Abstract. An $n \times n$ real matrix (not necessarily nonnegative) $A$ is $g$-doubly stochastic (generalized doubly stochastic) if all its row and column sums are one. The sets of all $g$-doubly stochastic and tridiagonal $g$-doubly stochastic matrices of order $n$ are denoted by $\Omega_n$ and $\Omega_{t,n}$, respectively. For $x, y \in \mathbb{R}^n$, it is said that $x$ is tridiagonal majorized by $y$ (written as $x \prec_{gt} y$) if there exists a tridiagonal $g$-doubly stochastic matrix $A$ such that $x = Ay$. This paper characterizes all strong linear preservers of $\prec_{gt}$ on $\mathbb{R}^n$ and $\mathbb{R}^n$.

Key words. Doubly stochastic matrix, g-Tridiagonal majorization, Strong linear preserver.

AMS subject classifications. 15A04, 15A21.

1. Introduction. Majorization is a topic of much interest in various areas of mathematics and statistics. In the recent years, this concept has been attended specially. Assume that $\mathbb{R}^n$ (respectively, $\mathbb{R}_n$) is the vector space of all real $n \times 1$ (respectively, $1 \times n$) vectors. Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a linear operator and let $\sim$ be a relation on $\mathbb{R}^n$. It is said that $T$ strongly preserves $\sim$, if, for all $x, y \in \mathbb{R}^n$,

$$x \sim y \iff T(x) \sim T(y).$$

An $n \times n$ nonnegative matrix $A$ is called doubly stochastic if all its row and column sums equal one. For $x, y \in \mathbb{R}^n$, it is said that $x$ is vector majorized by $y$ (written as $x \prec y$) if there exists a doubly stochastic matrix $D$ such that $x = Dy$.

In [1, 6], the authors characterized all strong linear preservers of $\prec$ on $\mathbb{R}^n$, as follows:

**Proposition 1.1.** Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a linear operator. Then $T$ strongly preserves $\prec$ if and only if there exist $\alpha, \beta \in \mathbb{R}$ and a permutation matrix $P$ such that $Tx = \alpha P x + \beta J x$ for all $x \in \mathbb{R}^n$ and $\alpha (\alpha + n \beta) \neq 0$, where $J$ is the $n \times n$ matrix with all entries equal one.

The following notation will be fixed throughout the paper: $M_n$ for the collection of all $n \times n$ real matrices, $\Omega_n$ for the set of all $n \times n$ $g$-doubly stochastic matrices,
for the set of all $n \times n$ tridiagonal g-doubly stochastic matrices, $\mathbf{J}$ and $\mathbf{e}$ for the matrix and the vector with all entries equal one, respectively (the size of $\mathbf{J}$ and $\mathbf{e}$ are understood from the content), $e_i$ for the $i^{th}$ element of the standard ordered basis of $\mathbb{R}^n$, and

$$A_\mu = \begin{pmatrix}
1 - \mu_1 & \mu_1 & 0 \\
\mu_1 & 1 - \mu_1 - \mu_2 & \mu_2 \\
0 & \ddots & \mu_{n-1} \\
0 & \mu_{n-1} & 1 - \mu_{n-1}
\end{pmatrix},$$

where $\mu = (\mu_1, \ldots, \mu_{n-1})^t \in \mathbb{R}^{n-1}$. It is easy to show that $\Omega_n^t = \{A_\mu : \mu \in \mathbb{R}^{n-1}\}$.

The notation $A^t$ stands for the transpose of a given matrix $A$. For a given vector $x \in \mathbb{R}^n$, $\text{tr}(x)$ is the sum of all components of $x$. For a given linear operator $T : \mathbb{R}^n \to \mathbb{R}^n$, the matrix representation of $T$ with respect to the standard ordered basis of $\mathbb{R}^n$, is denoted by $[T]$.

For $x, y \in \mathbb{R}^n$, it is said that $x$ is gs-majorized by $y$ (written as $x \prec_{gs} y$) if there exists an $n \times n$ g-doubly stochastic matrix $D$ such that $x = Dy$. The linear operators strongly preserving $\prec_{gs}$ on $\mathbb{R}^n$, have been characterized as follows; (see [2, 4, 3, 7] for more details).

**Proposition 1.2.** [3, Corollary 2.5] Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear operator. Then $T$ strongly preserves gs-majorization if and only if $T(x) = \alpha Dx$ for some nonzero scalar $\alpha \in \mathbb{R}$ and invertible matrix $D \in \Omega_n$.

