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# EIGENVALUE REGIONS AND REALIZING MONOTONE STOCHASTIC MATRICES*

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**Abstract.** Eigenvalues of stochastic matrices have been studied from two complementary perspectives. The individual eigenvalues are characterized through the well-established Karpelevich regions. The spectrum as a whole has also been analyzed, yielding powerful results such as the Johnson–Loewy–London inequalities. Current research now turns toward particular subsets of stochastic matrices, among others the doubly stochastic matrices. This paper studies spectral properties of monotone stochastic matrices, which are characterized by the fact that each row stochastically dominates the preceding one and arise in contexts such as intergenerational mobility, equal-input models, and credit-rating systems. This paper analyzes the dominance matrix associated with a monotone matrix, which is a nonnegative matrix that preserves the nontrivial eigenvalues. Properties are established, and the conditions are given under which a nonnegative matrix can be regarded as a dominance matrix. In analogy with the stochastic matrices, this study examines for the monotone stochastic matrices both the individual eigenvalues and the spectrum as a whole. Individually, the eigenvalue region for all $n \times n$ monotone matrices with $1 \leq n \leq 3$ is completely determined, and realizing matrices are provided. Collectively, the set of possible pairs of nontrivial eigenvalues arising from $3 \times 3$ monotone matrices is characterized, accompanied by realizing matrices. In both perspectives, the resulting regions are substantially smaller than those for general stochastic matrices. Finally, this paper proves a reduction theorem stating that, for $n \geq 3$, the eigenvalue region of $n \times n$ monotone matrices is contained within that of $(n-1) \times (n-1)$ stochastic matrices.

**Key words.** Nonnegative matrices, Stochastic matrices, Monotone stochastic matrices, Eigenvalues, Spectrum, Eigenvalue regions.

**AMS subject classifications.** 15B51, 15A18, 15A29, 15A42.

## 1. Introduction.

A stochastic matrix is a nonnegative matrix with each row sum equal to one. These matrices find their way into numerous applications in different domains, most famously in Markov chain theory [2]. In this field, stochastic matrices describe the transition probabilities of a given process over time. The spectrum of a stochastic matrix provides insights for the aforementioned Markov chains, among others the steady-state behavior, the long-term behavior, and the convergence properties [5, 15, 17, 18]. Characterizing the eigenvalue regions of stochastic matrices is a challenging problem with a long and extensive research history. The eigenvalues of such matrices can be examined from distinct perspectives.

On the one hand, the set of the individual eigenvalues can be considered. It was as early as 1938 that Kolmogorov first came up with this problem. Specifically, he examined the set $\mathcal{S}_n$ of $n \times n$ stochastic matrices and asked whether an exact description could be found for the region $\Theta_n = \{\lambda \in \mathbb{C} : \lambda \text{ is an eigenvalue of an } n \times n \text{ stochastic matrix}\}$, where $n$ is a fixed natural number. It was only in 1946 that Dmitriev and Dynkin [6] could give part of the solution to this problem, namely a complete description of the eigenvalue region for $\mathcal{S}_n$ up to $n = 5$. In 1951, Karpelevich [11] was finally able, by generalizing some ideas of Dmitriev and Dynkin, to give a complete description of the region consisting of all eigenvalues for $\mathcal{S}_n$ for any $n$.

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On the other hand, eigenvalues can be considered collectively. In 1978, Loewy and London [13] investigated when a specified set of complex numbers can be realized as the spectrum of a nonnegative matrix. They introduced the well-known Johnson–Loewy–London (JLL) inequalities as necessary conditions on the spectrum, which are formulated in terms of traces of matrix powers. For the specific case of $3 \times 3$ matrices, they presented conditions under which such spectra are realizable. The nonnegative inverse eigenvalue problem remains an active area of research, with recent developments including those in, e.g., [10].

Within this framework, doubly stochastic matrices, a subset of stochastic matrices with both row and column sums equal to one, are also a common topic of discussion. Research on the eigenvalue regions of these matrices is ongoing, with several results established, though much about the eigenvalues remains unknown. Conjectures and partial proofs appear in several publications, e.g., [14]. Besides (doubly) stochastic matrices, the eigenvalue regions of other matrix types, such as Metzler matrices [7] and Leslie matrices [12], are also studied.

This research focuses specifically on a subset of the $n \times n$ stochastic matrices $\mathcal{S}_n$, namely the set $\mathcal{M}_n$ of $n \times n$ monotone matrices. This terminology is not always used consistently in previous work. We adopt the definition provided by Daley [4]: A monotone matrix $M = (m_{ij})$ is a stochastic matrix in which each row is stochastically dominated by the next row, i.e., $\sum_{j=r}^{n} m_{lj} \geqslant \sum_{j=r}^{n} m_{kj} \forall l > k,\ \ \forall r \in \{1, \dots, n\}$. Monotone matrices appear in various contexts, such as intergenerational occupational mobility [3], equal-input modeling [1], and credit ratings based systems [9]. While some research has been conducted on these matrices and their eigenvalues, such as in [8], the scope remains relatively narrow.

