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# LENGTHS OF FAMILIES OF MATRICES ALLOWING A MINIMAL POLYNOMIAL OF GIVEN DEGREE[*]

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**Abstract.** Let $S$ be a family generating $\mathrm{Mat}_n(\mathbb{F})$ as an algebra over a field $\mathbb{F}$. If $\dim \mathbb{F}[A] = \delta > 1$ with some $A \in S$, then the products of length at most

$$(2n-\delta)\cdot\lfloor n/\delta\rfloor - (\lfloor n/\delta\rfloor^2 - 1)\cdot(\delta+1) + n - 2\cdot\lfloor(2\delta-1)/n\rfloor,$$

with multipliers in $S$, contain the full algebra $\mathrm{Mat}_n(\mathbb{F})$ in their $\mathbb{F}$-linear span.

**Key words.** Matrices, Generating sets, Growth.

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In this note, a set $S$ of $n \times n$ matrices over $\mathbb{F}$ is called *generating* if the $\mathbb{F}$-algebra generated by $S$ is the full matrix algebra $\mathrm{Mat}_n(\mathbb{F})$. I also write $\langle S\rangle$ to denote the $\mathbb{F}$-linear span of $S$. The set $S^k$ consists of all products $s_1 \cdot \ldots \cdot s_k$ with $s_1, \ldots, s_k \in S$, and I also assume $S^0 = \{I\}$, where $I$ is the $n \times n$ identity matrix. One can recall that the *length* of a family $S \subset \mathrm{Mat}_n(\mathbb{F})$ is the smallest integer $\ell = \ell(S)$ for which the $\mathbb{F}$-linear span of $(S \cup \{I\})^\ell$ is the algebra generated by $S$, see [2]. A recent article of Wang [5] reports the following main results, which give upper bounds on the length of a generating family $S$:

(W1) if $n/2 < \dim \mathbb{F}[A] = \delta \leqslant n$ holds for some $A \in S$, then $\ell(S) \leqslant 3n - 5$,

(W2) if $n/3 < \dim \mathbb{F}[A] = \delta \leqslant n/2$ holds for some $A \in S$, then $\ell(S) \leqslant 3.5n - 4$.

The arguments in [5] are based on Jordan forms and following results.

LEMMA 1 (see Theorem 4.1 in [2] and Corollary 7 in [3]). *Assume that $\langle (S \cup \{I\})^k\rangle$ contains a matrix of rank $r > 0$, for some $k$ and generating set $S \subset \mathrm{Mat}_n(\mathbb{F})$. Then, $\ell(S) \leqslant rn + n - r + k - 1$, and, in addition, if one has $r = 1$ and $k \geqslant 2$, then $\ell(S) \leqslant 2n + k - 4$.*

For reader's convenience, I present an apparently more clear account of the proofs of (W1, W2).

REMARK 2. *In considerations below, I always assume that $\mathbb{F}$ is algebraically closed, but, by basic further arguments, Theorems 3 and 6 are valid for any $\mathbb{F}$. Indeed, the condition $\langle (S \cup \{I\})^k\rangle = \mathrm{Mat}_n(\mathbb{F})$ is equivalent to the family $(S \cup \{I\})^k$ having $n^2$ linearly independent matrices, and the validity of this condition does not change if one looks at $S$ as a family of matrices over an extension of $\mathbb{F}$, see [2, 3, 4].*

THEOREM 3. *Let $S \subset \mathrm{Mat}_n(\mathbb{F})$ be a generating set, and $\dim \mathbb{F}[A] = \delta > 1$ with some $A \in S$ and $\delta \in \mathbb{Z}$. Then, $\ell(S) \leqslant (\lfloor n/\delta\rfloor + 1)\cdot(n-1) + \delta - 1$, and, if one has $\delta > n/2$ and $n \geqslant 3$, then also $\ell(S) \leqslant 2n + \delta - 5$.*

*Proof.* I take a polynomial $\pi$ of the degree $\delta - 1$ such that $\pi$ divides the minimal polynomial of $A$, and $\pi$ is divided by the minimal polynomials of all blocks of the rational canonical form of $A$, except for the largest

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such blocks. Then, $\operatorname{rk}\pi(A)$ is the count of the largest blocks, that is, the set $\mathbb{F}[A] \subset \langle (S \cup \{I\})^{\delta-1} \rangle$ has the matrix $\pi(A) \neq 0$ with rank at most $\lfloor n/\delta \rfloor$, and the result follows from Lemma 1. ∎

As pointed out by Wang, the known results were not sufficient to *address scenarios where the degrees of minimal polynomials within the generating set do not exceed half the order of the matrix algebra* [5, Subsection 3.2]. My argument further addresses this question so that all possible values of $\delta$ are included, and, also, various special cases of Theorem 3 include the main results of [5], as explained above.

