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THE CHARACTERIZATION OF PRIMITIVE SYMMETRIC LOOP-FREE SIGNED DIGRAPHS WITH THE MAXIMUM BASE*

LIHUA YOU† AND SHUYONG YI†

Abstract. In this paper, the primitive symmetric loop-free signed digraphs with the maximum base are characterized.

Key words. Matrix, Symmetric, Primitive, Non-powerful, Base, Signed digraph.

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1. Introduction. A sign pattern matrix is a matrix each of whose entries is a sign 1, -1 or 0. For a square sign pattern matrix M, notice that in the computations of the entries of the power M^k, an "ambiguous sign" may arise when we add a positive sign 1 to a negative sign -1. So a new symbol "#" was introduced in [8] to denote the ambiguous sign, the set $\Gamma = \{0, 1, -1, \#\}$ is defined as the generalized sign set and the addition and multiplication involving the symbol # are defined as follows:

$$(-1) + 1 = 1 + (-1) = \#; \quad a + \# = \# + a = \# \quad (\text{for all } a \in \Gamma)$$

$$0 \cdot \# = \# \cdot 0 = 0; \quad b \cdot \# = \# \cdot b = \# \quad (\text{for all } b \in \Gamma \setminus \{0\}).$$

In [8, 11], the matrices with entries in the set Γ are called generalized sign pattern matrices. The addition and multiplication of generalized sign pattern matrices are defined in the usual way, so that the sum and product of the generalized sign pattern matrices are still generalized sign pattern matrices. In this paper, we assume that all the matrix operations considered are operations of the matrices over Γ.

Definition 1.1. ([8]) A square generalized sign pattern matrix M is called *powerful* if each power of M has no # entry.

Definition 1.2. ([12]) Let M be a square generalized sign pattern matrix of order n and $M, M^2, M^3, \ldots$ be the sequence of powers of M. Suppose M^b is the first

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†School of Mathematical Sciences, South China Normal University, Guangzhou, 510631, P.R. China (ylhua@scnu.edu.cn, ysyhgnc@163.com). The first author is supported by National Natural Science Foundation of China (10901061, 11071088) and the Zhujiang Technology Star Foundation of Guangzhou (2011Z2200090).

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power that is repeated in the sequence. Namely, suppose b is the least positive integer such that there is a positive integer p such that

$$M^b = M^{b+p}. \tag{1.1}$$

Then b is called the *generalized base* (or simply *base*) of M, and is denoted by $b(M)$. The least positive integer p such that (1.1) holds for $b = b(M)$ is called the *generalized period* (or simply *period*) of M, and is denoted by $p(M)$.

We now introduce some graph theoretical concepts.

Let $D = (V, A)$ denote a digraph on n vertices. Loops are permitted, but no multiple arcs. A $u \to v$ *walk* in D is a sequence of vertices $u, u_1, \ldots, u_k = v$ and a sequence of arcs $e_1 = (u, u_1), e_2 = (u_1, u_2), \ldots, e_k = (u_{k-1}, v)$, where the vertices and the arcs are not necessarily distinct. A *closed walk* is a $u \to v$ walk where $u = v$. A *path* is a walk with distinct vertices. A *cycle* is a closed $u \to v$ walk with distinct vertices except for $u = v$. The *length* of a walk W is the number of arcs in W, denoted by $l(W)$. A k-cycle is a cycle of length k, denoted by C_k.

A *signed digraph* S is a digraph where each arc of S is assigned a sign 1 or -1. A *generalized signed digraph* S is a digraph where each arc of S is assigned a sign 1, -1 or $\#$.

The *sign* of the walk W in a (generalized) signed digraph, denoted by sgn$W$, is defined to be $\prod_{i=1}^{k} \text{sgn}(e_i)$, where $e_1, e_2, \ldots, e_k$ is the sequence of arcs of W.

Let $M = (m_{ij})$ be a square (generalized) sign pattern matrix of order n. The *associated digraph* $D(M) = (V, A)$ of M (possibly with loops) is defined to be the digraph with vertex set $V = \{1, 2, \ldots, n\}$ and arc set $A = \{(i, j) | m_{ij} \neq 0\}$. The *associated (generalized) signed digraph* $S(M)$ of M is obtained from $D(M)$ by assigning the sign of m_{ij} to each arc (i, j) in $D(M)$, and we say $D(M)$ is the *underlying digraph* of $S(M)$.

Let S be a (generalized) signed digraph on n vertices. Then there is a (generalized) sign pattern matrix M of order n whose associated (generalized) signed digraph $S(M)$ is S. We say that S is *powerful* if M is powerful. Also the base $b(S)$ and period $p(S)$ are defined to be those of M. Namely, we define $b(S) = b(M)$ and $p(S) = p(M)$.

A digraph D is said to be *strongly connected* if there exists a path from u to v for all $u, v \in V$, and D is called *primitive* if there is a positive integer k such that for each vertex x and each vertex y (not necessarily distinct) in D, there exists a walk of length k from x to y. The least such k is called the *primitive exponent* (or *exponent*) of D, denoted by $\exp(D)$. It is also well-known that a digraph D is primitive if and only if D is strongly connected and the greatest common divisor (g.c.d.) of the lengths of

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all the cycles of D is 1. A (generalized) signed digraph S is called *primitive* if the underlying digraph D is primitive, and in this case, we define $\exp(S) = \exp(D)$.

A digraph D is *symmetric* if for every arc (u, v) in D, the arc (v, u) is also in D. A (generalized) signed digraph S is called *combinatorially symmetric* (or *symmetric*) if the underlying digraph D is symmetric. A digraph D is loop-free if D has no loops. If a digraph D is symmetric and loop-free, we regard D as a simple graph.

