. As the invertibility of the considered matrices depends on their diagonal blocks, it suffices to prove that for arbitrary 2×2 matrices A, B , there exists an invertible matrix D such that $A + D$ and $B - D$ are invertible. Since X is invertible if and only if MXN is invertible for all invertible M, N , we can assume that A is in the canonical form. Consider then all possibilities.

Suppose that

$$A = \begin{bmatrix} 0 & * \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

and

$$D = \begin{bmatrix} x & z \\ 0 & y \end{bmatrix}.$$

If $c = 0$, without loss of generality, assume that $a = b = d = 0$. In this case, it suffices to set $x = y = 1$ and $z = 0$.

If $c \neq 0$, then we define x as an invertible element of F different from a , y as invertible element of F different from d , and for $z = b - c^{-1}(a - x)(d - y) + c^{-1}$, we obtain $\det(A + D) = xy \neq 0$, $\det(B - D) = 1$, and we are done. $\square$

We finish our decompositional considerations with the following lemma.

LEMMA 2.5. *Let F be a field. For every $M \in \mathcal{M}_{Cf}(F)$, there exist invertible matrices $A, B \in \mathcal{M}_{Cf}(F)$ such that $M = A + B$.*

Proof. Let us decompose M into a sum of three matrices $L = [\ell_{ij}]$, $D = [d_{ij}]$, $U = [u_{ij}]$ defined as follows

$$D = \begin{bmatrix} D_1 & & & & \\ & D_2 & & & \\ & & D_3 & & \\ & & & D_4 & \\ & & & & \ddots \end{bmatrix} \quad \text{with } D_i = \begin{bmatrix} m_{2i-1,2i-1} & m_{2i-1,2i} \\ m_{2i,2i-1} & m_{2i,2i} \end{bmatrix},$$

and

$$\ell_{ij} = \begin{cases} m_{ij} & \text{if } i - j \geq 2 \\ m_{ij} & \text{if } i - j = 1 \text{ and } 2 \nmid i \\ 0 & \text{otherwise,} \end{cases} \quad u_{ij} = \begin{cases} m_{ij} & \text{if } j - i \geq 2 \\ m_{ij} & \text{if } j - i = 1 \text{ and } 2 \mid i \\ 0 & \text{otherwise.} \end{cases}$$

From [8, Lemma 2.1], it follows that every block D_i in D can be expressed as a sum of two invertible matrices

A_i, B_i . Thus, let us put

$$\tilde{A} = \begin{bmatrix} A_1 & & & & \\ & A_2 & & & \\ & & A_3 & & \\ & & & A_4 & \\ & & & & \ddots \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} B_1 & & & & \\ & B_2 & & & \\ & & B_3 & & \\ & & & B_4 & \\ & & & & \ddots \end{bmatrix}.$$

Then, when we define A and B by the rule

$$A = \tilde{A} + L, \quad B = \tilde{B} + U,$$

we have $M = A + B$, and the matrices A and B are invertible. $\square$

Note that from the proof it follows that any matrix from $\mathcal{M}_{Cf}(F)$ can be decomposed into a sum of two invertible matrices from $\mathcal{M}_2^{\text{di}+u}(F)$ and similarly for $\mathcal{M}_2^{\text{di}+\ell}(F)$.

The inspiration and also certain foundation of our result and its proof is the analogous result for $n \times n$ matrices. Namely, the following theorem.

THEOREM 2.6. [8, Thm.4.1] *Let $n > 1$ be a natural number, and let $M_n(F)$ be the ring of all $n \times n$ matrices over a field F . Then, a map $\phi: M_n(F) \rightarrow M_n(F)$ is additive, if $\phi(A + B) = \phi(A) + \phi(B)$ for all $A, B \in \text{GL}_n(F)$.*

It should be underlined that the proof of the above theorem does not rely on the exact forms of the range of ϕ , and therefore, we can formulate the corollary.

COROLLARY 2.7. *Let $n > 1$ be a natural number, let $M_n(F)$ be the ring of all $n \times n$ matrices over a field F , and let $\mathcal{A}$ be an algebra. Then, a map $\phi: M_n(F) \rightarrow \mathcal{A}$ is additive if $\phi(A + B) = \phi(A) + \phi(B)$ for all $A, B \in \text{GL}_n(F)$.*

This corollary is vital for the next proof.

