Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 23, pp. 115-121, January 2012

ELA

STRONG LINEAR PRESERVERS OF G-TRIDIAGONAL MAJORIZATION ON $\mathbb{R}^N$*

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Abstract. An $n \times n$ real matrix (not necessarily nonnegative) A is g-doubly stochastic (generalized doubly stochastic) if all its row and column sums are one. The sets of all g-doubly stochastic and tridiagonal g-doubly stochastic matrices of order n are denoted by Ω_n and Ω_n^t, respectively. For $x, y \in \mathbb{R}^n$, it is said that x is g-tridiagonal majorized by y (written as $x \prec_{gt} y$) if there exists a tridiagonal g-doubly stochastic matrix A such that $x = Ay$. This paper characterizes all strong linear preservers of $\prec_{gt}$ on $\mathbb{R}^n$ and $\mathbb{R}_n$.

Key words. Doubly stochastic matrix, g-Tridiagonal majorization, Strong linear preserver.

AMS subject classifications. 15A04, 15A21.

1. Introduction. Majorization is a topic of much interest in various areas of mathematics and statistics. In the recent years, this concept has been attended specially. Assume that $\mathbb{R}^n$ (respectively, $\mathbb{R}_n$) is the vector space of all real $n \times 1$ (respectively, $1 \times n$) vectors. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear operator and let $\sim$ be a relation on $\mathbb{R}^n$. It is said that T strongly preserves $\sim$, if, for all $x, y \in \mathbb{R}^n$,

$$x \sim y \Longleftrightarrow T(x) \sim T(y).$$

An $n \times n$ nonnegative matrix A is called doubly stochastic if all its row and column sums equal one. For $x, y \in \mathbb{R}^n$, it is said that x is vector majorized by y (written as $x \prec y$) if there exists a doubly stochastic matrix D such that $x = Dy$.

In [1, 6], the authors characterized all strong linear preservers of $\prec$ on $\mathbb{R}^n$, as follows:

Proposition 1.1. *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear operator. Then T strongly preserves $\prec$ if and only if there exist $\alpha, \beta \in \mathbb{R}$ and a permutation matrix P such that $Tx = \alpha Px + \beta \boldsymbol{J}x$ for all $x \in \mathbb{R}^n$ and $\alpha(\alpha + n\beta) \neq 0$, where $\boldsymbol{J}$ is the $n \times n$ matrix with all entries equal one.*

The following notation will be fixed throughout the paper: $\mathbf{M}_n$ for the collection of all $n \times n$ real matrices, Ω_n for the set of all $n \times n$ g-doubly stochastic matrices,

*Received by the editors on July 29, 2011. Accepted for publication on December 18, 2011. Handling Editor: Bryan L. Shader.

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Electronic Journal of Linear Algebra ISSN 1081-3810
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ELA

Ω_n^t for the set of all $n \times n$ tridiagonal g-doubly stochastic matrices, $\mathbf{J}$ and $\mathbf{e}$ for the matrix and the vector with all entries equal one, respectively (the size of $\mathbf{J}$ and $\mathbf{e}$ are understood from the content), e_i for the i^{th} element of the standard ordered basis of $\mathbb{R}^n$, and

$$A_\mu = \begin{pmatrix} 1-\mu_1 & \mu_1 & & & 0 \\ \mu_1 & 1-\mu_1-\mu_2 & \mu_2 & & \\ & & & \ddots & \mu_{n-1} \\ 0 & & & \mu_{n-1} & 1-\mu_{n-1} \end{pmatrix},$$

where $\mu = (\mu_1, \ldots, \mu_{n-1})^t \in \mathbb{R}^{n-1}$. It is easy to show that $\Omega_n^t = \{A_\mu : \mu \in \mathbb{R}^{n-1}\}$. The notation A^t stands for the transpose of a given matrix A. For a given vector $x \in \mathbb{R}^n$, $\mathrm{tr}(x)$ is the sum of all components of x. For a given linear operator $T : \mathbb{R}^n \to \mathbb{R}^n$, the matrix representation of T with respect to the standard ordered basis of $\mathbb{R}^n$, is denoted by $[T]$.

For $x, y \in \mathbb{R}^n$, it is said that x is gs-majorized by y (written as $x \prec_{gs} y$) if there exists an $n \times n$ g-doubly stochastic matrix D such that $x = Dy$. The linear operators strongly preserving $\prec_{gs}$ on $\mathbb{R}^n$, have been characterized as follows; (see [2, 4, 3, 7] for more details).