Let $\mathcal{S}_n = \{S|S$ is a primitive symmetric signed digraph on n vertices$\}$, $\mathcal{S}_n^\star = \{S|S$ is a primitive symmetric loop-free signed digraph on n vertices$\}$. Let $\mathcal{E}_n = \{\exp(S)|S \in \mathcal{S}_n\}$, $\mathcal{E}_n^\star = \{\exp(S)|S \in \mathcal{S}_n^\star\}$, and $\mathcal{B}_n = \{b(S)|S \in \mathcal{S}_n\}$, $\mathcal{B}_n^\star = \{b(S)|S \in \mathcal{S}_n^\star\}$. The primitive exponent and exponent sets $\mathcal{E}_n$ and $\mathcal{E}_n^\star$ were discussed in [6, 7, 9, 10], and the base set $\mathcal{B}_n$ and $\mathcal{B}_n^\star$ were discussed in [4, 13].

THEOREM 1.3. ([10]) *Let D be a primitive symmetric digraph on n vertices. Then:*

(1) $\exp(D) \le 2n - 2$ *and the equality holds if and only if D is isomorphic to G_1, where $G_1 = (V, A), V = \{1, 2, \ldots, n\}, A = \{(i, i+1), (i+1, i)|1 \le i \le n-1\} \bigcup \{(1, 1)\}$.*
(2) $\mathcal{E}_n = \{1, 2, \ldots, 2n-2\} \backslash \mathcal{D}$ *where $\mathcal{D}$ is the set of odd numbers in $\{n, n+1, \ldots, 2n-2\}$.*

THEOREM 1.4. ([9]) *Let D be a primitive symmetric loop-free digraph on n vertices. Then:*

(1) $\exp(D) \le 2n - 4$.
(2) $\mathcal{E}_n^\star = \{2, 3, \ldots, 2n-4\} \backslash \mathcal{D}$ *where $\mathcal{D}$ is the set of odd numbers in $\{n-2, n-1, \ldots, 2n-5\}$.*

THEOREM 1.5. ([4, 5]) *Let S be a primitive symmetric signed digraph on n vertices. Then:*

(1) $b(S) \le 2n$ *and the equality holds if and only if S has at least one negative 2-cycle and D is isomorphic to G_1 where D is the underlying digraph of S.*
(2) $\mathcal{B}_n = \{1, 2, \ldots, 2n\}$.

THEOREM 1.6. ([13]) *Let S be a primitive symmetric loop-free signed digraph on n vertices. Then $b(S) \le 2n - 1$ and $\mathcal{B}_n^\star = \{2, \ldots, 2n-1\}$.*

A natural question is what primitive symmetric loop-free signed digraphs on n vertices attain this upper bound $2n - 1$? We answer this in Section 3.

2. Some preliminaries. In this section, we introduce some needed definitions, theorems and lemmas. Other definitions and results not in this article can be found

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in [1, 2, 3].

DEFINITION 2.1. ([12]) Two walks W_1 and W_2 in a signed digraph are called a *pair of SSSD walks*, if they have the same initial vertex, same terminal vertex, and same length, but they have different signs.

It is easy to see from the above relation between matrices and signed digraphs that a (generalized) sign pattern matrix M is powerful if and only if the associated (generalized) signed digraph $S(M)$ has no pairs of $SSSD$ walks. Thus, for a (generalized) signed digraph S, S is powerful if and only if S has no pairs of $SSSD$ walks.

In [12], You, Shao, and Shan obtained an important characterization of primitive non-powerful signed digraphs from the characterization of powerful irreducible sign pattern matrices (see [8]).

THEOREM 2.2. ([12]) *If S is a primitive signed digraph, then S is non-powerful if and only if S has a pair of cycles C' and C'' (say, with lengths p_1 and p_2, respectively) satisfying one of the following conditions:*

(A_1) *p_1 is odd, p_2 is even and $sgnC'' = -1$;*
(A_2) *Both p_1 and p_2 are odd and $sgnC' = -sgnC''$.*

A pair of cycles C' and C'' satisfying (A1) or (A2) is a "distinguished cycle pair". It is easy to check that if C' and C'' is a distinguished cycle pair with lengths p_1 and p_2, respectively, then the closed walks $W_1 = p_2C'$ (walk around C' by p_2 times) and $W_2 = p_1C''$ have the same length p_1p_2 and different signs:

$$(\mathrm{sgn}C')^{p_2} = -(\mathrm{sgn}C'')^{p_1}.$$

The following result can be used to determine the base.

THEOREM 2.3. [12] *Let S be a primitive non-powerful signed digraph. Then:*

(1) *There is an integer k such that there exists a pair of SSSD walks of length k from each vertex x to each vertex y in S.*
(2) *If there exists a pair of SSSD walks of length k from each vertex x to each vertex y, then there also exists a pair of SSSD walks of length $k+1$ from each vertex x to each vertex y in S.*
(3) *The minimal such k (as in (1)) is just $b(S)$-the base of S.*

In the rest of the paper, for an undirected walk W of graph G and two vertices x, y on W, let $Q_W(x \to y)$ be the shortest path from x to y on W. Let $Q(x \to y)$ be the shortest path from x to y on G. For a cycle C, if x and y are two (not necessarily distinct) vertices on C and P is a path from x to y along C, then $C\backslash P$ denotes the

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path or cycle from x to y along C obtained by deleting the edges of P.

The following lemmas will be useful.

LEMMA 2.4. *Let D be a symmetric digraph on n vertices. Suppose that there exist a cycle C and an odd cycle C' with lengths of $k \geq 1$ and $k' \geq 1$ in D such that $C \cap C' = \emptyset$. Let P be the shortest path from C to C', and for any $x \in D$, let P_1 (P_2) be the shortest path from x to C (C'). Then we have*

$$l(P_1) + l(P) + l(P_2) \leq 2(n - k - k' + 1) + \max\left\{\left[\frac{k}{2}\right], \frac{k'-1}{2}\right\}. \tag{2.1}$$

Proof. Suppose P intersects C (C') at v (v').

Case 1: $P_1 \cap C' = \emptyset$ and $P_2 \cap C = \emptyset$.

