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ON A GENERALIZATION OF THE JOHNSON–NEWMAN THEOREM TO MULTIPLE RANK-ONE PERTURBATIONS*

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Abstract. Wang and Zhao (Adv. Appl. Math. 173 (2026) 102994) generalized the classic Johnson–Newman theorem on simultaneous similarity of symmetric matrices from a single rank-one perturbation to multiple rank-one perturbations. However, their result applies only to specific rank-one perturbations, and the condition provided is computationally involved, relying on complex multivariate polynomials. We provide a simple proof of their result, leading to an improved version with a simplified condition that holds for arbitrary rank-one perturbations.

Key words. Rank-one perturbation, Simultaneous similarity, Weinstein–Aronszajn identity, Gram matrix, Orthogonal matrix, Nonnegative matrix.

AMS subject classifications. 05C50, 15A18.

1. Introduction.

Let $\mathcal{A} := (A_0, \ldots, A_m)$ and $\mathcal{B} := (B_0, \ldots, B_m)$ be two $(m+1)$-tuples of real symmetric matrices. We call $\mathcal{A}$ and $\mathcal{B}$ *entrywise similar* if for each $i \in \{0, 1, \ldots, m\}$, the corresponding matrices A_i and B_i are similar, i.e., there exists an orthogonal matrix Q_i such that $Q_i^{\mathrm{T}} A_i Q_i = B_i$. We call $\mathcal{A}$ and $\mathcal{B}$ *simultaneously similar* if there exists a common orthogonal matrix Q such that $Q^{\mathrm{T}} A_i Q = B_i$ for all $i \in \{0, 1, \ldots, m\}$. In a classic paper concerning cospectral graphs, Johnson and Newman [4] proved the following result on simultaneous similarity for the special case that $\mathcal{A} = (A, A + ee^{\mathrm{T}})$ and $\mathcal{B} = (B, B + ee^{\mathrm{T}})$, where e is the all-ones vector.

THEOREM 1.1 ([4]). *Let A and B be two $n \times n$ real symmetric matrices. Then, $(A, A + ee^{\mathrm{T}})$ and $(B, B + ee^{\mathrm{T}})$ are entrywise similar if and only if they are simultaneously similar via an orthogonal matrix Q such that $Q^{\mathrm{T}} A Q = B$ and $Q^{\mathrm{T}} e = e$.*

Assuming A and B are adjacency matrices of two graphs G and H, Wang [6] observed that if G is controllable, then the corresponding orthogonal matrix Q is unique and rational. Based on this fundamental observation, Wang and his collaborators have successfully developed the theory of generalized spectral characterizations of graphs; see, e.g., [2, 5, 6, 7].

In a recent paper concerning the spectral characterizations of regular graphs, Qiu et al. [5] found an extension of Theorem 1.1: the all-ones matrix ee^{T} in Theorem 1.1 can be replaced by any symmetric matrices of the form $\xi\xi^{\mathrm{T}}$, where $\xi \in \mathbb{R}^n \setminus \{0\}$.

THEOREM 1.2 ([5]). *Let A, B be two $n \times n$ real symmetric matrices and let α, β be two nonzero vectors in $\mathbb{R}^n$. Then, $(A, A + \alpha\alpha^{\mathrm{T}})$ and $(B, B + \beta\beta^{\mathrm{T}})$ are entrywise similar if and only if they are simultaneously similar via an orthogonal matrix Q such that $Q^{\mathrm{T}} \alpha = \beta$.*

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Note that both Theorems 1.1 and 1.2 only consider simultaneous similarity for 2-tuples, which play key roles in studying the problem of generalized spectral characterizations of graphs. Motivated by a refinement of generalized spectral characterization of graphs, the following natural problem was suggested in a recent paper of Wang and Zhao [10].