On the one hand, by examining eigenvalues individually, this work provides new insights into the eigenvalue region

$$\Xi_n := \left\{ \lambda \in \mathbb{C} \;\middle|\; \exists\, M \in \mathcal{M}_n \text{ such that } \lambda \in \sigma(M) \right\},$$

defined for arbitrary $n$, where $\sigma(M)$ denotes the spectrum of $M$. Similar to the (doubly) stochastic matrices, it also holds for the monotone matrices that $\Xi_n \subset \Xi_{n+1}$ for every $n \geq 1$. On the other hand, by examining the eigenvalues collectively, and knowing that the spectrum of a $3 \times 3$ monotone matrix is real [8], the set

$$\xi_3 := \left\{ (\lambda_2, \lambda_3) \in \mathbb{R}^2 \;\middle|\; \exists\, M \in \mathcal{M}_3 \text{ such that } \sigma(M) = \{1, \lambda_2, \lambda_3\} \text{ with } \lambda_2 \geq \lambda_3 \right\},$$

is fully characterized for $n = 3$.

This paper presents, in Section 2, several properties of the dominance matrix $D(M)$ associated with a monotone matrix $M$, together with an explicit construction. Section 3 provides a full determination of $\Xi_1$, $\Xi_2$, and $\Xi_3$, along with corresponding realizing matrices. In Section 4, the set $\xi_3$ is fully characterized and accompanied by realizing matrices. Section 5 proves a reduction theorem, which gives insight into the eigenvalue region for $\mathcal{M}_n$ for $n \geqslant 4$. Finally, in Section 6, conclusions and further research avenues are presented.

**2. Dominance matrix.** To analyze the eigenvalues of an $n \times n$ monotone matrix $M$, the dominance matrix $D(M)$ plays a central role. Its key advantage is that it reduces the problem to a, one order lower, nonnegative $(n-1) \times (n-1)$ matrix while retaining exactly the same nontrivial eigenvalues, i.e., $\sigma(D(M)) = \sigma(M) \setminus \{1\}$. The dominance matrix $D(M)$ is given by

$$(D(M))_{kl} = \sum_{j=1}^{l} m_{kj} - \sum_{j=1}^{l} m_{k+1,j}, \forall k, l \in \{1, \dots, n-1\}.$$

By the definition of a monotone matrix, it follows directly that $D(M) \geq 0$. More details can be found in [3].

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It is natural to ask whether the dominance matrix can be an arbitrary $(n-1)\times(n-1)$ nonnegative matrix. However, the additional constraints inherent to monotone matrices imply that this is not the case. Lemma 2.1 proves necessary conditions for the case $n=3$.

Given this, let $M=(m_{ij})$ be a $3\times 3$ monotone matrix. Its associated dominance matrix is

$$D(M)=\begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where

$$a = m_{11} - m_{21}, \tag{2.1}$$
$$b = (m_{11}+m_{12}) - (m_{21}+m_{22}), \tag{2.2}$$
$$c = m_{21} - m_{31}, \tag{2.3}$$
$$d = (m_{21}+m_{22}) - (m_{31}+m_{32}). \tag{2.4}$$

The nontrivial eigenvalues of $M$ are precisely the eigenvalues of $D(M)$, given explicitly by

$$\lambda_2 = \frac{a+d+\sqrt{(a-d)^2+4bc}}{2} \quad \text{and} \quad \lambda_3 = \frac{a+d-\sqrt{(a-d)^2+4bc}}{2}.$$

Further, the following properties can be deduced.

Lemma 2.1. *Let $D(M)$ be the dominance matrix defined above. Then,*

1. *Column- and antidiagonal sums are bounded by 1:*

$$a+c\le 1, \qquad b+c\le 1, \qquad b+d\le 1.$$

2. *Column- and antidiagonal products are bounded by $\frac{1}{4}$:*

$$ac\le \tfrac{1}{4}, \qquad bc\le \tfrac{1}{4}, \qquad bd\le \tfrac{1}{4}.$$

3. *Nonnegative trace:*

$$\mathrm{tr}[D(M)] = a+d\ge 0.$$

4. *Determinant bound:*

$$\det[D(M)] = ad-bc \ge -\tfrac{1}{4}.$$

*Proof.* 1. Using (2.2) and (2.3), we obtain $b+c=(m_{11}+m_{12})-(m_{22}+m_{31})\le m_{11}+m_{12}\le 1$, which establishes $b+c\le 1$. The arguments for $a+c\le 1$ and $b+d\le 1$ proceed in the same manner.
2. According to (2.2), an upper bound for $b$ can be obtained by choosing $m_{22}=0$ and $m_{11}+m_{12}=1$. In turn, according to (2.3), $c$ can be bounded upwards by setting $m_{31}=0$. In this way, we get $b\cdot c\le (1-m_{21})\cdot m_{21}$. The maximum of this last expression is reached for $m_{21}=1/2$ and is equal to $1/4$. So we obtain $bc\le\frac{1}{4}$. Similar arguments provide $ac\le\frac{1}{4}$ and $bd\le\frac{1}{4}$.
3. This follows from the fact that $D(M)\ge 0$ (see [8]).
4. From $bc\le\frac{1}{4}$ (proved in 2) follows that $\det[D(M)]=ad-bc\ge -bc\ge -\frac{1}{4}$. ∎