For $x, y \in \mathbb{R}^n$, it is said that $x$ is tridiagonally majorized by $y$ if there exists a tridiagonal doubly stochastic matrix $D$ such that $x = Dy$, see [5].

**Definition 1.3.** Let $x, y \in \mathbb{R}^n$. We say that $x$ is g-tridiagonally majorized by $y$ (written as $x \prec_{gt} y$) if there exists a tridiagonal g-doubly stochastic matrix $D$ such that $x = Dy$.

In the present paper, we find the structure of strong linear preservers of $\prec_{gt}$ on $\mathbb{R}^n$ and $\mathbb{R}_n$. In fact we will prove the following theorem:

**Theorem 1.4.** Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear operator. Then $T$ strongly preserves $\prec_{gt}$ if and only if there exist $a, b \in \mathbb{R}$ such that $(a - b)(a + (n - 1)b) \neq 0$ and $[T]$ is one of the following matrices

$$\begin{pmatrix}
  a & b & b & \ldots & b \\
  b & a & b & \ldots & b \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  b & b & b & \ldots & a
\end{pmatrix} \quad \text{or} \quad \begin{pmatrix}
  b & b & \ldots & b & a \\
  b & b & \ldots & a & b \\
  \vdots & \vdots & \ddots & \vdots & \vdots \\
  a & b & \ldots & b & b
\end{pmatrix}.$$
In other words, $T$ strongly preserves $\prec_{gt}$ if and only if there exist $\alpha, \beta \in \mathbb{R}$ such that $\alpha(\alpha + n\beta) \neq 0$ and $[T] = \alpha I + \beta J$ or $[T] = \alpha P + \beta J$, where $P$ is the backward identity matrix.

2. g-Tridiagonally majorization. In this section, we mention some properties of $\prec_{gt}$ on $\mathbb{R}^n$ and also we present some preliminaries to prove Theorem 1.3.

**Proposition 2.1.** [Lemma 3.6] Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a linear operator such that $T(x) = \alpha Dx + \beta Jx$ for some $\alpha, \beta \in \mathbb{R}$ and invertible matrix $D \in \Omega_n$. Then $T$ is invertible if and only if $\alpha(\alpha + n\beta) \neq 0$.

**Lemma 2.2.** Let $x, y \in \mathbb{R}^n$. If every two consecutive components of $y$ are distinct, then $x \prec_{gt} y$ if and only if $tr(x) = tr(y)$.

**Proof.** If $x \prec_{gt} y$, it is easy to see that $tr(x) = tr(y)$. Conversely, suppose that every two consecutive components of $y$ are distinct. For every $j (1 \leq j \leq n - 1)$, put $\mu_j = \sum_{i=j}^{n} \frac{x_i - y_i}{y_{j+1} - y_j}$. With a direct calculation it is easy to see that $x = A_\mu y$, where $\mu = (\mu_1, \ldots, \mu_{n-1})^t$, and hence $x \prec_{gt} y$. □

The following theorem gives an equivalent condition for $\prec_{gt}$ on $\mathbb{R}^n$.

**Theorem 2.3.** Let $x$ and $y$ be two distinct vectors in $\mathbb{R}^n$. Assume that $i_1 < i_2 < \cdots < i_k$ and $\{i_1, i_2, \ldots, i_k\} = \{j : 1 \leq j \leq n - 1, y_j = y_{j+1}\}$. Then $x \prec_{gt} y$ if and only if $\sum_{j=i_{l-1}+1}^{i_l} x_j = \sum_{j=i_{l-1}+1}^{i_l} y_j$ for every $l (1 \leq l \leq k + 1)$, where $i_{k+1} = n$ and $i_0 = 0$.