Problem 1.3 ([10]). *Let $\mathcal{A} = (A, A + \alpha_1\alpha_1^{\mathrm{T}}, \dots, A + \alpha_m\alpha_m^{\mathrm{T}})$ and $\mathcal{B} = (B, B + \beta_1\beta_1^{\mathrm{T}}, \dots, B + \beta_m\beta_m^{\mathrm{T}})$, where A, B are $n \times n$ real symmetric matrices, and α_i, β_i are nonzero vectors in $\mathbb{R}^n$ for all $i \in \{1, \dots, m\}$. Suppose that $\mathcal{A}$ and $\mathcal{B}$ are entrywise similar. Under what conditions can we guarantee that $\mathcal{A}$ and $\mathcal{B}$ are simultaneously similar via an orthogonal matrix Q such that $Q^{\mathrm{T}}AQ = B$ and $Q^{\mathrm{T}}\alpha_i = \beta_i$ for $i = 1, 2, \dots, m$?*

In the same paper, Wang and Zhao considered the special case that $\alpha_i = \beta_i = e_i$ for $i = 1, \dots, m$, where $e_1, \dots, e_m$ are some nonzero 0-1 vectors such that any two of them are orthogonal (equivalently, the positions of the ones are pairwise disjoint). For this case, they established the following theorem.

Theorem 1.4 ([10]). *Let $J_{i,j} = e_ie_j^{\mathrm{T}}$ for $1 \le i \le j \le m$. If $A + \sum s_{i,j}J_{i,j}$ and $B + \sum s_{i,j}J_{i,j}$ have the same characteristic polynomial for any $s_{i,j}$ where $1 \le i \le j \le m$, then there exists an orthogonal matrix Q such that $Q^{\mathrm{T}}AQ = B$ and $Q^{\mathrm{T}}e_i = e_i$ for $i = 1, \dots, m$.*

In a recent paper [11], under the additional assumption that A and B are adjacency matrices (or nonnegative matrices), Xu and Zhao simplify the condition of Theorem 1.4 significantly.

Theorem 1.5 ([11]). *Let $J_{i,i} = e_ie_i^{\mathrm{T}}$ for $1 \le i \le m$. Suppose that A and B are adjacency matrices of some graphs of the same order. If $A + \sum s_iJ_{i,i}$ and $B + \sum s_iJ_{i,i}$ have the same characteristic polynomial for any s_i where $1 \le i \le m$, then there exists an orthogonal matrix Q such that $Q^{\mathrm{T}}AQ = B$ and $Q^{\mathrm{T}}e_i = e_i$ for $i = 1, \dots, m$.*

Note that in Theorem 1.5, the summation contains exactly m items, whereas the corresponding summation in Theorem 1.4 contains as many as $\binom{m+1}{2}$ items. The main aim of this paper is to generalize Theorems 1.4 and 1.5 from the special case $\alpha_i = \beta_i = e_i$ to any vectors. The proof uses a technique introduced recently in [8] (which was originally developed to address a small gap in [1]).

Theorem 1.6. *Let A, B be two symmetric real matrices of order n and $\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_m$ be vectors in $\mathbb{R}^n$. Then the following two statements are equivalent:*

(i) *There exists an orthogonal matrix Q such that $Q^{\mathrm{T}}AQ = B$ and $Q^{\mathrm{T}}\alpha_i = \beta_i$ for $i = 1, \dots, m$.*

(ii) *The matrices $A + \left(\sum_{i\in S}\alpha_i\right)\left(\sum_{i\in S}\alpha_i\right)^{\mathrm{T}}$ and $B + \left(\sum_{i\in S}\beta_i\right)\left(\sum_{i\in S}\beta_i\right)^{\mathrm{T}}$ are similar for any subset $S \subset \{1, \dots, m\}$ with cardinality at most 2.*

We note that the second statement in Theorem 1.6 means that (1) $\mathcal{A} = (A, A + \alpha_1\alpha_1^{\mathrm{T}}, \dots, A + \alpha_m\alpha_m^{\mathrm{T}})$ and $\mathcal{B} = (B, B + \beta_1\beta_1^{\mathrm{T}}, \dots, B + \beta_m\beta_m^{\mathrm{T}})$ are entrywise similar, and (2) $A + (\alpha_i + \alpha_j)(\alpha_i + \alpha_j)^{\mathrm{T}}$ and $B + (\beta_i + \beta_j)(\beta_i + \beta_j)^{\mathrm{T}}$ are similar for any i, j with $1 \le i < j \le m$. Theorem 1.6 clearly improves upon Theorem 1.4. We make no assumption on the vectors α_i, β_i in Theorem 1.6. Moreover, the cospectrality condition in Theorem 1.4 is greatly simplified, and the new condition relies exclusively on symmetric matrices.