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The properties of Lemma 2.1 demonstrate that not every random nonnegative $2 \times 2$ matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is the dominance matrix of a $3 \times 3$ monotone matrix. However, the inequalities in Lemma 2.2 form a necessary and sufficient condition:

Lemma 2.2. *A nonnegative matrix* $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *arises as the dominance matrix* $D(M)$ *of a* $3 \times 3$ *monotone matrix* $M$ *if and only if there exist* $m_{11}, m_{33} \in [0,1]$ *such that*

$$\begin{cases} a + c \le m_{11}, \\ b + d \le m_{33}, \\ m_{11} + m_{33} \le 1 + b + d, \\ m_{11} + m_{33} \le 1 + a + d, \\ m_{11} + m_{33} \le 1 + a + c. \end{cases}$$

*Proof.* $\Longrightarrow$ Assume that $M = (m_{ij})$ is a $3 \times 3$ monotone matrix with $D(M) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Using (2.1)–(2.4), together with the stochasticity of $M$, i.e., $m_{11} + m_{12} = 1 - m_{13}$, $m_{21} + m_{22} = 1 - m_{23}$, and $m_{31} + m_{32} = 1 - m_{33}$, we obtain

$$m_{21} = m_{11} - a, \tag{2.5}$$
$$m_{31} = m_{11} - a - c, \tag{2.6}$$
$$m_{13} = m_{33} - b - d, \tag{2.7}$$
$$m_{23} = m_{33} - d. \tag{2.8}$$

Since $M \ge 0$, relations (2.6) and (2.7) immediately yield the first two inequalities in the system. Moreover, from $m_{12} = 1 - m_{11} - m_{13}$ and (2.7), we deduce

$$m_{11} + m_{33} \le 1 + b + d,$$

which provides one of the remaining inequalities. The remaining two follow by analogous considerations.

$\Longleftarrow$ Conversely, suppose that the inequalities in the statement of the lemma are satisfied. Then, one may explicitly construct the following $3 \times 3$ monotone matrix:

$$M = \begin{pmatrix} m_{11} & 1 - (m_{11} + m_{33} - b - d) & m_{33} - b - d \\ m_{11} - a & 1 - (m_{11} - a + m_{33} - d) & m_{33} - d \\ m_{11} - a - c & 1 - (m_{11} - a - c + m_{33}) & m_{33} \end{pmatrix},$$

with dominance matrix $D(M) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. ∎

Observe that each monotone matrix determines a unique dominance matrix, but different monotone matrices can have the same dominance matrix. For example, for the monotone matrices $M_1 = \begin{pmatrix} 0,3 & 0,7 & 0 \\ 0,2 & 0,7 & 0,1 \\ 0,1 & 0,7 & 0,2 \end{pmatrix}$

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and $M_2 = \begin{pmatrix} 0,4 & 0,6 & 0 \\ 0,3 & 0,6 & 0,1 \\ 0,2 & 0,6 & 0,2 \end{pmatrix}$ holds that $D(M) = \begin{pmatrix} 0,1 & 0,1 \\ 0,1 & 0,1 \end{pmatrix}$. This is not an issue for our purposes, since we are concerned only with the existence of a monotone matrix possessing a prescribed eigenvalue, rather than with the multiplicity of such matrices.

## 3. Eigenvalue regions $\Xi_1, \Xi_2$, and $\Xi_3$.

In analogy with the Karpelevich regions $\Theta_n$ [11], we consider the eigenvalues individually in this section and determine the eigenvalue regions $\Xi_n$ for $1 \leqslant n \leqslant 3$. Because monotonicity imposes further restrictions on the matrix entries, we expect the resulting eigenvalue regions to be narrower than those that arise for the stochastic matrices.

THEOREM 3.1. $\Xi_1 = \{1\}$.

*Proof.* This statement is trivial since a $1 \times 1$ monotone matrix can only be the matrix $(1)$ with eigenvalue $\lambda = 1$. ∎

In this trivial case, it holds that $\Xi_1 = \Theta_1$.

THEOREM 3.2. $\Xi_2 = [0, 1]$.

*Proof.* Let $M$ be a $2 \times 2$ monotone matrix with $\sigma(M) = \{\lambda_1 = 1, \lambda_2\}$, then $\lambda_2 \geqslant 0$ because $\mathrm{tr}(M) \geqslant 1$ (see [8]). Hence, $\Xi_2 \subseteq [0, 1]$. Furthermore, for $\lambda_2 \in [0, 1]$, the monotone matrix $\begin{pmatrix} \lambda_2 & 1 - \lambda_2 \\ 0 & 1 \end{pmatrix}$ has $\lambda_2$ as eigenvalue. This concludes the proof. ∎

We get, in this case, a first reduction, namely that $\Xi_2 = [0, 1] \subset [-1, 1] = \Theta_2$.