Proposition 1.2. [3, Corollary 2.5] *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear operator. Then T strongly preserves gs-majorization if and only if $T(x) = \alpha Dx$ for some nonzero scalar $\alpha \in \mathbb{R}$ and invertible matrix $D \in \Omega_n$.*

For $x, y \in \mathbb{R}^n$, it is said that x is tridiagonally majorized by y if there exists a tridiagonal doubly stochastic matrix D such that $x = Dy$, see [5].

Definition 1.3. Let $x, y \in \mathbb{R}^n$. We say that x is g-tridiagonally majorized by y (written as $x \prec_{gt} y$) if there exists a tridiagonal g-doubly stochastic matrix D such that $x = Dy$.

In the present paper, we find the structure of strong linear preservers of $\prec_{gt}$ on $\mathbb{R}^n$ and $\mathbb{R}_n$. In fact we will prove the following theorem:

Theorem 1.4. *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear operator. Then T strongly preserves $\prec_{gt}$ if and only if there exist $a, b \in \mathbb{R}$ such that $(a-b)(a+(n-1)b) \neq 0$ and $[T]$ is one of the following matrices*

$$\begin{pmatrix} a & b & b & \cdots & b \\ b & a & b & \cdots & b \\ \vdots & \vdots & \vdots & & \vdots \\ b & b & b & \cdots & a \end{pmatrix} \quad \textit{or} \quad \begin{pmatrix} b & b & \cdots & b & a \\ b & b & \cdots & a & b \\ \vdots & \vdots & & \vdots & \vdots \\ a & b & \cdots & b & b \end{pmatrix}.$$

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A publication of the International Linear Algebra Society
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In other words, T strongly preserves $\prec_{gt}$ if and only if there exist $\alpha, \beta \in \mathbb{R}$ such that $\alpha(\alpha + n\beta) \neq 0$ and $[T] = \alpha I + \beta \mathbf{J}$ or $[T] = \alpha P + \beta \mathbf{J}$, where P is the backward identity matrix.

2. g-Tridiagonally majorization.

In this section, we mention some properties of $\prec_{gt}$ on $\mathbb{R}^n$ and also we present some preliminaries to prove Theorem 1.4.

PROPOSITION 2.1. [3, Lemma 3.6] *Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a linear operator such that $T(x) = \alpha Dx + \beta \boldsymbol{J}x$ for some $\alpha, \beta \in \mathbb{R}$ and invertible matrix $D \in \Omega_n$. Then T is invertible if and only if $\alpha(\alpha + n\beta) \neq 0$.*

LEMMA 2.2. *Let $x, y \in \mathbb{R}^n$. If every two consecutive components of y are distinct, then $x \prec_{gt} y$ if and only if $tr(x) = tr(y)$.*

Proof. If $x \prec_{gt} y$, it is easy to see that $tr(x) = tr(y)$. Conversely, suppose that every two consecutive components of y are distinct. For every j $(1 \leq j \leq n-1)$, put $\mu_j = \frac{\sum_{i=1}^{j}(x_i - y_i)}{y_{j+1} - y_j}$. With a direct calculation it is easy to see that $x = A_\mu y$, where $\mu = (\mu_1, \ldots, \mu_{n-1})^t$, and hence $x \prec_{gt} y$. ☐

The following theorem gives an equivalent condition for $\prec_{gt}$ on $\mathbb{R}^n$.

THEOREM 2.3. *Let x and y be two distinct vectors in $\mathbb{R}^n$. Assume that $i_1 < i_2 < \cdots < i_k$ and $\{i_1, i_2, \ldots, i_k\} = \{j : 1 \leq j \leq n-1,\ y_j = y_{j+1}\}$. Then $x \prec_{gt} y$ if and only if $\sum_{j=i_{l-1}+1}^{i_l} x_j = \sum_{j=i_{l-1}+1}^{i_l} y_j$ for every l $(1 \leq l \leq k+1)$, where $i_{k+1} = n$ and $i_0 = 0$.*