**Proof.** If $x \prec_{gt} y$, then there exists $A_\mu \in \Omega_n^k$ such that $x = A_\mu y$. Consequently, for every $j (1 \leq j \leq n)$, $x_j = \mu_j - 1 (y_j - y_{j+1}) + \mu_j (y_{j+1} - y_j) + y_j$, where $y_0 = y_n = \mu_0 - 1 = \mu_n = 0$. For every $j \in \{i_1, i_2, \ldots, i_k\}$, $y_j = y_{j+1}$ then $\sum_{j=i_{l-1}+1}^{i_l} x_j = \sum_{j=i_{l-1}+1}^{i_l} y_j$, for every $l (1 \leq l \leq k + 1)$. Conversely, put $y^1 = (y_1, y_2, \ldots, y_1)^t$ and $x^1 = (x_1, x_2, \ldots, x_1)^t$. Then every two consecutive components of $y^1$ are distinct. Since $\sum_{j=1}^{i_1} x_j = \sum_{j=1}^{i_1} y_j$, $x^1 \prec_{gt} y^1$ by Lemma 2.2. Then there exists $A_1 \in \Omega_{i_1}^1$ such that $x^1 = A_1 y^1$. Now, for every $l (2 \leq l \leq k + 1)$ put $x^l = (x_{i_{l-1}+1}, x_{i_{l-1}+2}, \ldots, x_n)^t$ and $y^l = (y_{i_{l-1}+1}, y_{i_{l-1}+2}, \ldots, y_n)^t$. Since $\sum_{j=i_{l-1}+1}^{i_l} x_j = \sum_{j=i_{l-1}+1}^{i_l} y_j$, $x^l \prec_{gt} y^l$ by Lemma 2.2. Then there exists $A_l \in \Omega_{i_{l-1}+1}^1$ such that $x^l = A_l y^l$. Put $A := \oplus_{l=1}^{k} A_l$, it follows that $A \in \Omega_n^k$ and $x = Ay$, therefore $x \prec_{gt} y$. □

**Lemma 2.4.** Let $y \in \mathbb{R}^n$. Assume that $i_1 < i_2 < \cdots < i_k$ and $\{i_1, i_2, \ldots, i_k\} = \{j : 1 \leq j \leq n - 1, y_j = y_{j+1}\}$. Then $H_y := \{x \in \mathbb{R}^n : x \prec_{gt} y\}$ is an affine set with dimension $n - (k + 1)$.

**Proof.** By Theorem 2.3 it follows that: $H_y = \{x \in \mathbb{R}^n : \sum_{j=i_{l-1}+1}^{i_l} x_j = \sum_{j=i_{l-1}+1}^{i_l} y_j, \forall l \in \{1, \ldots, k + 1\}\}$,
where \( i_{k+1} = n \) and \( i_0 = 0 \). If \( \lambda \in \mathbb{R}, x, z \in H_y \), it is clear that \( \lambda x + (1 - \lambda)z \in H_y \), so \( H_y \) is an affine set. Since every \( x \in H_y \) have to satisfy \( k + 1 \) equations, it is easy to see that \( \dim H_y = n - (k + 1) \).

**Corollary 2.5.** Let \( y \in \mathbb{R}^n \). Then \( \dim H_y = 0 \) if and only \( y \in \text{Span}\{e\} \).

**Proposition 2.6.** Let \( T : \mathbb{R}^n \to \mathbb{R}^n \) be a linear operator. If \( T \) strongly preserves \( \prec_{gt} \), then the following statements are true:

(i) \( T \) is invertible.

(ii) \( \text{tr}(Te_i) = \text{tr}(Te_j) \), for every \( i, j \in \{1, \ldots, n\} \).

(iii) \( Te \in \text{Span}\{e\} \).

(iv) \( [T] \) is a multiple of a \( g \)-doubly stochastic matrix.

**Proof.** (i) Suppose \( T(x) = 0 \). Since \( T \) is linear, \( T(0) = 0 = T(x) \). Then it is obvious that \( T(x) = T(0) \). Therefore, \( x \prec_{gt} 0 \) because \( T \) strongly preserves \( gt \)-majorization. Then, there exists an \( R \in \mathcal{O}_n^+ \) such that \( x = R0 \). So, \( x = 0 \), and hence \( T \) is invertible. (ii) Using Theorem 2.3, \( e_j \prec_{gt} e_{j+1} \) for every \( j, 1 \leq j \leq n-1 \). Then \( Te_j \prec_{gt} Te_{j+1} \) for every \( j, 1 \leq j \leq n-1 \) and hence \( \text{tr}(Te_i) = \text{tr}(Te_j) \), for every \( i, j \in \{1, \ldots, n\} \). (iii) Since \( T \) is invertible, there exists \( a \in \mathbb{R}^n \) such that \( Ta = e \). By Corollary 2.3, \( \dim(H_a) = \dim(H_{Te_a}) = 0 \) and hence \( Te \in \text{Span}\{e\} \). (v) It is clear that by (ii) and (iii), \( [T] \) is a multiple of a \( g \)-doubly stochastic matrix.