THEOREM 3.3. $\Xi_3 = [-1/2, 1]$.

*Proof.* **Step 1: proof of $\Xi_3 \subseteq [-1/2, 1]$.**

Let $M$ be a $3 \times 3$ monotone matrix and $D(M)$ its dominance matrix. In order to prove $\Xi_3 \subseteq [-1/2, 1]$, we want to know how small an eigenvalue can be. Therefore, we are going to minimize the smallest eigenvalue (see Section 2):

$$\begin{aligned} \lambda_3 &= \frac{a + d - \sqrt{(a-d)^2 + 4bc}}{2} \\ &\geqslant \frac{a + d - \sqrt{(a+d)^2 + 4bc}}{2}, && (3.1) \\ &\geqslant \frac{a + d - \sqrt{(a + d + 2\sqrt{bc})^2}}{2}, && (3.2) \\ &= -\sqrt{bc}. && (3.3) \end{aligned}$$

Above, inequalities (3.1) and (3.2) follow from the fact that $a$ and $d$ are nonnegative. It follows from equation (3.3) that, in order to investigate a lower bound for the eigenvalues, it suffices to find an upper bound of $bc$. From Lemma 2.1(2), this can be $1/4$. So we obtain that $\lambda_3 \geqslant -1/2$, and as we know that an eigenvalue of a stochastic matrix is at most 1, this concludes the first step.

**Step 2:** $\Xi_3 = [-1/2, 1]$

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So we already know that the region $\Xi_3$ is located in the interval $[-1/2, 1]$. However, the last interval turns out to be exactly the monotone eigenvalue region $\Xi_3$. This can be seen by the following 2 constructions of realizing matrices.

**Realizing matrices type 1—covering** $[0, 1]$

For each $\alpha \in [0, 1]$, we construct the following monotone matrix

$$\begin{pmatrix} 0 & \alpha & 1-\alpha \\ 0 & \alpha & 1-\alpha \\ 0 & 0 & 1 \end{pmatrix}.$$

Because the spectrum of a triangular matrix is its diagonal, it follows:

$$\lambda_1 = 1, \lambda_2 = \alpha \text{ and } \lambda_3 = 0.$$

For $\alpha \in [0, 1]$, the eigenvalue $\lambda_2 = \alpha$ traverses the line segment $[0, 1]$.

**Realizing matrices type 2—covering** $[-1/2, 0]$

For each $\alpha \in [0, 1/2]$, we construct the following monotone matrix

$$\begin{pmatrix} 1/2-\sqrt{1/4-\alpha^2} & 1/2+\sqrt{1/4-\alpha^2} & 0 \\ 1/2-\sqrt{1/4-\alpha^2} & 0 & 1/2+\sqrt{1/4-\alpha^2} \\ 0 & 1/2-\sqrt{1/4-\alpha^2} & 1/2+\sqrt{1/4-\alpha^2} \end{pmatrix},$$

with characteristic equation

$$-\lambda^3 + \lambda^2 + \alpha^2\lambda - \alpha^2 = -(\lambda-1)(\lambda-\alpha)(\lambda+\alpha),$$

from which the eigenvalues below follow:

$$\lambda_1 = 1, \lambda_2 = \alpha \text{ and } \lambda_3 = -\alpha.$$

For $\alpha \in [0, \frac{1}{2}]$, the eigenvalues $\lambda_2 = \alpha$ and $\lambda_3 = -\alpha$ traverse, respectively, line segments $[0, 1/2]$ and $[-1/2, 0]$. ∎

The eigenvalues of the realizing matrices of type 1 and 2 are presented in Figure 1 where the union results in the eigenvalue region $\Xi_3 = [-1/2, 1]$. Figure 1 also illustrates the eigenvalue region $\Theta_3$ for $3 \times 3$ stochastic matrices, namely $\Theta_3 = [-1, -\frac{1}{2}) \cup \text{conv}\{1, e^{\frac{i2\pi}{3}}, e^{\frac{i4\pi}{3}}\}$, where $\text{conv}\{\cdot\}$ denotes the convex hull. We see that $\Xi_3$ represents a significant contraction of $\Theta_3$. For instance, in contrast to $\Theta_3$, the region $\Xi_3$ does not consist of complex eigenvalues neither includes $[-1, -\frac{1}{2})$. It is worth noting that this is a remarkable feature, as the eigenvalue region for $3 \times 3$ doubly stochastic matrices coincides with that of stochastic matrices and therefore does not exhibit such a reduction.