PROPOSITION 2.8. *Let $n > 1$ be a natural number, F a field of at least three elements and $\mathcal{A}$ an algebra. If a map $\phi: \mathcal{M}_{Cf}(F) \rightarrow \mathcal{A}$ satisfies the condition $\phi(A + B) = \phi(A) + \phi(B)$ for all invertible A, B , then ϕ satisfies the condition $\phi(U_1 + U_2) = \phi(U_1) + \phi(U_2)$ for all pairs $U_1, U_2 \in \mathcal{M}_n^{\text{di}+u}(F)$.*

Proof. Let us prove the claim in two steps.

- 1. The map ϕ satisfies $\phi(U_1 + U_2) = \phi(U_1) + \phi(U_2)$ for all $U_1, U_2 \in \mathcal{M}_n^{\text{di}+u}(F)$ with U_2 invertible.**

From Lemma 2.3, we know that for all $U_1 \in \mathcal{M}_n^{\text{di}+u}(F)$, $U_2 \in \mathcal{M}_n^{\text{di}+u}(F)$, where U_2 is invertible, there exists invertible $C \in \mathcal{M}_n^{\text{di}+u}(F)$ such that $U_1 + C, U_2 - C$ are both invertible. Then,

$$\begin{aligned} \phi(U_1 + U_2) &= \phi(U_1 + C + U_2 - C) = \phi(U_1 + C) + \phi(U_2 - C) \\ &= \phi(U_1 + C) + \phi(U_2) - \phi(C) = \phi(U_1) + \phi(U_2). \end{aligned}$$

- 2. The map ϕ satisfies $\phi(U_1 + U_2) = \phi(U_1) + \phi(U_2)$ for all $U_1, U_2 \in \mathcal{M}_n^{\text{di}+u}(F)$.**

By Lemma 2.5, there exist invertible $A, B \in \mathcal{M}_n^{\text{di}+u}(F)$ such that $U_2 = A + B$. Then, by the preceding step

$$\begin{aligned} \phi(U_1 + U_2) &= \phi(U_1 + A + B) = \phi(U_1 + A) + \phi(B) = \phi(U_1) + \phi(A) + \phi(B) \\ &= \phi(U_1) + \phi(A + B) = \phi(U_1) + \phi(U_2). \end{aligned}$$

By similar arguments, we obtain

COROLLARY 2.9. *Let $n > 1$ be a natural number, F a field of at least three elements, and $\mathcal{A}$ an algebra. If a map $\phi: \mathcal{M}_{Cf}(F) \rightarrow \mathcal{A}$ satisfies the condition $\phi(A + B) = \phi(A) + \phi(B)$ for all invertible A, B , then ϕ satisfies the condition $\phi(A + B) = \phi(A) + \phi(B)$ for all pairs $A, B \in \mathcal{M}_n^{\text{di}+\ell}(F)$.*

Now we can prove Theorem 1.2.

Proof of Thm. 1.2. Again, the proof involves two claims.

1. For all $U \in \mathcal{M}_2^{\text{di}+u}(F)$, $L \in \mathcal{M}_2^{\text{di}+\ell}(F)$, we have $\phi(U + L) = \phi(U) + \phi(L)$.

By Corollary 2.4, there exists $D \in \mathcal{M}_2^{\text{di}}(F)$ such that both matrices $U + D$ and $L - D$ are invertible. Thus, by (1.2), fact that $U + D \in \mathcal{M}_2^{\text{di}+u}(F)$, $L - D \in \mathcal{M}_2^{\text{di}+\ell}(F)$, Proposition 2.9, and Corollary 2.4, we have

$$\begin{aligned}\phi(L + U) &= \phi(U + D + L - D) = \phi(U + D) + \phi(L - D) \\ &= \phi(U) + \phi(D) + \phi(L) - \phi(D) = \phi(U) + \phi(L).\end{aligned}$$