We also obtain an improvement of Theorem 1.5 for nonnegative matrices and vectors, which we state as the following theorem.

Theorem 1.7. *Let A, B be two nonnegative symmetric real matrices of order n and α_i, β_i be nonnegative vectors in $\mathbb{R}^n$ for $i = 1, \dots, m$. Then the following two statements are equivalent:*

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(i) *There exists an orthogonal matrix Q such that $Q^{\mathrm{T}}AQ = B$ and $Q^{\mathrm{T}}\alpha_i = \beta_i$ for $i = 1, \dots, m$.*

(ii) *The matrices $A + \sum_{i\in S} \alpha_i\alpha_i^{\mathrm{T}}$ and $B + \sum_{i\in S} \beta_i\beta_i^{\mathrm{T}}$ are similar for any subset $S \subset \{1, \dots, m\}$ with cardinality at most 2.*

2. Proofs of Theorems 1.6 and 1.7. Let A be an $n \times n$ real symmetric matrix and $\lambda_1, \dots, \lambda_s$ be all its distinct eigenvalues with multiplicities $r_1, \dots, r_s$, respectively. Let P_i be any $n \times r_i$ matrix whose columns consist of an orthonormal basis of $\mathcal{E}_{\lambda_i}(A)$, the eigenspace of A corresponding to λ_i. Then, A has the spectral decomposition

$$A = \lambda_1 P_1 P_1^{\mathrm{T}} + \cdots + \lambda_s P_s P_s^{\mathrm{T}}.$$

We note that $P_i P_i^{\mathrm{T}}$ is well defined although P_i is not unique. Indeed, if $\tilde{P}_i$ consists of another orthonormal basis of $\mathcal{E}_{\lambda_i}(A)$, then we must have $\tilde{P}_i = P_i Q$ for some orthogonal matrix Q, which clearly implies $\tilde{P}_i\tilde{P}_i^{\mathrm{T}} = P_i P_i^{\mathrm{T}}$.

The following result can be proved using spectral decomposition and some standard skills. See, e.g., [5] for details.

Lemma 2.1. *Let A and B be two similar real symmetric matrices with spectral decomposition $A = \sum_{k=1}^{s} \lambda_k P_k P_k^{\mathrm{T}}$ and $B = \sum_{k=1}^{s} \lambda_k R_k R_k^{\mathrm{T}}$, respectively. For $\alpha, \beta \in \mathbb{R}^n$, if $A + \alpha\alpha^{\mathrm{T}}$ and $B + \beta\beta^{\mathrm{T}}$ are similar, then $||P_k^{\mathrm{T}}\alpha|| = ||R_k^{\mathrm{T}}\beta||$ for $k = 1, \dots, s$.*

A direct consequence of Lemma 2.1 is the following:

Proposition 2.2. *Let A and B be two similar real symmetric matrices with spectral decomposition $A = \sum_{k=1}^{s} \lambda_k P_k P_k^{\mathrm{T}}$ and $B = \sum_{k=1}^{s} \lambda_k R_k R_k^{\mathrm{T}}$, respectively. For $\alpha, \tilde{\alpha}, \beta, \tilde{\beta} \in \mathbb{R}^n$, if $(A + \alpha\alpha^{\mathrm{T}}, A + \tilde{\alpha}\tilde{\alpha}^{\mathrm{T}}, A + (\alpha + \tilde{\alpha})(\alpha + \tilde{\alpha})^{\mathrm{T}})$ and $(B + \beta\beta^{\mathrm{T}}, B + \tilde{\beta}\tilde{\beta}^{\mathrm{T}}, B + (\beta + \tilde{\beta})(\beta + \tilde{\beta})^{\mathrm{T}})$ are entrywise similar, then $\langle P_k^{\mathrm{T}}\alpha, P_k^{\mathrm{T}}\tilde{\alpha}\rangle = \langle R_k^{\mathrm{T}}\beta, R_k^{\mathrm{T}}\tilde{\beta}\rangle$ for $k = 1, \dots, s$.*