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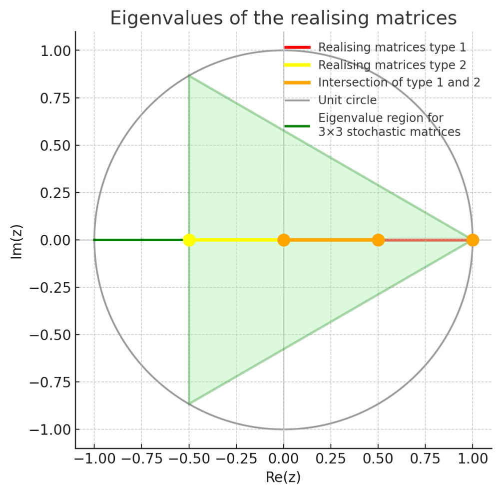


FIGURE 1. *Eigenvalues of the realizing matrices.*

**4. Possible eigenvalue pairs $(\lambda_2, \lambda_3)$ for $\mathcal{M}_3$.** In analogy with the JLL-inequalities [13], we investigate in this section the set $\xi_3$ of possible eigenvalue pairs $(\lambda_2, \lambda_3)$ arising from a $3 \times 3$ monotone matrix $M$. Since $\lambda_1 = 1$ holds universally, it is excluded from consideration. Although the previous section established that all eigenvalues of a $3 \times 3$ monotone matrix must lie in the interval $[-1/2, 1]$, this bound does not identify which pairs can occur simultaneously. For instance, $\lambda_2 = 1$ and $\lambda_3 = -\frac{1}{2}$ both belong to $[-\frac{1}{2}, 1]$ but their combination is not possible, since $\lambda_2\lambda_3 = -\frac{1}{2}$, which contradicts Lemma 2.1(4). Our aim is therefore to characterize the region $\xi_3$ in the $(\lambda_2, \lambda_3)$-plane. By similar arguments to [21], we get:

LEMMA 4.1. *The region $\xi_3$ is star-convex with respect to the origin.*

Thus, determining the boundary of $\xi_3$ is sufficient to obtain the complete region.

Let $D(M) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be the dominance matrix as introduced in Section 2. For particular subsets of $\lambda_2$ and $\lambda_3$, additional information about $\mathrm{tr}[D(M)]$ can be derived.

LEMMA 4.2. *If $\lambda_2 \geq \frac{1+\sqrt{5}}{4}$ and $\lambda_3 \leq 0$, then $\frac{1}{2} \leq \mathrm{tr}[D(M)] = a + d \leq 1$.*

*Proof.* Because $\lambda_2 \leq 1$ and $\lambda_3 \leq 0$, we immediately obtain $\mathrm{tr}[D(M)] = \lambda_2 + \lambda_3 \leq 1$. Moreover, for $\lambda_2 \geq \frac{1+\sqrt{5}}{4}$, the smallest admissible value of $\lambda_3$ is $-\frac{1}{1+\sqrt{5}}$, since Lemma 2.1(4) implies $\det[D(M)] = \lambda_2\lambda_3 \geq -\frac{1}{4}$. Evaluating the trace at these extremal values yields $\frac{1+\sqrt{5}}{4} - \frac{1}{1+\sqrt{5}} = \frac{1}{2}$, and therefore, $\mathrm{tr}[D(M)] \geq \frac{1}{2}$ throughout this region. ∎

In fact, a more explicit connection can be derived between $\mathrm{tr}[D(M)]$ and $\det[D(M)]$.

LEMMA 4.3. *If $\lambda_2 \geq \frac{1+\sqrt{5}}{4}$ and $\lambda_3 \leq 0$, then*

$$(\mathrm{tr}[D(M)])^2 - \mathrm{tr}[D(M)] - \det[D(M)] \leq 0.$$

*Proof.* Substituting $\mathrm{tr}[D(M)] = a + d$ and $\det[D(M)] = ad - bc$ into $(\mathrm{tr}[D(M)])^2 - \mathrm{tr}[D(M)] - \det[D(M)] \leq 0$ and rearranging terms results in the equivalent inequality

$$ad + bc - a(1 - a) - d(1 - d) \leq 0. \tag{4.1}$$

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Using $d \le 1-a$ from Lemma 4.2 and $b \le 1-d$, as established in Lemma 2.1(1), we conclude that inequality (4.1) holds whenever $c \le d$. Consequently, the remainder of the proof may focus on the case $c > d$.

By the bound $b \le 1-c$ (from Lemma 2.1(1)), we obtain

$$ad + bc - a(1-a) - d(1-d) \le G(c),$$

where $G(c) := a^2 + d^2 + ad - a - d + c - c^2$ is considered for fixed $a$ and $d$. Since we are in the case $c > d$ and Lemma 2.1(1) ensures $c \le 1-a$, the admissible range is $c \in (d,\, 1-a]$. Our goal is therefore to show that $G(c) \le 0$ for all $c \in (d,\, 1-a]$.

Analyzing $G(c)$ reduces to studying the function $c \mapsto c - c^2$, which is concave with a unique maximum at $c = \frac{1}{2}$, increasing on $[0,\, \frac{1}{2})$ and decreasing on $(\frac{1}{2},\, 1]$. We distinguish three subcases.