Subcase 1.1: $(P_1 \cup P_2) \cap P = \emptyset$.

It is easy to see that $l(P_1) + l(P) + l(P_2) \leq 2(n - k - k' + 1) \leq 2(n - k - k' + 1) + \max\left\{\left[\frac{k}{2}\right], \frac{k'-1}{2}\right\}$.

Subcase 1.2: $(P_1 \cup P_2) \cap P \neq \emptyset$.

We have $P_1 \cap P \neq \emptyset$ or $P_2 \cap P \neq \emptyset$. Without loss of generality, we may assume $P_1 \cap P \neq \emptyset$. Suppose z is the first vertex on $P_1 \cap P$. Then $l(P_1) + l(P) + l(P_2) \leq l(Q_{P_1}(x \to z)) + l(Q_P(z \to v)) + l(P) + l(Q_{P_1}(x \to z)) + l(Q_P(z \to v')) = 2(l(P) + l(Q_{P_1}(x \to z))) \leq 2(n - k - k' + 1)$.

Case 2: $P_1 \cap C' \neq \emptyset$.

Suppose z is the first vertex on $P_1 \cap C'$. We have $l(P_1) + l(P) + l(P_2) \leq (l(Q_{P_1}(x \to z)) + l(Q_{C'}(z \to v')) + l(P)) + l(P) + l(Q_{P_1}(x \to z)) = 2(l(P) + l(Q_{P_1}(x \to z))) + l(Q_{C'}(z \to v')) \leq 2(n - k - k' + 1) + \frac{k'-1}{2}$.

Case 3: $P_2 \cap C \neq \emptyset$.

Suppose z is the first vertex on $P_2 \cap C$. We have $l(P_1) + l(P) + l(P_2) \leq l(Q_{P_2}(x \to z)) + l(P) + (l(Q_{P_2}(x \to z)) + l(Q_C(z \to v)) + l(P)) = 2(l(P) + l(Q_{P_2}(x \to z))) + l(Q_C(z \to v)) \leq 2(n - k - k' + 1) + \left[\frac{k}{2}\right]$.

Combining the above three cases, we see that (2.1) holds. $\square$

LEMMA 2.5. *Let D be a symmetric digraph on n vertices. Suppose that there exist a cycle C and an odd cycle C' with lengths of $k \geq 1$ and $k' \geq 1$ in D such that $C \cap C' = \emptyset$. Let P be the shortest path from C to C', $d(x, y)$ be the distance from x to y. Then for any two vertices $x, y \in D$, there exist $x' \in C, y' \in C'$ or $x' \in C', y' \in C$*

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such that

$$d(x,x') + l(P) + d(y,y') \le 2(n-k-k'+1) + \max\left\{\left[\frac{k}{2}\right], \frac{k'-1}{2}\right\}. \tag{2.2}$$

Proof. Note that $l(P) \le n-k-k'+1$. Thus, we only need to consider the following three cases.

Case 1: $x \in C$ or $y \in C$. Without loss of generality, we may assume $x \in C$.

Take $x' = x$, and for any $y \in D$, there exists $y' \in C'$ such that $d(y,y') \le \left[\frac{k}{2}\right] + n - k - k' + 1$. So $d(x,x') + l(P) + d(y,y') \le 2(n-k-k'+1) + \left[\frac{k}{2}\right] \le 2(n-k-k'+1) + \max\left\{\left[\frac{k}{2}\right], \frac{k'-1}{2}\right\}$.

Case 2: $x \in C'$ or $y \in C'$. Without loss of generality, we may assume $x \in C'$.

Taking $x' = x$, and for any $y \in D$, there exists $y' \in C$ such that $d(y,y') \le \frac{k'-1}{2} + n - k - k' + 1$. So $d(x,x') + l(P) + d(y,y') \le 2(n-k-k'+1) + \frac{k'-1}{2} \le 2(n-k-k'+1) + \max\left\{\left[\frac{k}{2}\right], \frac{k'-1}{2}\right\}$.

Case 3: $x \notin C \cup C'$ and $y \notin C \cup C'$.

Let P_1 and P_1' be the shortest path from x to C and C' respectively, and let P_2 and P_2' be the shortest path from y to C and C' respectively. Assume the result does not hold. Then we have

$$l(P_1) + l(P) + l(P_2') > 2(n-k-k'+1) + \max\left\{\left[\frac{k}{2}\right], \frac{k'-1}{2}\right\},$$

and

$$l(P_1') + l(P) + l(P_2) > 2(n-k-k'+1) + \max\left\{\left[\frac{k}{2}\right], \frac{k'-1}{2}\right\}.$$

Therefore, $l(P_1)+l(P_1')+2l(P)+l(P_2)+l(P_2') > 4(n-k-k'+1)+2\max\left\{\left[\frac{k}{2}\right], \frac{k'-1}{2}\right\}$.

On the other hand, by Lemma 2.4, we have

$$l(P_1) + l(P) + l(P_1') \le 2(n-k-k'+1) + \max\left\{\left[\frac{k}{2}\right], \frac{k'-1}{2}\right\},$$

and

$$l(P_2) + l(P) + l(P_2') \le 2(n-k-k'+1) + \max\left\{\left[\frac{k}{2}\right], \frac{k'-1}{2}\right\}.$$

So, $l(P_1)+l(P_1')+2l(P)+l(P_2)+l(P_2') \le 4(n-k-k'+1)+2\max\left\{\left[\frac{k}{2}\right], \frac{k'-1}{2}\right\}$; this is a contradiction.

Combining the above three cases, we obtain (2.2). $\square$

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3. Characterization of the primitive symmetric loop-free signed digraphs with the maximum base. It was shown in [8] that if a primitive signed digraph S is powerful, then $b(S) = \exp(D)$, where D is the underlying digraph of S. So for a primitive powerful symmetric (loop-free) signed digraph, Theorems 1.3 and 1.4 give the results, and if S is a primitive symmetric (loop-free) signed digraph on n vertices with base $2n-1$, S must be non-powerful.