Now, we prove the main theorem of this paper. Every linear operator \( T : \mathbb{R}^n \to \mathbb{R}^n \) strongly preserves \( \prec_{gt} \) if and only if \( \alpha T : \mathbb{R}^n \to \mathbb{R}^n \) strongly preserves \( \prec_{gt} \) for all \( \alpha \in \mathbb{R} \setminus \{0\} \). So in the following proof, we assume without loss of generality that \( \text{tr}(Te_i) = \cdots = \text{tr}(Te_n) = 1 \).

**Proof of Theorem 1.2.** Let \( A = [a_{ij}] = [T] \). If \( n \leq 2 \), then the concepts \( \prec_{gt} \) and \( \prec_{gs} \) are the same on \( \mathbb{R}^n \), and hence the proof is complete by Proposition 1.2.

We assume without loss of generality that \( n \geq 3 \). The fact that the conditions (i) and (ii) are sufficient for \( T \) to be a strong linear preserver of \( \prec_{gt} \) is easy to prove. So we prove the necessity of the conditions. Suppose that \( T \) strongly preserves \( \prec_{gt} \), then \( T \) is invertible, by Proposition 2.6. Put \( \Phi := \{x \in \mathbb{R}^n : x \prec_{gt} e_1\} \). From Theorem 2.3, we have \( \Phi = \{x \in \mathbb{R}^n : x_1 + x_2 = 1, x_3 = \cdots = x_n = 0\} \) and \( \dim \Phi = 1 \). Since \( T \) is a strong linear preserver of \( \prec_{gt} \), \( T(\Phi) = \{Tx \in \mathbb{R}^n : x \prec_{gt} e_1\} \). By invertibility of \( T \), we have \( \dim \Phi = \dim T(\Phi) = 1 \).

Since \( Te_1 \in T(\Phi) \) and \( \dim T(\Phi) = 1 \), \( Te_1 \) has \( (n - 1) \) equal consecutive components and hence \( Te_1 = (a_1, b, b, \ldots, b)^t \) or \( Te_1 = (b, b, a_{n,1})^t \) for some \( b \in \mathbb{R} \). Put \( \Psi := \{x \in \mathbb{R}^n : x \prec_{gt} e_n\} \). From Theorem 2.3, we have \( \Psi = \{x \in \mathbb{R}^n : x_{n-1} + x_n = 1, x_{n-2} = \cdots = x_1 = 0\} \) and \( \dim \Psi = 1 \). With a similar argument as above we may establish \( Te_n = (a_1, c, \ldots, c)^t \) or \( Te_n = (c, \ldots, c, a_{n,n})^t \), for some \( c \in \mathbb{R} \). Now, we consider all possible forms of \( Te_1 \) and \( Te_n \).
Let $T(e_1) = (b, \ldots, b, a_{n,1})^t$. We have

\[ e_2 \prec_{gt} e_1 \Rightarrow T e_2 \prec_{gt} T e_1 \]
\[ \Rightarrow (a_{1,2}, a_{2,2}, \ldots, a_{n,2})^t \prec_{gt} (b, \ldots, b, a_{n,1})^t \]
\[ \Rightarrow a_{n-2,2} = \cdots = a_{2,2} = a_{1,2} = b \]
\[ \Rightarrow T e_2 = (b, \ldots, b, a_{n-1,2}, a_{n,2})^t. \]

For every $j$ $(1 \leq j \leq n - 1)$, $e_{j+1} \prec_{gt} e_j$. So with a similar argument as above

\[ T e_j = (b, \ldots, b, a_{n-j+1,j}, \ldots, a_{n,j})^t. \]

It follows that

\[ A = \begin{pmatrix}
  b & \ldots & b & a_{1,n} \\
  \vdots & \ddots & \vdots & \vdots \\
  b & \ldots & b & a_{n,1} \\
  a_{n,1} & \ast & \cdots & \ast
\end{pmatrix}, \quad (1) \]

Let $T e_1 = (a_{1,1}, b, \ldots, b)^t$. Similarly one may show that

\[ A = \begin{pmatrix}
  a_{1,1} & \ast \\
  b & \ddots \\
  \vdots & \ddots & \ddots \\
  b & \ldots & b & a_{n,n}
\end{pmatrix}. \quad (2) \]