Proof. If $x \prec_{gt} y$, then there exists $A_\mu \in \Omega_n^t$ such that $x = A_\mu y$. Consequently, for every j $(1 \leq j \leq n)$, $x_j = \mu_{j-1}(y_{j-1} - y_j) + \mu_j(y_{j+1} - y_j) + y_j$, where $y_0 = \mu_0 = y_{n+1} = \mu_n = 0$. For every $j \in \{i_1, i_2, \ldots, i_k\}$, $y_j = y_{j+1}$ then $\sum_{j=i_{l-1}+1}^{i_l} x_j = \sum_{j=i_{l-1}+1}^{i_l} y_j$, for every l $(1 \leq l \leq k+1)$. Conversely, put $y^1 = (y_1, y_2, \ldots, y_{i_1})^t$ and $x^1 = (x_1, x_2, \ldots, x_{i_1})^t$. Then every two consecutive components of y^1 are distinct. Since $\sum_{j=1}^{i_1} x_j = \sum_{j=1}^{i_1} y_j$, $x^1 \prec_{gt} y^1$ by Lemma 2.2. Then there exists $A_1 \in \Omega_{i_1}^t$ such that $x^1 = A_1 y^1$. Now, for every l $(2 \leq l \leq k+1)$ put $x^l = (x_{i_{l-1}+1}, x_{i_{l-1}+2}, \ldots, x_{i_l})^t$ and $y^l = (y_{i_{l-1}+1}, y_{i_{l-1}+2}, \ldots, y_{i_l})^t$. Since $\sum_{j=i_{l-1}+1}^{i_l} x_j = \sum_{j=i_{l-1}+1}^{i_l} y_j$, $x^l \prec_{gt} y^l$ by Lemma 2.2. Then there exists $A_l \in \Omega_{i_l - i_{l-1}}^t$ such that $x^l = A_l y^l$. Put $A := \oplus_{j=1}^{k} A_j$, it follows that $A \in \Omega_n^t$ and $x = Ay$, therefore $x \prec_{gt} y$. ☐

LEMMA 2.4. *Let $y \in \mathbb{R}^n$. Assume that $i_1 < i_2 < \cdots < i_k$ and $\{i_1, i_2, \ldots, i_k\} = \{j : 1 \leq j \leq n-1,\ y_j = y_{j+1}\}$. Then $H_y := \{x \in \mathbb{R}^n : x \prec_{gt} y\}$ is an affine set with dimension $n - (k+1)$.*

Proof. By Theorem 2.3, it follows that:

$$H_y = \{x \in \mathbb{R}^n : \textstyle\sum_{j=i_{l-1}+1}^{i_l} x_j = \sum_{j=i_{l-1}+1}^{i_l} y_j, \forall\ l \in \{1, \ldots, k+1\}\},$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 23, pp. 115-121, January 2012

ELA

where $i_{k+1} = n$ and $i_0 = 0$. If $\lambda \in \mathbb{R}$, $x, z \in H_y$, it is clear that $\lambda x + (1-\lambda)z \in H_y$, so H_y is an affine set. Since every $x \in H_y$ have to satisfy $k+1$ equations, it is easy to see that $\dim H_y = n - (k+1)$. ∎

Corollary 2.5. *Let $y \in \mathbb{R}^n$. Then $\dim H_y = 0$ if and only $y \in \mathrm{Span}\{\boldsymbol{e}\}$.*

Proposition 2.6. *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear operator. If T strongly preserves $\prec_{gt}$, then the following statements are true:*
(i) T is invertible.
(ii) $\mathrm{tr}(Te_i) = \mathrm{tr}(Te_j)$, for every $i, j \in \{1, \ldots, n\}$.
(iii) $T\boldsymbol{e} \in \mathrm{Span}\{\boldsymbol{e}\}$.
(v) $[T]$ is a multiple of a g-doubly stochastic matrix.