In fact, my recent article [4] contains a development of the approach of Lemma 1, and its results lead to much strengthened versions of Theorem 3 and main results of [5]. Indeed, a family $(a_1, \ldots, a_k)$ of matrices in $\operatorname{Mat}_n(\mathbb{F})$ is called *$p$-increasing* if $\dim((\operatorname{row} a_1) + \ldots + (\operatorname{row} a_t)) = \dim((\operatorname{row} a_1) + \ldots + (\operatorname{row} a_{t-1})) + p$ holds with every $t \in \{1, \ldots, k\}$, where $\operatorname{row} a$ stands for the $\mathbb{F}$-linear span of the rows of $a$.

LEMMA 4 (see Lemma 7 in [4]). *Let $R$, $S \subset \operatorname{Mat}_n(\mathbb{F})$. Assume $R$ is $p$-increasing, $S$ is generating, and every matrix $s \in \langle S \rangle$ satisfies* $\dim \mathbb{F}[s] \leqslant \delta$ *with fixed $\delta > 1$. Then, for any integer $k \geqslant pn - \lceil pn/\delta \rceil$, one has $\langle (S \cup \{I\})^k \cdot R \rangle = \langle \operatorname{Mat}_n(\mathbb{F}) \cdot R \rangle$.*

LEMMA 5 (see Observation 9 in [4]). *Assume $A \in \operatorname{Mat}_n(\mathbb{F})$ and* $\dim \mathbb{F}[A] = \delta > 1$. *If $n = p\delta + r$ with $p \in \mathbb{Z}$ and $r \in \{0, 1, \ldots, \delta - 1\}$, then the set $\mathbb{F}[A]$ contains either (C1) a $p$-increasing family $R$ of $\delta - r$ matrices or (C2) a single matrix $M$ with rank $\rho \in \{1, 2, \ldots, p - 1\}$.*

I proceed with a further improvement of the result of Theorem 3. It is particularly meaningful if the ratio $n/\delta$ remains bounded, and the case $\delta = o(n/\log n)$ follows from the known bounds [2, 3].

THEOREM 6. *Assume $S \subset \operatorname{Mat}_n(\mathbb{F})$ is a generating set, and one has* $\dim \mathbb{F}[A] = \delta > 1$, *for some $A \in S$. Then $\ell(S) \leqslant (2n - \delta) \cdot \lfloor n/\delta \rfloor - (\lfloor n/\delta \rfloor^2 - 1) \cdot (\delta + 1) - 2 \cdot \lfloor (2\delta - 1)/n \rfloor + n$.*

*Proof.* As my bound is non-increasing over $\delta$, I can assume that every matrix $s \in \langle S \rangle$ gives $\dim \mathbb{F}[s] \leqslant \delta$. For $(p, r)$ in Lemma 5, if case (C1) appears, I use Lemma 4 to get $\langle \operatorname{Mat}_n(\mathbb{F}) \cdot R \rangle \subset \langle (S \cup \{I\})^t \rangle$ with $t = (\delta - 1) + (pn - \lceil pn/\delta \rceil)$. Since $R$ is $p$-increasing, the sum of row spaces of its matrices has dimension $(\delta - r)p$, and I obtain $\langle \operatorname{Mat}_n(\mathbb{F}) \cdot R \cdot (S \cup \{I\})^\tau \rangle = \operatorname{Mat}_n(\mathbb{F})$ with $\tau = n - (\delta - r)p$. This gives $\ell(S) \leqslant t + \tau$ and implies the desired bound. The case (C2) gives $\langle (S \cup \{I\})^\phi \cdot M \cdot (S \cup \{I\})^\varphi \rangle = \operatorname{Mat}_n(\mathbb{F})$ with $\phi = \rho n - \lceil \rho n/\delta \rceil$ and $\varphi = n - \rho$, and the resulting bound $\phi + (\delta - 1) + \varphi$ is smaller than the one in the formulation. ∎

Assuming (W1), one gets $\ell(S) \leqslant 3n - \delta - 2 \leqslant 2.5\,(n - 1)$ and improves on the $3n - 5$ bound [5]. Also, the regime (W2) gives $\ell(S) \leqslant 5n - 5\delta - 3 \leqslant 10(n - 1)/3$, which is stronger than $7n/2 - 4$ in [5]. As a note added at proof stage, I recall that Wang's result (W1) is related to an earlier statement in [1, Theorem 3.1], and my work gives a further improvement and independent proof of that statement, too.

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