Proof. By Lemma 2.1, we have $||P_k^{\mathrm{T}}\alpha|| = ||R_k^{\mathrm{T}}\beta||$, $||P_k^{\mathrm{T}}\tilde{\alpha}|| = ||R_k^{\mathrm{T}}\tilde{\beta}||$, and $||P_k^{\mathrm{T}}(\alpha + \tilde{\alpha})|| = ||R_k^{\mathrm{T}}(\beta + \tilde{\beta})||$ for $k = 1, \dots, s$. Using the identity $||u + v||^2 = ||u||^2 + ||v||^2 + 2\langle u, v\rangle$, the last equality is equivalent to

$$||P_k^{\mathrm{T}}\alpha||^2 + ||P_k^{\mathrm{T}}\tilde{\alpha}||^2 + 2\langle P_k^{\mathrm{T}}\alpha, P_k^{\mathrm{T}}\tilde{\alpha}\rangle = ||R_k^{\mathrm{T}}\beta||^2 + ||R_k^{\mathrm{T}}\tilde{\beta}||^2 + 2\langle R_k^{\mathrm{T}}\beta, R_k^{\mathrm{T}}\tilde{\beta}\rangle,$$

and hence the proposition follows. □

With a little work, we can obtain an additional version of Proposition 2.2 for nonnegative case.

Proposition 2.3. *Let A and B be two similar real symmetric nonnegative matrices with spectral decomposition $A = \sum_{k=1}^{s} \lambda_k P_k P_k^{\mathrm{T}}$ and $B = \sum_{k=1}^{s} \lambda_k R_k R_k^{\mathrm{T}}$, respectively. Let $\alpha, \tilde{\alpha}, \beta, \tilde{\beta}$ be nonnegative vectors in $\mathbb{R}^n$. If $(A + \alpha\alpha^{\mathrm{T}}, A + \tilde{\alpha}\tilde{\alpha}^{\mathrm{T}}, A + \alpha\alpha^{\mathrm{T}} + \tilde{\alpha}\tilde{\alpha}^{\mathrm{T}})$ and $(B + \beta\beta^{\mathrm{T}}, B + \tilde{\beta}\tilde{\beta}^{\mathrm{T}}, B + \beta\beta^{\mathrm{T}} + \tilde{\beta}\tilde{\beta}^{\mathrm{T}})$ are entrywise similar, then $\langle P_k^{\mathrm{T}}\alpha, P_k^{\mathrm{T}}\tilde{\alpha}\rangle = \langle R_k^{\mathrm{T}}\beta, R_k^{\mathrm{T}}\tilde{\beta}\rangle$ for $k = 1, \dots, s$.*

Proof. Using the Weinstein–Aronszajn identity $|I_p - M_{p\times q}N_{q\times p}| = |I_q - N_{q\times p}M_{p\times q}|$, we obtain

$$\begin{aligned}\left|xI - (A + \alpha\alpha^{\mathrm{T}} + \tilde{\alpha}\tilde{\alpha}^{\mathrm{T}})\right| &= |xI - A| \cdot \left| I_n - (xI - A)^{-1}(\alpha, \tilde{\alpha}) \begin{pmatrix} \alpha^{\mathrm{T}} \\ \tilde{\alpha}^{\mathrm{T}} \end{pmatrix} \right| \\ &= |xI - A| \cdot \left| I_2 - \begin{pmatrix} \alpha^{\mathrm{T}} \\ \tilde{\alpha}^{\mathrm{T}} \end{pmatrix} (xI - A)^{-1}(\alpha, \tilde{\alpha}) \right|.\end{aligned}$$

By the spectral decomposition $A = \sum_{k=1}^{s} \lambda_k P_k P_k^{\mathrm{T}}$, we see that

$$(xI - A)^{-1} = \sum_{k=1}^{s} \frac{1}{x - \lambda_k} P_k P_k^{\mathrm{T}},$$

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and hence

$$\left|I_2 - \begin{pmatrix}\alpha^{\mathrm{T}}\\ \tilde{\alpha}^{\mathrm{T}}\end{pmatrix}(xI-A)^{-1}(\alpha,\tilde{\alpha})\right| = \begin{vmatrix} 1-\sum_{k=1}^{s}\frac{||P_k^{\mathrm{T}}\alpha||^2}{x-\lambda_k} & -\sum_{k=1}^{s}\frac{\langle P_k^{\mathrm{T}}\alpha, P_k^{\mathrm{T}}\tilde{\alpha}\rangle}{x-\lambda_k} \\ -\sum_{k=1}^{s}\frac{\langle P_k^{\mathrm{T}}\alpha, P_k^{\mathrm{T}}\tilde{\alpha}\rangle}{x-\lambda_k} & 1-\sum_{k=1}^{s}\frac{||P_k^{\mathrm{T}}\tilde{\alpha}||^2}{x-\lambda_k}\end{vmatrix}.$$