In case of $1 - a \le \frac{1}{2}$, it holds that

$$G(c) \le G(1-a) = d(a+d-1) \le 0,$$

since $d \ge 0$ and Lemma 4.2 ensures $a + d \le 1$.

In case of $1 - a > \frac{1}{2}$ and $d \ge \frac{1}{2}$, we obtain

$$G(c) \le G(d) = a(a+d-1) \le 0.$$

In case of $d < \frac{1}{2} < 1 - a$, it can be observed that

$$G(c) \le G\left(\frac{1}{2}\right) = a^2 - a + d^2 - d + ad + \frac{1}{4} := P(a,d).$$

Since $1-a > \frac{1}{2}$, it follows that $a < \frac{1}{2}$. Combined with $d < \frac{1}{2}$ and Lemma 4.2, this implies that the domain of $P(a,d)$ is the triangular region bounded by $a \leqslant \frac{1}{2}$, $d \leqslant \frac{1}{2}$, and $a + d \ge \frac{1}{2}$. On this compact set, the function $P(a,d)$ attains its maximum, which is 0, on the vertices of the triangle. It follows that $G(c) \leqslant P(a,d) \leqslant 0$.

From the three subcases above, we conclude that $G(c) \le 0$ for all $c \in (d,\, 1-a]$, which completes the proof. $\square$

The following theorem provides a complete characterization of $\xi_3$ in the $(\lambda_2, \lambda_3)$-plane, as illustrated in Figure 2.

THEOREM 4.4. *$\xi_3$ is the region bounded by the following curves:*

- $C_1 : \lambda_2 = \lambda_3$ *for* $0 \le \lambda_2 \le 1$,
- $C_2 : \lambda_2 = -\lambda_3$ *for* $0 \le \lambda_2 \le \frac{1}{2}$,
- $C_3 : \lambda_2 = 1$ *for* $0 \le \lambda_3 \le 1$,
- $C_4 : \lambda_2 \cdot \lambda_3 = -\frac{1}{4}$ *for* $\frac{1}{2} \le \lambda_2 \le \frac{1+\sqrt{5}}{4}$,
- $C_5 : \lambda_2^2 + \lambda_2\lambda_3 + \lambda_3^2 - \lambda_2 - \lambda_3 \le 0$ *for* $\frac{1+\sqrt{5}}{4} \le \lambda_2 \le 1$ *and* $\lambda_3 \le 0$.

*Proof.* By Lemma 4.1, it suffices to determine the boundary of $\xi_3$. We first establish the boundary curves and subsequently show that each of them is realized. Thus, we begin by identifying the boundaries, which are given below:

- $C_1$ follows directly from the imposed condition $\lambda_2 \ge \lambda_3$.

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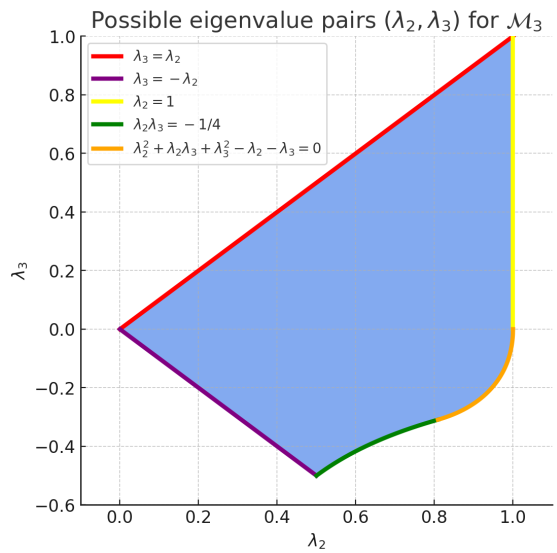


FIGURE 2. *Possible eigenvalue pairs* $(\lambda_2, \lambda_3)$ *for* $\mathcal{M}_3$*, with* $\lambda_2 \geqslant \lambda_3$.

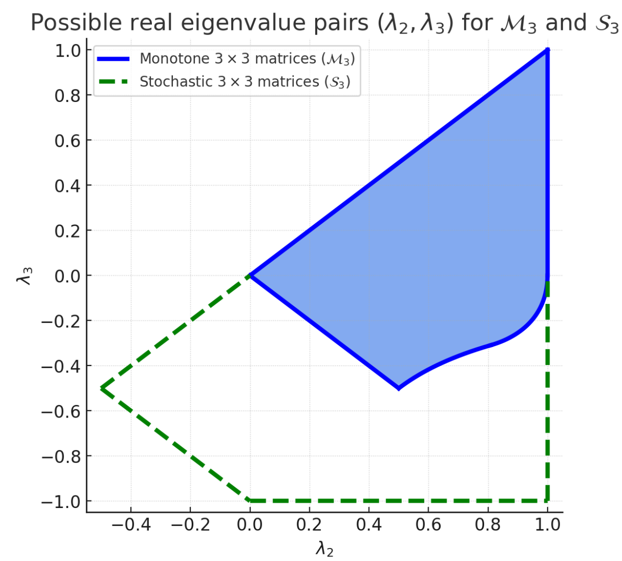


FIGURE 3. *Possible real eigenvalue pairs* $(\lambda_2, \lambda_3)$ *for* $\mathcal{M}_3$ *and* $\mathcal{S}_3$*, with* $\lambda_2 \geqslant \lambda_3$.