Proof. (i) Suppose $T(x) = 0$. Since T is linear, $T(0) = 0 = T(x)$. Then it is obvious that $T(x) \prec_{gt} T(0)$. Therefore, $x \prec_{gt} 0$ because T strongly preserves gt-majorization. Then, there exists an $R \in \Omega_n^t$ such that $x = R0$. So, $x = 0$, and hence T is invertible. (ii) Using Theorem 2.3, $e_j \prec_{gt} e_{j+1}$ for every j $(1 \le j \le n-1)$. Then $Te_j \prec_{gt} Te_{j+1}$ for every j $(1 \le j \le n-1)$ and hence $\mathrm{tr}(Te_i) = \mathrm{tr}(Te_j)$, for every $i, j \in \{1, \ldots, n\}$. (iii) Since T is invertible, there exists $a \in \mathbb{R}^n$ such that $Ta = \mathbf{e}$. By Corollary 2.5, $dim(H_a) = dim(H_{Ta}) = 0$ and hence $T\mathbf{e} \in \mathrm{Span}\{\mathbf{e}\}$. (v) It is clear that by (ii) and (iii), $[T]$ is a multiple of a g-doubly stochastic matrix. ∎

Now, we prove the main theorem of this paper. Every linear operator $T : \mathbb{R}^n \to \mathbb{R}^n$ strongly preserves $\prec_{gt}$ if and only if $\alpha T : \mathbb{R}^n \to \mathbb{R}^n$ strongly preserves $\prec_{gt}$ for all $\alpha \in \mathbb{R} \setminus \{0\}$. So in the following proof, we assume without loss of generality that $tr(Te_1) = \cdots = tr(Te_n) = 1$.

Proof of Theorem 1.4. Let $A = [a_{ij}] = [T]$. If $n \le 2$, then the concepts $\prec_{gt}$ and $\prec_{gs}$ are the same on $\mathbb{R}^n$, and hence the proof is complete by Proposition 1.2. We assume without loss of generality that $n \geqslant 3$. The fact that the conditions (i) and (ii) are sufficient for T to be a strong linear preserver of $\prec_{gt}$ is easy to prove. So we prove the necessity of the conditions. Suppose that T strongly preserves $\prec_{gt}$, then T is invertible, by Proposition 2.6. Put $\Phi := \{x \in \mathbb{R}^n : x \prec_{gt} e_1\}$. From Theorem 2.3, we have $\Phi = \{x \in \mathbb{R}^n : x_1 + x_2 = 1,\ x_3 = \cdots = x_n = 0\}$ and $\dim \Phi = 1$. Since T is a strong linear preserver of $\prec_{gt}$, $T(\Phi) = \{Tx \in \mathbb{R}^n : x \prec_{gt} e_1\} = \{Tx \in \mathbb{R}^n : Tx \prec_{gt} Te_1\}$. By invertibility of T, we have $\dim \Phi = \dim T(\Phi) = 1$. Since $Te_1 \in T(\Phi)$ and $\dim T(\Phi) = 1$, Te_1 has $(n-1)$ equal consecutive components and hence $Te_1 = (a_{1,1}, b, \ldots, b)^t$ or $Te_1 = (b, \ldots, b, a_{n,1})^t$ for some $b \in \mathbb{R}$. Put $\Psi := \{x \in \mathbb{R}^n : x \prec_{gt} e_n\}$. From Theorem 2.3, we have $\Psi = \{x \in \mathbb{R}^n : x_{n-1} + x_n = 1,\ x_{n-2} = \cdots = x_1 = 0\}$ and $\dim \Psi = 1$. With a similar argument as above we may establish $Te_n = (a_{1,n}, c, \ldots, c)^t$ or $Te_n = (c, \ldots, c, a_{n,n})^t$, for some $c \in \mathbb{R}$. Now, we consider all possible forms of Te_1 and Te_n.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 23, pp. 115-121, January 2012

Let $T(e_1) = (b, \dots, b, a_{n,1})^t$. We have

$$\begin{aligned} e_2 \prec_{gt} e_1 &\Rightarrow Te_2 \prec_{gt} Te_1 \\ &\Rightarrow (a_{1,2}, a_{2,2}, \dots, a_{n,2})^t \prec_{gt} (b, \dots, b, a_{n,1})^t \\ &\Rightarrow a_{n-2,2} = \cdots = a_{2,2} = a_{1,2} = b \\ &\Rightarrow Te_2 = (b, \dots, b, a_{n-1,2}, a_{n,2})^t. \end{aligned}$$

For every j $(1 \leq j \leq n-1)$, $e_{j+1} \prec_{gt} e_j$. So with a similar argument as above $Te_j = (b, \dots, b, a_{n-j+1,j}, \dots, a_{n,j})$. It follows that

$$A = \begin{pmatrix} b & \cdots & b & a_{1,n} \\ \vdots & & & \\ b & & & * \\ a_{n,1} & & & \end{pmatrix}. \tag{1}$$