Let $\mathcal{S}_n^\star = \{S|S$ is a primitive symmetric loop-free signed digraph on n vertices$\}$. For a cycle C in a (generalized) signed digraph S, if $\text{sgn}C = 1$ (or -1), then we call C a *positive* (or *negative*) *cycle*.

Let $n \geq 4$, l $(3 \leq l \leq n)$ be odd, and let $D_l = (V, A)$ be a digraph on n vertices with vertex set $V = \{1, 2, \ldots, n\}$ and arc set $A = \{(i, i+1), (i+1, i)|1 \leq i \leq n-1\} \cup \{(1, l), (l, 1)\}$. Clearly, D_l is a primitive symmetric loop-free digraph.

Lemma 3.1. *Let $n \geq 4$, l $(3 \leq l \leq n)$ be odd, and let SD_l be a signed digraph with D_l as its underlying digraph, where every 2-cycle in SD_l is negative. Then*

(1) *$SD_l \in \mathcal{S}_n^\star$ and SD_l is non-powerful.*
(2) $b(SD_l) = 2n-1$.

Proof. (1) It follows from Theorem 2.2 and the definitions.

(2) It is obvious that $b(SD_l) \leq 2n-1$ by Theorem 1.5. Since there are no $SSSD$ walks of even length $2n-2$ from n to n, $b(SD_l) \geq 2n-1$. Combining the two inequalities, we obtain $b(SD_l) = 2n-1$. ☐

Lemma 3.2. *Suppose $S \in \mathcal{S}_n^\star$. If there exists a vertex v in $V(S)$ such that v is contained in a positive 2-cycle C' and a negative 2-cycle C'', then $b(S) \leq 2n-2$.*

Proof. Since there exist a positive 2-cycle C' and a negative 2-cycle C'' in S, S is non-powerful, C', C'' is a "distinguished cycle pair", there exists a pair of $SSSD$ walks of length 2 from v to v.

Since S is primitive, there exists an odd cycle $C = v_1v_2\cdots v_lv_1$ with length l (≥ 3) in S. Let x and y be any two (not necessarily distinct) vertices in $V(S)$.

Let P be the shortest path from v to C and let P intersect C at v'. Suppose there are k vertices on P where $k \geq 1$. Then $P \cup C$ has $k+l-1$ vertices.

Let P_1 be the shortest path from x to $P \cup C$ and let P_1 intersect $P \cup C$ at x' where $0 \leq l(P_1) \leq n-k-l+1$, let P_2 be the shortest path from y to $P \cup C$ and let P_2 intersect $P \cup C$ at y' where $0 \leq l(P_2) \leq n-k-l+1$.

We consider the following three cases.

Case 1: $x' \in P$, $y' \in P$.

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Set $a = l(Q_P(x' \to v))$, $b = l(Q_P(v \to y'))$ and

$$W = \begin{cases} P_1 + Q_P(x' \to v) + P + Q_P(v' \to y') + P_2, & \text{if } a \le b; \\ P_1 + Q_P(x' \to v') + P + Q_P(v \to y') + P_2, & \text{otherwise.} \end{cases}$$

Then $l(W) \le (n-k-l+1) + (k-1) + (k-1) + (n-k-l+1) = 2n-2l$. If $l(W)$ is even, we set $W_1 = W$. Otherwise, we set $W_1 = W + C$. Therefore, $l(W_1)$ is even, and $l(W_1) \le 2n - l$, thus $W_1 + C'$ and $W_1 + C''$ are a pair of $SSSD$ walks from x to y with even length $\le 2n-l+2 \le 2n-1$. Therefore, there exists a pair of $SSSD$ walks of length $2n-2$ from x to y.

Case 2: Either x' or y' belongs to P. Without loss of generality, we may assume $x' \in P$ and $y' \notin P$.

Set $w = l(P_1) + l(Q_P(x' \to v)) + l(P) + l(Q_C(v' \to y')) + l(P_2)$ and

$$W = \begin{cases} P_1 + Q_P(x' \to v) + P + Q_C(v' \to y') + P_2, & \text{if } w \text{ is even;} \\ P_1 + Q_P(x' \to v) + P + C \setminus Q_C(v' \to y') + P_2, & \text{otherwise.} \end{cases}$$

Then $l(W)$ is even, and $l(W) \le (n-l-k+1) + (k-1) + (k-1) + (l-1) + (n-l-k+1) = 2n-l-1$, thus $W + C'$ and $W + C''$ are a pair of $SSSD$ walks from x to y with even length $\le 2n-l+1 \le 2n-2$. Therefore, there exists a pair of $SSSD$ walks of length $2n-2$ from x to y.

Case 3: $x' \notin P$, $y' \notin P$.

Subcase 3.1: If $v' \in Q_C(x' \to y')$.

Set $w = l(P_1) + l(Q_C(x' \to y')) + l(P_2)$ and

$$W = \begin{cases} P_1 + Q_C(x' \to y') + 2P + P_2, & \text{if } w \text{ is even;} \\ P_1 + Q_C(x' \to y') + 2P + C + P_2, & \text{otherwise.} \end{cases}$$

Then $l(W)$ is even, and $l(W) \le (n-l-k+1) + \frac{(l-1)}{2} + 2(k-1) + l + (n-l-k+1) = 2n - \frac{l}{2} - \frac{1}{2}$, thus $W + C'$ and $W + C''$ are a pair of $SSSD$ walks with even length $\le 2n - l + \frac{3}{2} < 2n-1$. Therefore, there exists a pair of $SSSD$ walks of length $2n-2$ from x to y.

Subcase 3.2: If $v' \notin Q_C(x' \to y')$.

Set $w = l(P_1) + l(Q_C(x' \to y')) + l(P_2)$.