2. For all $A, B \in \mathcal{M}_{Cf}(F)$, we have $\phi(A + B) = \phi(A) + \phi(B)$.

By Lemma 2.5, there exists invertible matrices $U_1, U_2 \in \mathcal{M}_2^{\text{di}+u}(F)$, $L_1, L_2 \in \mathcal{M}_2^{\text{di}+\ell}(F)$ such that $A = U_1 + L_1$, $B = U_2 + L_2$. Then, by (1.2), the previous point, Proposition 2.9 and Corollary 2.4

$$\begin{aligned}\phi(A + B) &= \phi(U_1 + L_1 + U_2 + L_2) = \phi(U_1 + U_2) + \phi(L_1 + L_2) \\ &= \phi(U_1) + \phi(U_2) + \phi(L_1) + \phi(L_2) = \phi(U_1 + L_1) + \phi(U_2 + L_2) = \phi(A) + \phi(B).\end{aligned}$$

2.2. Proof of Theorem 1.3. The idea of the proof for the triangular case is similar to the one for the ring $\mathcal{M}_n^{\text{di}+u}(F)$.

LEMMA 2.10. *Let F be a field of at least three elements. For every $M \in \mathcal{T}_\infty(F)$, there exist invertible matrices $A, B \in \mathcal{T}_\infty(F)$ such that $M = A + B$.*

Proof. If F has at least three elements, then every element $a \in F$ can be written as a sum of two invertible elements $\tilde{a}, \hat{a} \in F$. Namely

$$a = \begin{cases} 1 + (-1) & \text{if } a = 0 \\ 1 + (a - 1) & \text{if } a \neq 0, 1 \\ b + (1 - b) & \text{for } b \neq 0, 1, \text{ if } a = 1. \end{cases}$$

Hence, for all $i \in \mathbb{N}$, there exist invertible $\tilde{m}_{ii}, \hat{m}_{ii}$ such that $m_{ii} = \tilde{m}_{ii} + \hat{m}_{ii}$. Define then $A = [a_{ij}]$, $B = [b_{ij}]$ as follows

$$a_{ij} = \begin{cases} \tilde{m}_{ii} & \text{if } j = i \\ m_{ij} & \text{otherwise,} \end{cases} \quad b_{ij} = \begin{cases} \hat{m}_{ii} & \text{if } j = i \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, both matrices A and B are invertible and $M = A + B$. □

In the following lemma, it turns out that in order to obtain all the same properties, we will have to assume that a field contain at least four elements.

LEMMA 2.11. *Let F be a field of at least four elements. For any matrix $M \in \mathcal{T}_\infty(F)$ and any invertible matrix $A \in \mathcal{T}_\infty(F)$, there exists an invertible matrix $B \in \mathcal{T}_\infty(F)$ such that $M - B$ and $B + A$ are invertible.*

Proof. Suppose that $M = [m_{ij}]$ is an arbitrary triangular matrix and $A = [a_{ij}]$ is arbitrary invertible triangular matrix. Since invertibility of a triangular matrix depends only on its main diagonal, it suffices to construct the main diagonal of desired matrix B . The matrices B , $M - B$, $B + A$ are invertible if and only if we have

$$b_{ii} \neq 0, m_{ii}, -a_{ii} \quad \text{for all } i \in \mathbb{N}.$$

Clearly, as $|F| \geq 4$, it is possible to fulfill this condition. $\square$

Now we can prove Theorem 1.3.

Proof of Thm. 1.3. 1. If $\phi : \mathcal{T}_\infty(F) \rightarrow \mathcal{T}_\infty(F)$ satisfies (1.1) for all invertible matrices, then

$$\phi(A + M) = \phi(A) + \phi(M) \quad \text{for all invertible } A \in \mathcal{T}_\infty(F) \text{ and all } M \in \mathcal{T}_\infty(F).$$

Let $A \in \mathcal{T}_\infty(F)$ be invertible and let $M \in \mathcal{T}_\infty(F)$ be arbitrary matrix. By Lemma 2.11, there exists invertible $B \in \mathcal{T}_\infty(F)$ such that $M - B$ and $B + A$ are also invertible. Then, by (1.1)

$$\begin{aligned} \phi(M + A) &= \phi((M - B) + (B + A)) = \phi(M - B) + \phi(B + A) = \phi(M - B) + \phi(B) + \phi(A) \\ &= \phi(M - B + B) + \phi(A) = \phi(M) + \phi(A). \end{aligned}$$

2. If $\phi : \mathcal{T}_\infty(F) \rightarrow \mathcal{T}_\infty(F)$ satisfies (1.1) for all invertible matrices, then $\phi(M + N) = \phi(M) + \phi(N)$ for all $M, N \in \mathcal{T}_\infty(F)$.