It follows that

$$\left|xI-(A+\alpha\alpha^{\mathrm{T}}+\tilde{\alpha}\tilde{\alpha}^{\mathrm{T}})\right| = |xI-A|\cdot\begin{vmatrix} 1-\sum_{k=1}^{s}\frac{||P_k^{\mathrm{T}}\alpha||^2}{x-\lambda_k} & -\sum_{k=1}^{s}\frac{\langle P_k^{\mathrm{T}}\alpha, P_k^{\mathrm{T}}\tilde{\alpha}\rangle}{x-\lambda_k} \\ -\sum_{k=1}^{s}\frac{\langle P_k^{\mathrm{T}}\alpha, P_k^{\mathrm{T}}\tilde{\alpha}\rangle}{x-\lambda_k} & 1-\sum_{k=1}^{s}\frac{||P_k^{\mathrm{T}}\tilde{\alpha}||^2}{x-\lambda_k}\end{vmatrix}. \tag{2.1}$$

Similarly, we have

$$\left|xI-(B+\beta\beta^{\mathrm{T}}+\tilde{\beta}\tilde{\beta}^{\mathrm{T}})\right| = |xI-B|\cdot\begin{vmatrix} 1-\sum_{k=1}^{s}\frac{||R_k^{\mathrm{T}}\beta||^2}{x-\lambda_k} & -\sum_{k=1}^{s}\frac{\langle R_k^{\mathrm{T}}\beta, R_k^{\mathrm{T}}\tilde{\beta}\rangle}{x-\lambda_k} \\ -\sum_{k=1}^{s}\frac{\langle R_k^{\mathrm{T}}\beta, R_k^{\mathrm{T}}\tilde{\beta}\rangle}{x-\lambda_k} & 1-\sum_{k=1}^{s}\frac{||R_k^{\mathrm{T}}\tilde{\beta}||^2}{x-\lambda_k}\end{vmatrix}. \tag{2.2}$$

It is not difficult to see from the similarity assumption of Proposition 2.3 together with Eqs. (2.1) and (2.2) that the two determinants of order 2 are equal. Moreover, by Lemma 2.1, we see that $||P_k^{\mathrm{T}}\alpha|| = ||R_k^{\mathrm{T}}\beta||$ and $||P_k^{\mathrm{T}}\tilde{\alpha}|| = ||R_k^{\mathrm{T}}\tilde{\beta}||$ for $k=1,\ldots,s$. It follows that

$$\left(\sum_{k=1}^{s}\frac{\langle P_k^{\mathrm{T}}\alpha, P_k^{\mathrm{T}}\tilde{\alpha}\rangle}{x-\lambda_k}\right)^2 = \left(\sum_{k=1}^{s}\frac{\langle R_k^{\mathrm{T}}\beta, R_k^{\mathrm{T}}\tilde{\beta}\rangle}{x-\lambda_k}\right)^2.$$

This means that either $\langle P_k^{\mathrm{T}}\alpha, P_k^{\mathrm{T}}\tilde{\alpha}\rangle = \langle R_k^{\mathrm{T}}\beta, R_k^{\mathrm{T}}\tilde{\beta}\rangle$ for all k, or $\langle P_k^{\mathrm{T}}\alpha, P_k^{\mathrm{T}}\tilde{\alpha}\rangle = -\langle R_k^{\mathrm{T}}\beta, R_k^{\mathrm{T}}\tilde{\beta}\rangle$ for all k. Thus, to complete the proof of Proposition 2.3, it suffices to establish the following assertion.