If w is odd, then set $W = P_1 + Q_C(x' \to y') + C + 2P + P_2$.

If w is even, then set $a = l(Q_C(x' \to v'))$, $b = l(Q_C(y' \to v'))$ and

$$W = \begin{cases} P_1 + 2Q_C(x' \to v') + Q_C(x' \to y') + 2P + P_2, & \text{if } a \le b; \\ P_1 + Q_C(x' \to y') + 2Q_C(y' \to v') + 2P + P_2, & \text{otherwise.} \end{cases}$$

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Then $l(W)$ is even, and $l(W) \le (n-l-k+1)+\frac{(l-1)}{2}+l+2(k-1)+(n-l-k+1) = 2n-\frac{l}{2}-\frac{1}{2}$. Thus, $W+C'$ and $W+C''$ are a pair of $SSSD$ walks with even length $\le 2n-l+\frac{3}{2} < 2n-1$. Therefore, there exists a pair of $SSSD$ walks of length $2n-2$ from x to y.

Combining the three cases, we have $b(S) \le 2n-2$ by Theorem 2.3. $\square$

LEMMA 3.3. *Suppose $S \in \mathcal{S}_n^\star$ and S is non-powerful. If every 2-cycle is positive, then $b(S) \le 2n-2$.*

Proof. Since S is primitive and non-powerful, Theorem 2.2 applies.

Let x and y be any two (not necessarily distinct) vertices in S.

Case 1: If (A_1) of Theorem 2.2 holds.

In this case, there exist an odd cycle $C_l(l \ge 3)$ and an even cycle $C_k(k \ge 4)$ such that $\mathrm{sgn} C_k = -1$.

Subcase 1.1: $C_l \cap C_k = \emptyset$.

Let P be the shortest path from C_l to C_k and P intersect $C_l(C_k)$ at $v(v')$. By Lemma 2.5, there exist $x' \in C_l, y' \in C_k$ or $x' \in C_k, y' \in C_l$ such that (2.2) holds. Without loss of generality, suppose there exist $x' \in C_l, y' \in C_k$ such that (2.2) holds. For convenience, let P_1 be the shortest path from x to x' and P_2 be the shortest path from y to y'.

Set $w = l(P_1) + l(Q_{C_l}(x' \to v)) + l(P) + l(Q_{C_k}(v' \to y')) + l(P_2)$ and

$$W_1 = \begin{cases} P_1 + Q_{C_l}(x' \to v) + P + Q_{C_k}(v' \to y') + P_2, & \text{if } w \text{ is even;} \\ P_1 + C_l \setminus Q_{C_l}(x' \to v) + P + Q_{C_k}(v' \to y') + P_2, & \text{otherwise.} \end{cases}$$

and

$$W_2 = \begin{cases} P_1 + Q_{C_l}(x' \to v) + P + C_k \setminus Q_{C_k}(v' \to y') + P_2, & \text{if } w \text{ is even;} \\ P_1 + C_l \setminus Q_{C_l}(x' \to v) + P + C_k \setminus Q_{C_k}(v' \to y') + P_2, & \text{otherwise.} \end{cases}$$

Then $l(W_1)$ and $l(W_2)$ are even because $l(Q_{C_k}(v' \to y'))$ and $l(C_k \setminus Q_{C_k}(v' \to y'))$ have the same parity, $\mathrm{sgn} Q_{C_k}(v' \to y') = -\mathrm{sgn} C_k \setminus Q_{C_k}(v' \to y')$ and $\mathrm{sgn}(W_1) = -\mathrm{sgn}(W_2)$ because $\mathrm{sgn} C_k = -1$ and every 2-cycle is positive.

So $W_2, W_3 = W_1 + \frac{l(W_2)-l(W_1)}{2} C_2$ are a pair of $SSSD$ walks from x to y with even length $l(W_2) \le 2(n-k-l+1) + \max\left\{\frac{l-1}{2}, \frac{k}{2}\right\} + l + k < 2n-2$ by (2.2), and thus, there exists a pair of $SSSD$ walks of length $2n-2$ from x to y.

Subcase 1.2: $C_l \cap C_k \ne \emptyset$.

Suppose $C_l \cup C_k$ has k' vertices. Let P_1 (P_2) be the shortest path from x (y) to

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$C_l \cup C_k$ and P_1 (P_2) intersect $C_l \cup C_k$ at x' (y') where $0 \le l(P_i) \le n - k'$, $i = 1, 2$.

Subcase 1.2.1: $x' \in C_l$ and $y' \in C_l$.

Suppose $z \in C_l \cap C_k$, without loss of generality, we suppose $z \in Q_{C_l}(x' \to y')$.

Set $w = l(P_1) + l(Q_{C_l}(x' \to y')) + l(P_2)$, $a = l(Q_{C_l}(x' \to z))$, $b = l(Q_{C_l}(y' \to z))$ and

$$W = \begin{cases} P_1 + Q_{C_l}(x' \to y') + P_2, & \text{if } w \text{ is even;} \\ P_1 + C_l \setminus Q_{C_l}(x' \to y') + 2Q_{C_l}(x' \to z) + P_2, & \text{if } w \text{ is odd and } a \le b; \\ P_1 + C_l \setminus Q_{C_l}(x' \to y') + 2Q_{C_l}(y' \to z) + P_2, & \text{if } w \text{ is odd and } a > b. \end{cases}$$

Then $W_1 = W + C_k$ and $W_2 = W + \frac{k}{2}C_2$ are a pair of $SSSD$ walks from x to y with even length $l(W_1) = l(W_2) \le 2(n-k') + |C_l| + |C_k| = 2(n-k') + |C_l \cup C_k| + |C_l \cap C_k| \le 2(n-k') + k' + (k'-1) = 2n-1$. Thus, there exists a pair of $SSSD$ walks of length $2n-2$ from x to y.

Subcase 1.2.2: $x' \in C_l$ and $y' \in C_k$.