Let $T e_n = (c, \ldots, c, a_{n,n})^t$. We have

\[ e_{n-1} \prec_{gt} e_n \Rightarrow T e_{n-1} \prec_{gt} T e_n \]
\[ \Rightarrow (a_{1,n-1}, a_{2,n-1}, \ldots, a_{n,n-1})^t \prec_{gt} (c, \ldots, c, a_{n,n})^t \]
\[ \Rightarrow a_{1,n-1} = a_{2,n-1} = \cdots = a_{n-2,n-1} = c \]
\[ \Rightarrow T e_{n-1} = (c, \ldots, c, a_{n-1,n-1}, a_{n,n-1})^t. \]

For every $i$ $(2 \leq i \leq n - 3)$, $e_{n-1} \prec_{gt} e_{n-i+1}$, so with an argument same as the above

\[ T e_i = (c, \ldots, c, a_{i,i}, \ldots, a_{n,i})^t. \]

It follows that

\[ A = \begin{pmatrix}
  a_{1,1} & c & \cdots & c \\
  \ast & \ddots & \vdots & \vdots \\
  \ast & \ddots & c & \ast \\
  a_{n,1} & c & \cdots & c
\end{pmatrix}, \quad (1)^* \]

Let $T e_n = (a_{1,n}, c, \ldots, c)^t$. Similarly one may show that

\[ A = \begin{pmatrix}
  \ast & a_{1,n} \\
  \ast & \ddots & \ddots \\
  a_{n,1} & c & \cdots & c
\end{pmatrix}. \quad (2)^* \]
Since \( n \geq 3 \), and \( T \) is invertible the only possible cases are: (1), (2)* and (2), (1)*. In view of Theorem 2.3, \( A \) is a multiple of a g-doubly stochastic matrix. Therefore \( A \) has one of the following forms:

\[
\begin{pmatrix}
  a & b & b & \cdots & b \\
  b & a & b & \cdots & b \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  b & b & b & \cdots & a \\
\end{pmatrix}
\quad \text{or} \quad
\begin{pmatrix}
  b & b & \cdots & b & a \\
  b & b & \cdots & a & b \\
  \vdots & \vdots & \ddots & \vdots & \vdots \\
  a & b & \cdots & b & b \\
\end{pmatrix}
\]

Using Proposition 2.1 to obtain \((a - b)(a + (n - 1)b) \neq 0\) in each case, these as done.

**Corollary 2.7.** Let \( P \in M_n \) be a permutation matrix. Then the linear operator \( T : \mathbb{R}^n \to \mathbb{R}^n \), defined by \( T(x) = Px \), strongly preserves \( \triangleleft_{gt} \) if and only if \( P \) is identity or backward identity matrix.

Now, we consider the \( gt \)-majorization on \( \mathbb{R}_n \). Let \( x, y \in \mathbb{R}_n \). We say that \( x \) is g-tridiagonally majorized by \( y \) (written as \( x \triangleleft_{gt} y \)) if there exists a tridiagonal g-doubly stochastic matrix \( D \) such that \( x = yD \). Since the transpose of every tridiagonal g-doubly stochastic matrix is tridiagonal g-doubly stochastic too, we have \( x \triangleleft_{gt} y \) if and only if \( x^t \triangleleft_{gt} y^t \) for every \( x, y \in \mathbb{R}_n \).

**Corollary 2.8.** Let \( T : \mathbb{R}_n \to \mathbb{R}_n \) be a linear operator. Then \( T \) strongly preserves \( \triangleleft_{gt} \) if and only if there exist \( \alpha, \beta \in \mathbb{R} \) such that \( \alpha(\alpha + n\beta) \neq 0 \) and \( Tx = \alpha xP + \beta xJ \) for all \( x \in \mathbb{R}_n \), where \( P \) is the identity or the backward identity matrix.

**Proof.** Define \( S : \mathbb{R}^n \to \mathbb{R}^n \) by \( Sx = [T(x^t)]^t \) for all \( x \in \mathbb{R}^n \). It is easy to see that \( S \) strongly preserves \( \triangleleft_{gt} \) and hence Theorem 1.3 is applicable to \( S \).

**Acknowledgments.** The authors would like to thank an anonymous referee for helpful comments and remarks.

**REFERENCES**