Claim: If $\langle P_k^{\mathrm{T}}\alpha, P_k^{\mathrm{T}}\tilde{\alpha}\rangle = -\langle R_k^{\mathrm{T}}\beta, R_k^{\mathrm{T}}\tilde{\beta}\rangle$ for all $k\in\{1,\ldots,s\}$, then

$$\langle P_k^{\mathrm{T}}\alpha, P_k^{\mathrm{T}}\tilde{\alpha}\rangle = \langle R_k^{\mathrm{T}}\beta, R_k^{\mathrm{T}}\tilde{\beta}\rangle = 0 \quad \text{for all } k.$$

Let i be any integer in $\{0,1,\ldots,s-1\}$. From the spectral decomposition of A, we have

$$\alpha^{\mathrm{T}}A^i\tilde{\alpha} = \alpha^{\mathrm{T}}\left(\sum_{k=1}^{s}\lambda_k^i P_k P_k^{\mathrm{T}}\right)\tilde{\alpha} = \sum_{k=1}^{s}\lambda_k^i\langle P_k^{\mathrm{T}}\alpha, P_k^{\mathrm{T}}\tilde{\alpha}\rangle. \tag{2.3}$$

Similarly, $\beta^{\mathrm{T}}B^i\tilde{\beta} = \sum_{k=1}^{s}\lambda_k^i\langle R_k^{\mathrm{T}}\beta, R_k^{\mathrm{T}}\tilde{\beta}\rangle$. It follows from the condition of the Claim that $\alpha^{\mathrm{T}}A^i\tilde{\alpha} = -\beta^{\mathrm{T}}B^i\tilde{\beta}$. However, as the matrices A, B, and the vectors $\alpha,\tilde{\alpha},\beta,\tilde{\beta}$ are nonnegative, we must have $\alpha^{\mathrm{T}}A^i\tilde{\alpha}, \beta^{\mathrm{T}}B^i\tilde{\beta}\geq 0$. This implies that all the scalars $\alpha^{\mathrm{T}}A^i\tilde{\alpha}$ and $\beta^{\mathrm{T}}B^i\tilde{\beta}$ are zero. It follows from Eq. (2.3) that

$$\sum_{k=1}^{s}\lambda_k^i\langle P_k^{\mathrm{T}}\alpha, P_k^{\mathrm{T}}\tilde{\alpha}\rangle = 0, \quad i=0,1,\ldots,s-1. \tag{2.4}$$

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As the Vandermonde matrix generated by $\lambda_1, \ldots, \lambda_s$ is invertible, we see from Eq. (2.4) that $\langle P_k^{\mathrm{T}}\alpha, P_k^{\mathrm{T}}\tilde{\alpha}\rangle = 0$ for all $k \in \{1, \ldots, s\}$. As $\beta^{\mathrm{T}}B^i\tilde{\beta} = 0$ for all i, the same argument indicates that $\langle R_k^{\mathrm{T}}\beta, R_k^{\mathrm{T}}\tilde{\beta}\rangle = 0$ for all $k \in \{1, \ldots, s\}$. This proves the Claim and hence completes the proof of Proposition 2.3. ∎

For a set of vectors $v_1, v_2, \ldots, v_m$ in $\mathbb{R}^n$, the *Gram matrix* is the real symmetric matrix

$$\begin{pmatrix} v_1^{\mathrm{T}}v_1 & v_1^{\mathrm{T}}v_2 & \ldots & v_1^{\mathrm{T}}v_m \\ v_2^{\mathrm{T}}v_1 & v_2^{\mathrm{T}}v_2 & \ldots & v_2^{\mathrm{T}}v_m \\ \vdots & \vdots & \vdots & \vdots \\ v_m^{\mathrm{T}}v_1 & v_m^{\mathrm{T}}v_2 & \ldots & v_m^{\mathrm{T}}v_m \end{pmatrix}. \tag{2.5}$$

We need the following basic result in Linear Algebra; see, e.g., [3, Theorem 7.3.11].