Let $M, N \in \mathcal{T}_\infty(F)$ be arbitrary matrices. By Lemma 2.10, there exist invertible $A, B \in \mathcal{T}_\infty(F)$ such that $N = A + B$. Then,

$$\begin{aligned} \phi(M + N) &= \phi(M + A + B) = \phi(M + A) + \phi(B) = \phi(M) + \phi(A) + \phi(B) \\ &= \phi(M) + \phi(A + B) = \phi(M) + \phi(N), \end{aligned}$$

which finishes the proof. $\square$

3. Closing comments. Let us repeat that we have proved Theorem 1.2 for the matrices defined over fields of at least three elements, whereas Theorem 1.3 for matrices over fields of at least four elements. We are not sure whether (and how) all the lacking cases can be completed. However, for sure in the case of triangular matrices, over two element field theorem analogous to Theorem 1.3 does not hold.

EXAMPLE 3.1. Let $|F| = 2$. Define $\phi : \mathcal{T}_\infty(F) \rightarrow \mathcal{T}_\infty(F)$ as follows

$$\phi(A) = B = [b_{ij}], \quad b_{ij} = \begin{cases} 0 & \text{if } j \neq i \\ 0 & \text{if } j = i \text{ and } a_{ii} = 0 \text{ for all } i \\ 1 & \text{if } j = i \text{ and } a_{ii} = 1 \text{ for all } i \\ 1 & \text{if } j = i \text{ and there exist } k, \ell \text{ such that } a_{kk} = 1, a_{\ell\ell} = 0, \\ & \text{and } i \text{ is the first nonzero element of the main diagonal of } A, \\ 0 & \text{otherwise.} \end{cases}$$

Then,

$$0 = I + I = \phi(I + U) + \phi(I + \tilde{U}) = \phi(I + U + I + \tilde{U}) = \phi(U + \tilde{U}) = 0,$$

for all strictly upper triangular $U, \tilde{U}$, that is, ϕ is additive on invertible matrices. Yet, denoting by e_{ii} , the matrix with 1 in the position (i, i) and zeros elsewhere, we have

$$e_{11} = \phi(e_{11} + e_{22}) \neq \phi(e_{11}) + \phi(e_{22}) = e_{11} + e_{22},$$

so ϕ is not additive on $\mathcal{T}_\infty(F)$.

REFERENCES

- [1] M. Bernkopf. A history of infinite matrices. *Arch. Hist. Exact Sci.* 4(4):308–358, 1968.
- [2] D.P. Bossaller and S.R. López. On the associativity of infinite matrix multiplication. *Am. Math. Mon.*, 126(1):41–52, 2019.
- [3] W.L. Chooi and K.H. Kwa. Additive maps of rank r tensors and symmetric tensors. *Linear Multilinear Algebra*, 67(6):1269–1293, 2019.
- [4] H. Liu and X. Xu. Additive maps on invertible upper triangular matrices. *J. Jilin Univ., Sci.*, 55(1):79–81, 2017.
- [5] H. Liu, X. Xu, and H. Yu. Additive maps on rank- s upper triangular matrices. *Aequat. Math.*, 99:1507–1520, 2025.
- [6] X. Xu, C. Li, and J. Zhu. Maps determined by rank- s matrices for relatively small s . *Aequat. Math.*, 91:391–400, 2017.
- [7] X. Xu and H. Liu. Additive maps on rank- s matrices. *Linear Multilinear Algebra*, 65(4):806–812, 2017.
- [8] X. Xu, Y. Pei, and X. Yi. Additive maps on invertible matrices. *Linear Multilinear Algebra*, 64(7):1283–1294, 2016.