Suppose $z \in C_l \cap C_k$. Set $w = l(P_1) + l(Q_{C_l}(x' \to z)) + l(Q_{C_k}(z \to y')) + l(P_2)$ and

$$W_1 = \begin{cases} P_1 + Q_{C_l}(x' \to z) + Q_{C_k}(z \to y') + P_2, & \text{if } w \text{ is even;} \\ P_1 + C_l \setminus Q_{C_l}(x' \to z) + Q_{C_k}(z \to y') + P_2, & \text{otherwise.} \end{cases}$$

$$W_2 = \begin{cases} P_1 + Q_{C_l}(x' \to z) + C_k \setminus Q_{C_k}(z \to y') + P_2, & \text{if } w \text{ is even;} \\ P_1 + C_l \setminus Q_{C_l}(x' \to z) + C_k \setminus Q_{C_k}(z \to y') + P_2, & \text{otherwise.} \end{cases}$$

Then $\mathrm{sgn} W_1 = -\mathrm{sgn} W_2$, $l(W_1)$ and $l(W_2)$ are even lengths with $l(W_1) \le l(W_2)$. So W_2 and $W_3 = W_1 + \frac{l(W_2) - l(W_1)}{2} C_2$ are a pair of $SSSD$ walks from x to y with even length $l(W_2) \le 2(n-k') + |C_l| + |C_k| \le 2n-1$, and thus, there exists a pair of $SSSD$ walks of length $2n-2$ from x to y.

Subcase 1.2.3: $x' \in C_k$ and $y' \in C_k$. Suppose $z \in C_l \cap C_k$, without loss of generality, we suppose $z \in Q_{C_k}(x' \to y')$.

Set $w = l(P_1) + l(Q_{C_k}(x' \to y')) + l(P_2)$, $a = l(Q_{C_k}(x' \to z))$, $b = l(Q_{C_k}(y' \to z))$ and

$$W_1 = \begin{cases} P_1 + Q_{C_k}(x' \to y') + P_2, & \text{if } w \text{ is even;} \\ P_1 + Q_{C_k}(x' \to y') + C_l + P_2, & \text{otherwise.} \end{cases}$$

$$W_2 = \begin{cases} P_1 + C_k \setminus Q_{C_k}(x' \to y') + P_2, & w \text{ is even;} \\ P_1 + C_k \setminus Q_{C_k}(x' \to y') + 2Q_{C_k}(x' \to z) + C_l + P_2, & w \text{ is odd, } a \le b; \\ P_1 + C_k \setminus Q_{C_k}(x' \to y') + 2Q_{C_k}(y' \to z) + C_l + P_2, & w \text{ is odd, } a > b. \end{cases}$$

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Then $\mathrm{sgn}W_1 = -\mathrm{sgn}W_2$, $l(W_1)$ and $l(W_2)$ are even lengths with $l(W_1) \le l(W_2)$. So W_2 and $W_3 = W_1 + \frac{l(W_2)-l(W_1)}{2} C_2$ are a pair of $SSSD$ walks from x to y with even length $l(W_2) \le 2(n-k') + |C_l| + |C_k| \le 2n-1$. Thus, there exists a pair of $SSSD$ walks of length $2n-2$ from x to y.

Case 2: If (A_2) of Theorem 2.2 holds.

In this case, there exist two odd cycles have different signs. Suppose C_l and C_k are two odd cycles such that $\mathrm{sgn}C_l = -\mathrm{sgn}C_k$ and the sum $l+k$ is the least length where $l, k (\ge 3)$ are odd.

Subcase 2.1: $C_l \cap C_k = \emptyset$.

Without loss of generality, we assume $\mathrm{sgn}C_l = 1$ and $\mathrm{sgn}C_k = -1$.

Let P, P_1, P_2, v, v', x', y' be defined as Subcase 1.1. Set $w = l(P_1) + l(Q_{C_l}(x' \to v)) + l(P) + l(Q_{C_k}(v' \to y')) + l(P_2)$, and

$$W_1 = \begin{cases} P_1 + Q_{C_l}(x' \to v) + P + Q_{C_k}(v' \to y') + P_2, & \text{if } w \text{ is even;} \\ P_1 + C_l \setminus Q_{C_l}(x' \to v) + P + Q_{C_k}(v' \to y') + P_2, & \text{otherwise.} \end{cases}$$

and

$$W_2 = \begin{cases} P_1 + C_l \setminus Q_{C_l}(x' \to v) + P + C_k \setminus Q_{C_k}(v' \to y') + P_2, & \text{if } w \text{ is even;} \\ P_1 + Q_{C_l}(x' \to v) + P + C_k \setminus Q_{C_k}(v' \to y') + P_2, & \text{otherwise.} \end{cases}$$

Then $l(W_1)$, $l(W_2)$ are even lengths, and $\mathrm{sgn}W_1 = -\mathrm{sgn}W_2$ since $\mathrm{sgn}Q_{C_k}(v' \to y') = -\mathrm{sgn}C_k \setminus Q_{C_k}(v' \to y')$. So there exist a pair of $SSSD$ walks from x to y with even length no more than $\max\{l(W_1), l(W_2)\} \le 2(n-l-k+1) + \max\left\{\frac{l-1}{2}, \frac{k-1}{2}\right\} + l + k < 2n-2$ by (2.2). Thus, there exists a pair of $SSSD$ walks of length $2n-2$ from x to y.

Subcase 2.2: $C_l \cap C_k \ne \emptyset$.

Suppose $l \le k$ and $C_l \cup C_k$ has k' vertices. We claim

$$|C_l \cap C_k| \le k' - 1. \tag{3.1}$$

If $|C_l \cap C_k| > k'-1$, then $|C_l \cap C_k| = k' = |C_l \cup C_k|$, and thus, C_l, C_k have the same vertices with $l = k = k'$.