LEMMA 2.4. *Let $\{v_1, \ldots, v_m\}$ and $\{w_1, \ldots, w_m\}$ be two sets of vectors in $\mathbb{R}^n$. If they have the same Gram matrix, then there exists an orthogonal matrix Q such that $Qv_i = w_i$ for $i \in \{1, \ldots, m\}$.*

Proof of Theorem 1.6. The implication (i) $\Longrightarrow$ (ii) is straightforward, so we only prove (ii) $\Longrightarrow$ (i). Note that A and B are similar. Let $A = \sum_{k=1}^{s}\lambda_k P_k P_k^{\mathrm{T}}$ and $B = \sum_{k=1}^{s}\lambda_k R_k R_k^{\mathrm{T}}$ be the spectral decompositions. By Lemma 2.1 and Proposition 2.2, we find that $\{P_k^{\mathrm{T}}\alpha_1, \ldots, P_k^{\mathrm{T}}\alpha_m\}$ and $\{R_k^{\mathrm{T}}\beta_1, \ldots, R_k^{\mathrm{T}}\beta_m\}$ have the same Gram matrix, for each $k \in \{1, \ldots, s\}$. It follows from Lemma 2.4 that, for each k, there exists an orthogonal matrix Q_k such that $Q_k(P_k^{\mathrm{T}}\alpha_i) = R_k^{\mathrm{T}}\beta_i$ for $i \in \{1, \ldots, m\}$. Written in the matrix form, we have

$$\begin{pmatrix} Q_1P_1^{\mathrm{T}} \\ \vdots \\ Q_sP_s^{\mathrm{T}} \end{pmatrix}(\alpha_1, \ldots, \alpha_m) = \begin{pmatrix} R_1^{\mathrm{T}} \\ \vdots \\ R_s^{\mathrm{T}} \end{pmatrix}(\beta_1, \ldots, \beta_m). \tag{2.6}$$

Let

$$Q = (P_1Q_1^{\mathrm{T}}, \ldots, P_sQ_s^{\mathrm{T}})\begin{pmatrix} R_1^{\mathrm{T}} \\ \vdots \\ R_s^{\mathrm{T}} \end{pmatrix}. \tag{2.7}$$

It is easy to see that both matrices on the right-hand side of Eq. (2.7) are orthogonal matrices and hence Q is also orthogonal. By Eq. (2.6), we easily see that $Q^{\mathrm{T}}(\alpha_1, \ldots, \alpha_m) = (\beta_1, \ldots, \beta_m)$. Finally, noting that

$$(P_1Q_1^{\mathrm{T}}, \ldots, P_sQ_s^{\mathrm{T}})^{\mathrm{T}}A(P_1Q_1^{\mathrm{T}}, \ldots, P_sQ_s^{\mathrm{T}}) = \mathrm{diag}(\lambda_1 I_{r_1}, \ldots, \lambda_s I_{r_s})$$

and

$$(R_1, \ldots, R_s)^{\mathrm{T}}B(R_1, \ldots, R_s) = \mathrm{diag}(\lambda_1 I_{r_1}, \ldots, \lambda_s I_{r_s}),$$

we obtain

$$\begin{aligned} Q^{\mathrm{T}}AQ &= (R_1, \ldots, R_s)(P_1Q_1^{\mathrm{T}}, \ldots, P_sQ_s^{\mathrm{T}})^{\mathrm{T}}A(P_1Q_1^{\mathrm{T}}, \ldots, P_sQ_s^{\mathrm{T}})\begin{pmatrix} R_1^{\mathrm{T}} \\ \vdots \\ R_s^{\mathrm{T}} \end{pmatrix} \\ &= (R_1, \ldots, R_s)\mathrm{diag}(\lambda_1 I_{r_1}, \ldots, \lambda_s I_{r_s})\begin{pmatrix} R_1^{\mathrm{T}} \\ \vdots \\ R_s^{\mathrm{T}} \end{pmatrix} \\ &= B. \end{aligned}$$

This completes the proof of Theorem 1.6. ∎

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Proof of Theorem 1.7. Theorem 1.7 can be proved in exactly the same way as Theorem 1.6. Indeed, by Lemma 2.1 and Proposition 2.3, we find that $\{P_k^{\mathrm{T}}\alpha_1,\ldots,P_k^{\mathrm{T}}\alpha_m\}$ and $\{R_k^{\mathrm{T}}\beta_1,\ldots,R_k^{\mathrm{T}}\beta_m\}$ have the same Gram matrix for each $k\in\{1,\ldots,s\}$. Now the remaining proof of Theorem 1.6 clearly works for Theorem 1.7 and hence we are done. ∎

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