Let $Te_1 = (a_{1,1}, b, \dots, b)^t$. Similarly one may show that

$$A = \begin{pmatrix} a_{1,1} & & & * \\ b & \ddots & & \\ \vdots & & & \\ b & \cdots & b & a_{n,n} \end{pmatrix}. \tag{2}$$

Let $Te_n = (c, \dots, c, a_{n,n})^t$. We have

$$\begin{aligned} e_{n-1} \prec_{gt} e_n &\Rightarrow Te_{n-1} \prec_{gt} Te_n \\ &\Rightarrow (a_{1,n-1}, a_{2,n-1}, \dots, a_{n,n-1})^t \prec_{gt} (c, \dots, c, a_{nn})^t \\ &\Rightarrow a_{1,n-1} = a_{2,n-1} = \cdots = a_{n-2,n-1} = c \\ &\Rightarrow Te_{n-1} = (c, \dots, c, a_{n-1,n-1}, a_{n,n-1})^t. \end{aligned}$$

For every i $(2 \leq i \leq n-3)$, $e_{n-i} \prec_{gt} e_{n-i+1}$, so with an argument same as the above $Te_i = (c, \dots, c, a_{i,i}, \dots, a_{n,i})^t$. It follows that

$$A = \begin{pmatrix} a_{1,1} & c & \cdots & c \\ & \ddots & & \vdots \\ * & & & c \\ & & & a_{n,n} \end{pmatrix}. \tag{1*}$$

Let $Te_n = (a_{1,n}, c, \dots, c)^t$. Similarly one may show that

$$A = \begin{pmatrix} * & & & a_{1,n} \\ & & & c \\ & & & \vdots \\ a_{n,1} & c & \cdots & c \end{pmatrix}. \tag{2*}$$

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ELA

Since $n \geq 3$, and T is invertible the only possible cases are: $(1), (2)^\star$ and $(2), (1)^\star$. In view of Theorem 2.3, A is a multiple of a g-doubly stochastic matrix. Therefore A has one of the following forms:

$$\begin{pmatrix} a & b & b & \cdots & b \\ b & a & b & \cdots & b \\ \vdots & \vdots & \vdots & & \vdots \\ b & b & b & \cdots & a \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} b & b & \cdots & b & a \\ b & b & \cdots & a & b \\ \vdots & \vdots & & \vdots & \vdots \\ a & b & \cdots & b & b \end{pmatrix}.$$

Using Proposition 2.1 to obtain $(a-b)(a+(n-1)b) \neq 0$ in each case, these as done. $\square$

Corollary 2.7. *Let $P \in \mathbf{M}_n$ be a permutation matrix. Then the linear operator $T : \mathbb{R}^n \to \mathbb{R}^n$, defined by $T(x) = Px$, strongly preserves $\prec_{gt}$ if and only if P is identity or backward identity matrix.*

Now, we consider the gt-majorization on $\mathbb{R}_n$. Let $x, y \in \mathbb{R}_n$. We say that x is g-tridiagonally majorized by y (written as $x \prec_{rgt} y$) if there exists a tridiagonal g-doubly stochastic matrix D such that $x = yD$. Since the transpose of every tridiagonal g-doubly stochastic matrix is tridiagonal g-doubly stochastic too, we have $x \prec_{rgt} y$ if and only if $x^t \prec_{gt} y^t$ for every $x, y \in \mathbb{R}_n$.

Corollary 2.8. *Let $T : \mathbb{R}_n \to \mathbb{R}_n$ be a linear operator. Then T strongly preserves $\prec_{rgt}$ if and only if there exist $\alpha, \beta \in \mathbb{R}$ such that $\alpha(\alpha + n\beta) \neq 0$ and $Tx = \alpha xP + \beta x\mathbf{J}$ for all $x \in \mathbb{R}_n$, where P is the identity or the backward identity matrix.*

Proof. Define $S : \mathbb{R}^n \to \mathbb{R}^n$ by $Sx = [T(x^t)]^t$ for all $x \in \mathbb{R}^n$. It is easy to see that S strongly preserves $\prec_{gt}$ and hence Theorem 1.4 is applicable to S. $\square$

Acknowledgments. The authors would like to thank an anonymous referee for helpful comments and remarks.

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Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 23, pp. 115-121, January 2012

ELA

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