Suppose $C_l = v_1 v_2 \cdots v_l v_1$, take $\overleftarrow{C_l} = v_1 v_l v_{l-1} \cdots v_2 v_1$, then $C_k \ne C_l$ since they have different signs and $C_k \ne \overleftarrow{C_l}$ since all 2-cycle are positive. But the vertices $v_1, v_2, \ldots, v_l$ are on the cycle C_k, we must have $(v_i, v_j) \in C_k$ or $(v_i, v_j) \in \overleftarrow{C_k}$ where $(v_i, v_j) \notin C_l$, $(v_i, v_j) \notin \overleftarrow{C_l}$ and $1 \le i < j \le l$. Then $C' = v_i v_{i+1} \cdots v_j v_i$ and $C'' = v_1 v_2 \cdots v_i v_j v_{j+1} \cdots v_l v_1$ are two cycles in S with the sum $l(C') + l(C'') = l + 2$

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is odd, thus $l(C')$ or $l(C'')$ is odd and $3 \leq l(C'), l(C'') \leq l-1$. Without loss of generality, we assume $l(C')$ is odd, so C' and C_l (or C' and C_k) have different signs and the sum $l(C')+l \leq l+k$ (or $l(C')+k \leq l+k$); this is a contradiction.

So (3.1) holds, and thus, we have

$$|C_l|+|C_k| = |C_l \cup C_k| + |C_l \cap C_k| \leq 2k'-1. \tag{3.2}$$

Let $P_1(P_2)$ be the shortest path from $x(y)$ to $C_l \cup C_k$ and P_1 (P_2) intersect $C_l \cup C_k$ at x' (y') where $0 \leq l(P_i) \leq n-k'$, $i=1,2$.

Subcase 2.2.1: $x' \in C_l$ and $y' \in C_l$.

Let $z \in C_l \cap C_k$, without loss of generality, we suppose $z \in Q_{C_l}(x' \to y')$.

Set $w = l(P_1)+l(Q_{C_l}(x' \to y'))+l(P_2)$, $a = l(Q_{C_l}(x' \to z))$, $b = l(Q_{C_l}(y' \to z))$ and

$$W = \begin{cases} P_1 + Q_{C_l}(x' \to y') + P_2, & w \text{ is odd;} \\ P_1 + C_l \setminus Q_{C_l}(x' \to y') + 2Q_{C_l}(x' \to z) + P_2, & w \text{ is even, } a \leq b; \\ P_1 + C_l \setminus Q_{C_l}(x' \to y') + 2Q_{C_l}(y' \to z) + P_2, & w \text{ is even, } a > b. \end{cases}$$

Let $W_1 = W + C_k$ and $W_2 = W + C_l + \frac{k-l}{2}C_2$. Then W_1 and W_2 are a pair of $SSSD$ walks from x to y with even length $\leq 2(n-k')+|C_l|+|C_k| \leq 2n-1$ by (3.2). So there exists a pair of $SSSD$ walks of length $2n-2$ from x to y.

Subcase 2.2.2: $x' \in C_l$ and $y' \in C_k$.

Suppose $z \in C_l \cap C_k$. Set $w = l(P_1)+l(Q_{C_l}(x' \to z))+l(Q_{C_k}(z \to y'))+l(P_2)$, and

$$W_1 = \begin{cases} P_1 + Q_{C_l}(x' \to z) + Q_{C_k}(z \to y') + P_2, & \text{if } w \text{ is even;} \\ P_1 + C_l \setminus Q_{C_l}(x' \to z) + Q_{C_k}(z \to y') + P_2, & \text{otherwise.} \end{cases}$$

and

$$W_2 = \begin{cases} P_1 + C_l \setminus Q_{C_l}(x' \to z) + C_k \setminus Q_{C_k}(z \to y') + P_2, & \text{if } w \text{ is even;} \\ P_1 + Q_{C_l}(x' \to z) + C_k \setminus Q_{C_k}(z \to y') + P_2, & \text{otherwise.} \end{cases}$$

Then $\mathrm{sgn}W_1 = -\mathrm{sgn}W_2$, $l(W_1)$ and $l(W_2)$ are even lengths. So there exists a pair of $SSSD$ walks from x to y with even length $\leq \max\{l(W_1), l(W_2)\} \leq 2(n-k')+|C_l|+|C_k| \leq 2n-1$ by (3.2). Thus, there exists a pair of $SSSD$ walks of length $2n-2$ from x to y.

Subcase 2.2.3: $x' \in C_k$ and $y' \in C_k$.

Let $z \in C_l \cap C_k$, without loss of generality, we suppose $z \in Q_{C_k}(x' \to y')$.

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Subcase 2.2.3.1: $\text{sgn}C_k = -1$ and $\text{sgn}C_l = 1$.

Set $w = l(P_1)+l(Q_{C_k}(x' \to y'))+l(P_2)$, $a = l(Q_{C_k}(x' \to z))$, $b = l(Q_{C_k}(y' \to z))$, and

$$W_1 = \begin{cases} P_1 + Q_{C_k}(x' \to y') + P_2, & \text{if } w \text{ is even;} \\ P_1 + Q_{C_k}(x' \to y') + C_l + P_2, & \text{otherwise.} \end{cases}$$

and

$$W_2 = \begin{cases} P_1 + C_k \setminus Q_{C_k}(x' \to y') + P_2, & w \text{ is odd;} \\ P_1 + C_k \setminus Q_{C_k}(x' \to y') + 2Q_{C_k}(x' \to z) + C_l + P_2, & w \text{ is even, } a \le b; \\ P_1 + C_k \setminus Q_{C_k}(x' \to y') + 2Q_{C_k}(y' \to z) + C_l + P_2, & w \text{ is even, } a > b. \end{cases}$$

Subcase 2.2.3.2: $\text{sgn}C_k = 1$ and $\text{sgn}C_l = -1$.