- From Lemma 2.1(3), we know that $\mathrm{tr}[D(M)] = \lambda_2 + \lambda_3 \geq 0$, which gives $C_2$.
- Since for a stochastic matrix, the modulus of each of the eigenvalues is at most 1, $C_3$ follows trivially.
- From Lemma 2.1(4) follows immediately that $\det[D(M)] = \lambda_2\lambda_3 \geq -\frac{1}{4}$, and hence $C_4$.
- For $\lambda_2 \geq \frac{1+\sqrt{5}}{4}$ and $\lambda_3 \leq 0$, Lemma 4.3 yields $(\mathrm{tr}[D(M)])^2 - \mathrm{tr}[D(M)] - \det[D(M)] \leq 0$. Substituting $\mathrm{tr}[D(M)] = \lambda_2 + \lambda_3$ and $\det[D(M)] = \lambda_2\lambda_3$ into this expression gives $C_5$.

Combining these boundary conditions shows that the region is precisely bounded by the curves $C_i$ specified in this theorem. This is illustrated in Fig. 2.

What remains is to verify that each of these bounds is realized. This can be confirmed by the matrices presented in Table 1. ☐

We note that the realizing matrices corresponding to the boundary of $\xi_3$, shown in Table 1, induce realizing matrices for the entire region through the construction in [21]. Figure 3 additionally gives the region of all possible real eigenvalue pairs $(\lambda_2, \lambda_3)$ arising from $\mathcal{S}_3$, determined by the constraints $-1 \leq \lambda_3 \leq \lambda_2 \leq 1$ and the JLL-inequality $\mathrm{tr}(S) = 1 + \lambda_2 + \lambda_3 \geq 0$, which is a necessary and sufficient condition for $3 \times 3$ stochastic matrices with real spectrum [20]. We mention that complex eigenvalues can also occur for $3 \times 3$ stochastic matrices, but we are only considering real eigenvalues here. In comparison with the stochastic matrices with real spectrum, the region $\xi_3$ represents a substantial contraction.

**5. Reduction theorem for $n \geqslant 4$.** From Section 3, we already have a complete determination of $\Xi_1$, $\Xi_2$, and $\Xi_3$. For $n \geqslant 4$, generalized arguments analogous to those in Lemma 2.1 show that the dominance matrix has column sums less than 1 and a nonnegative trace. Further, $\Xi_n$ is star-convex with respect to the origin for $n \geqslant 4$, so it suffices to determine the boundary.

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Table 1
*Realizing matrices for $C_1$–$C_5$*

| Curve | Realizing matrices | Parameter range |
|---|---|---|
| $C_1$ | $\begin{pmatrix} \frac{1+2\alpha}{3} & \frac{1-\alpha}{3} & \frac{1-\alpha}{3} \\ \frac{1-\alpha}{3} & \frac{1+2\alpha}{3} & \frac{1-\alpha}{3} \\ \frac{1-\alpha}{3} & \frac{1-\alpha}{3} & \frac{1+2\alpha}{3} \end{pmatrix}$ | $\alpha \in [0,1]$ |
| $C_2$ | $\begin{pmatrix} \alpha & 1-\alpha & 0 \\ \alpha & 1-2\alpha & \alpha \\ 0 & 1-\alpha & \alpha \end{pmatrix}$ | $\alpha \in \left[0, \frac{1}{2}\right]$ |
| $C_3$ | $\begin{pmatrix} \alpha & 1-\alpha & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ | $\alpha \in [0,1]$ |
| $C_4$ | $\begin{pmatrix} \frac{4\alpha^2+2\alpha-1}{4\alpha} & \frac{1-4\alpha^2+2\alpha}{4\alpha} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ | $\alpha \in \left[\frac{1}{2}, \frac{1+\sqrt{5}}{4}\right]$ |
| $C_5$ | $\begin{pmatrix} \frac{1+\alpha-\sqrt{1+2\alpha-3\alpha^2}}{2} & \frac{1-\alpha+\sqrt{1+2\alpha-3\alpha^2}}{2} & 0 \\ \frac{1+\alpha-\sqrt{1+2\alpha-3\alpha^2}}{2} & 0 & \frac{1-\alpha+\sqrt{1+2\alpha-3\alpha^2}}{2} \\ 0 & 0 & 1 \end{pmatrix}$ | $\alpha \in \left[\frac{1+\sqrt{5}}{4}, 1\right]$ |

Moreover, we prove the Reduction Theorem 5.2. According to this theorem, we can reduce the eigenvalue region $\Xi_n$ for $\mathcal{M}_n$ to a subset of the eigenvalue region $\Theta_{n-1}$ for $\mathcal{S}_{n-1}$. The proof of this theorem is build on the following lemma.