Set $w = l(P_1)+l(Q_{C_k}(x' \to y'))+l(P_2)$, $a = l(Q_{C_k}(x' \to z))$, $b = l(Q_{C_k}(y' \to z))$, and

$$W_1 = \begin{cases} P_1 + Q_{C_k}(x' \to y') + P_2, & \text{if } w \text{ is even;} \\ P_1 + C_k \setminus Q_{C_k}(x' \to y') + P_2, & \text{otherwise.} \end{cases}$$

and

$$W_2 = \begin{cases} P_1 + C_k \setminus Q_{C_k}(x' \to y') + 2Q_{C_k}(x' \to z) + C_l + P_2, & w \text{ is even, } a \le b; \\ P_1 + C_k \setminus Q_{C_k}(x' \to y') + 2Q_{C_k}(y' \to z) + C_l + P_2, & w \text{ is even, } a > b. \\ P_1 + Q_{C_k}(x' \to y') + 2Q_{C_k}(x' \to z) + C_l + P_2, & w \text{ is odd, } a \le b; \\ P_1 + Q_{C_k}(x' \to y') + 2Q_{C_k}(y' \to z) + C_l + P_2, & w \text{ is odd, } a > b. \end{cases}$$

In both Subcase 2.2.3.1 and Subcase 2.2.3.2, we have $\text{sgn}W_1 = -\text{sgn}W_2$, $l(W_1)$ and $l(W_2)$ are even lengths. So there exists a pair of $SSSD$ walks from x to y with even length $\le \max\{l(W_1), l(W_2)\} \le 2(n-k') + |C_l| + |C_k| \le 2n-1$ by (3.2). Thus, there exists a pair of $SSSD$ walks of length $2n-2$ from x to y.

From the above arguments, we have $b(S) \le 2n-2$ by Theorem 2.3. □

By Lemmas 3.2 and 3.3, we can obtain the following corollary.

Corollary 3.4. *Suppose $S \in \mathcal{S}_n^\star$ and $b(S) = 2n-1$, then every 2-cycle in S is negative.*

Now we characterize the primitive symmetric loop-free signed digraphs with the maximum base as follows.

Theorem 3.5. *Suppose $S \in \mathcal{S}_n^\star$. Then $b(S) = 2n-1$ if and only if $S \in SD_L = \{SD_l | 3 \le l \le n$ and l is odd $\}$.*

Proof. Sufficiency follows easily from Lemma 3.1.

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ELA

Necessity: Since S is primitive, there exists at least one odd cycle in S. Suppose $C = v_1v_2\cdots v_lv_1$ is an odd cycle with the shortest length l in S, then C and $\overleftarrow{C} = v_1v_l\cdots v_2v_1$ is a pair of $SSSD$ walks because all 2-cycles in S are negative by Corollary 3.4 and l is odd.

Since $b(S) = 2n-1$, there exist two vertices x, y such that there are no $SSSD$ walks of length $2n-2$ from x to y. Let P_1 (P_2) be the shortest path from x (y) to C and intersect C at $x'(y')$ where $0 \le l(P_i) \le n-l$, $i = 1, 2$. Now we prove $l(P_1) = n-l$.

If $l = n$, $l(P_1) = n-l$ holds clearly. If $l < n$, we suppose $l(P_1) \le n-l-1$, set $w = l(P_1) + l(Q_C(x' \to y')) + l(P_2)$ and

$$W = \begin{cases} P_1 + Q_C(x' \to y') + P_2, & \text{if } w \text{ is odd;} \\ P_1 + C \setminus Q_C(x' \to y') + P_2, & \text{otherwise.} \end{cases}$$

Let $W_1 = W + C$ and $W_2 = W + \overleftarrow{C}$, then W_1 and W_2 are a pair of $SSSD$ walks from x to y with even lengths $l(W_1) = l(W_2) \le (n-l-1)+l+(n-l)+l = 2n-1$. Then there exists a pair of $SSSD$ walks of length $2n-2$ from x to y; this is a contradiction.

Therefore, $l(P_1) = n-l$. Similarly, we can show $l(P_2) = n-l$ and $x = y$. It implies that for any $v \in S$ and $v \notin C$, we have $v \in P_1$ and $v \in P_2$. Thus, we assume $P_1 = x_1x_2\cdots x_{n-l}x'$ and $P_2 = x_1x_2\cdots x_{n-l}y'$ where $x_1 = x = y$.

Now we prove $x' = y'$. Suppose $x' \neq y'$. Then $1 \le l(Q_C(x' \to y')) \le \frac{l-1}{2}$, $\frac{l+1}{2} \le l(C \setminus Q_C(x' \to y')) \le l-1$. Let W, W_1, W_2 defined as above. Thus, W_1 and W_2 are a pair of $SSSD$ walks from x to y with even lengths $l(W_1) = l(W_2) \le (n-l) + (l-1) + (n-l) + l = 2n-1$. Then there exists a pair of $SSSD$ walks of length $2n-2$ from x to y; this is a contradiction. Thus, $x' = y'$ ($\in V(C)$), denoted by v_l.

Now we see that D_l is isomorphic to the subgraph of D where D is the underlying digraph of S. In fact, D_l is isomorphic to D. For this purpose, we only need to show D has no more arcs.

Firstly, there are no more arcs between vertex x_{n-l} and vertex v_i ($1 \le i \le l$) by the same reason why $x' = y'$. Secondly, there are no more arcs between vertex x_j ($1 \le j \le n-l-1$) and vertex v_i ($1 \le i \le l$) and there are no more arcs between vertex x_j ($1 \le j \le n-l$) and vertex x_i ($1 \le i \le n-l$) because the path $P_1 = x_1x_2\cdots x_{n-l}v_l$ is the shortest path from x_1 to C. Finally, there are no more arcs between vertex v_j ($1 \le j \le l$) and vertex v_i ($1 \le i \le l$) because the cycle C is the shortest odd cycle in S.

Thus, D_l is isomorphic to D and $S \in SD_L$ because all 2-cycles in S are negative by Corollary 3.4. □

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