Lemma 5.1 ([16]). *If $A$ is an $n \times n$ nonnegative matrix with positive maximal eigenvalue $r$ and a corresponding positive eigenvector $x = (x_1, x_2, \ldots, x_n)$, then $(1/r) \cdot D^{-1}AD$ is a stochastic matrix, with the diagonal matrix $D = \operatorname{diag}(x_1, x_2, \ldots, x_n)$. Moreover, if $\sigma(A) = \{\lambda_1, \ldots, \lambda_n\}$ and $\sigma((1/r) \cdot D^{-1}AD) = \{\mu_1, \ldots, \mu_n\}$ are the spectra of, respectively, $A$ and $(1/r) \cdot D^{-1}AD$, then $\mu_i = \lambda_i / r$ for $i \in \{1, \ldots, n\}$.*

Theorem 5.2 (Reduction Theorem). *For every $n \geqslant 3$: $\Xi_n \subseteq \Theta_{n-1}$.*

*Proof.* Let $M$ be an $n \times n$ monotone matrix and $\sigma(M) = \{1, \lambda_2, \ldots, \lambda_n\}$, for $n \geqslant 3$, with $D(M)$ its $(n-1)\times(n-1)$ dominance matrix. It is clear that $1 \in \Theta_{n-1}$ because 1 is an eigenvalue of every $(n-1)\times(n-1)$ stochastic matrix. Our aim is to prove that $\sigma(M) = \{1, \lambda_2, \ldots, \lambda_n\} \subseteq \Theta_{n-1}$, so it suffices to proof that $\{\lambda_2, \ldots, \lambda_n\} \subseteq \Theta_{n-1}$. In order to do so, we use the dominance matrix $D(M)$ with $\sigma(D(M)) = \{\lambda_2, \ldots, \lambda_n\}$.

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The dominance matrix $D(M)$ can be transformed into a block upper triangular matrix (known as the Perron–Frobenius normal form), so we can assume that $D(M)$ is a nonnegative irreducible matrix. It follows from the Perron–Frobenius theorem for irreducible matrices [19] that $D(M)$ has a strictly positive maximal eigenvalue $r = \lambda_2$ and a strictly positive maximal eigenvector. It follows from Lemma 5.1 that $S = (1/\lambda_2) \cdot D^{-1} \cdot D(M) \cdot D$ is an $(n-1) \times (n-1)$ stochastic matrix. If $\sigma(S) = \{\mu_1, \ldots, \mu_{n-1}\}$, then we have the following link between the eigenvalues of $S$ and $D(M)$:

$$\mu_i = \lambda_{i+1}/\lambda_2 \text{ for } i \in \{1, \ldots, n-1\}.$$

Since $\lambda_2$ is an eigenvalue of the stochastic matrix $M$, it necessarily holds that $\lambda_2 \leq 1$. Because each $\mu_i$ lies in the region $\Theta_{n-1}$ and, moreover, this region is star-convex, it follows that also $\lambda_{i+1} = \lambda_2 \mu_i$ lies in $\Theta_{n-1}$. Thus, $\sigma(D(M)) = \{\lambda_2, \ldots, \lambda_n\} \subseteq \Theta_{n-1}$, which completes the proof. ∎

**6. Conclusions and further research.** This paper presents a first step in analyzing the eigenvalue regions for monotone stochastic matrices. We establish fundamental properties of the dominance matrix $D(M)$ and characterize the conditions under which a nonnegative matrix qualifies as a dominance matrix. The eigenvalues of monotone stochastic matrices are examined from two perspectives.

First, viewed individually, we determine the eigenvalue region for $n \times n$ monotone stochastic matrices for $1 \leq n \leq 3$, together with the realizing matrices of these regions. Second, viewed collectively, we characterize the set $\xi_3$ of all possible eigenvalue pairs $(\lambda_2, \lambda_3)$ arising from $3 \times 3$ monotone stochastic matrices, again accompanied by corresponding realizing matrices. In both approaches, the regions for monotone stochastic matrices are markedly smaller than those obtained for general stochastic matrices. Moreover, we prove a reduction theorem which shows that, for $n \geq 3$, the eigenvalue region for $n \times n$ monotone stochastic matrices is contained within that of $(n-1) \times (n-1)$ stochastic matrices.

The eigenvalue regions for $\mathcal{M}_n$ are completely determined for $1 \leq n \leq 3$. However, for $n \geq 4$, the problem remains unsolved. Although this paper introduces the reduction theorem 5.2 to provide additional restrictions, a comprehensive examination is necessary to establish an exact characterization.

Knowledge about the eigenvalues of monotone stochastic matrices holds potential applications in Markov theory. Consequently, future research should focus on linking the spectral properties of monotone stochastic matrices to their corresponding Markov chains, as eigenvalues offer critical insights into both the long-term behavior and the rate of convergence.

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