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FREE PLANE CURVES WITH A LINEAR JACOBIAN SYZYGY*

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Abstract. The study of planar free curves is a very active area of research, but a structural study of such a class is missing. We give a complete classification of the possible generators of the Jacobian syzygy module of a plane free curve under the assumption that one of them is linear. Specifically, we prove that, up to similarities, there are two possible forms for the Hilbert–Burch matrix. Our strategy relies on a translation of the problem into the accurate study of the geometry of maximal segments of a suitable triangle with integer points. Following this description, we are able to determine explicitly the equations of free curves and the associated Hilbert–Burch matrices.

Key words. Free curves, Jacobian syzygies, Hilbert–Burch matrix, Combinatorial commutative algebra.

AMS subject classifications. 13D02, 13P99, 14H45, 15A21.

1. Introduction. Let R be the ring of polynomials $\mathbb{C}[x, y, z]$ and consider $g \in R$, homogeneous of degree n. To the polynomial g, we can associate the projective curve $V(g) \subset \mathbb{P}^2$ and the R-module $\operatorname{Syz}(J_g)$ of Jacobian relations, where $J_g = \langle \partial_x g, \partial_y g, \partial_z g \rangle$. The curve $V(g)$ is called *free* if the module $\operatorname{Syz}(J_g)$ is free. The latter condition is equivalent to the projective dimension of R/J_g being 2; hence, $\operatorname{Syz}(J_g)$ has two generators.

The study of free curves is a very active area of research, also in connection with Terao's conjecture for line arrangements, asserting that the freeness of a line arrangement only depends on the combinatorics of the intersection lattice (see [19, Conjecture 4.138]).

Although many examples of free curves have been exhibited, see for instance [5, 6, 8, 10, 11, 12, 13, 14, 15], a structural study of such a class of curves is missing.

In the present paper, we shall give a complete classification of the possible generators of $\operatorname{Syz}(J_g)$ under the assumption that one of them is given by a triple of linear forms. Equivalently, we will explore the locus of Hilbert–Burch matrices associated with free curves, having the property that their three 2×2 minors yield a vector proportional to the gradient of a homogeneous form. Thanks to the exactness of a suitable de Rham sequence, this is equivalent to the curl of the three minors being zero. We translate this condition into a series of bihomogeneous equations in the coefficients of the polynomials appearing in the Hilbert–Burch matrix, which we study accurately with methods of combinatorial commutative algebra.

Our main result is the following. Up to similarities, there are two possible forms for the Hilbert–Burch matrix. The first one is the most interesting for us and corresponds to the case where the matrix is of the form

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$$\begin{pmatrix} ax & D(y,z) \\ by & E(x,y,z) \\ cz & F(x,y,z) \end{pmatrix},$$

where $a \neq 0, b, c \in \mathbb{C}$, and D, E, F are homogeneous forms of degree $n-2$. The core of the paper is a careful analysis of the curl equations. Useful tools are the set $\mathcal{T} \subset \mathbb{N}^3$ of integer points in the convex hull of $(n,0,0)$, $(0,n,0)$, and $(0,0,n)$, and $\mathcal{T}'$, namely $\mathcal{T}$ with the three vertices removed. Our work allows one to explicitly express the solutions of the curl equations in terms of maximal segments of $\mathcal{T}'$. In this way, we obtain Proposition 5.1, which precisely determines the polynomials D, E, and F. We then select those solutions corresponding to reduced curves that are not unions of concurrent lines (Lemma 6.5 and Theorem 6.6). Finally, we gather our findings in Table 1, which collects (up to permutations of the variables) all the possible free curves with a linear Jacobian syzygy in the considered case.
The second possible form for the Hilbert–Burch matrix is

$$\begin{pmatrix} y & D(x,z) \\ z & E(x,y,z) \\ 0 & F(x,y,z) \end{pmatrix},$$

where D, E, F are homogeneous forms of degree $n-2$. This case yields a single class of free curves, i.e., unions of hyperosculating conics, possibly together with their common tangent, the so-called *Płoski curves*. This class has been thoroughly studied, see [7, 21, 22].

Free curves with a linear Jacobian syzygy belong to the wider class of curves with a positive dimensional automorphism group, and they have been classified in [1] and [20, Proposition 4.4]. Our results specify precisely which curves are the free ones and allow one to describe the corresponding Hilbert–Burch matrices; see Section 8.3 for more details.

Finally, as an application, we give a characterization of free curves with a linear Jacobian syzygy and only quasi-homogeneous singularities, see Theorem 8.4.

The organization of the paper is as follows. Section 2 sets up the notation and assumptions that we are going to use throughout the paper. Section 3 describes, up to similarities, the two possible cases for the Hilbert–Burch matrices of the syzygy modules of free curves. Section 4 introduces the triangle $\mathcal{T}$ described above and the relations between its maximal segments and the curl equations. Section 5 provides further restrictions on the solutions since we neglect unions of concurrent lines. Section 6 concludes the study of the general case precisely selecting the polynomials we are interested in. Section 7 addresses the second case for the Hilbert–Burch matrix. Section 8 concludes the paper presenting some applications regarding quasi-homogeneous singularities and nearly free curves.

2. Notation and preliminaries.

We set $R := \mathbb{C}[x,y,z]$, and we denote by $C = V(g)$ a reduced plane curve of degree n defined by a homogeneous polynomial $g \in R_n$ in the complex projective plane $\mathbb{P}^2 = \mathrm{Proj}(R)$.

We denote by $\mathrm{Syz}(J_g)$ the graded R-module of all Jacobian relations for g, that is,

$$\mathrm{Syz}(J_g) := \left\{(a,b,c) \in R^3 \mid a\,\partial_x g + b\,\partial_y g + c\,\partial_z g = 0\right\}.$$

It is well known that the three partials $\partial_x g$, $\partial_y g$, and $\partial_z g$ are linearly dependent if and only if C is a union of lines passing through a point in $\mathbb{P}^2$ (i.e., a union of concurrent lines). So, we make the following assumption.

Assumption 2.1. *From now on, we assume that the three partials $\partial_x g$, $\partial_y g$, and $\partial_z g$ are linearly independent. Moreover, if g_{red} generates $\sqrt{(g)}$, then*

$$\mathrm{Syz}(J_g) \cong \mathrm{Syz}(J_{g_{\mathrm{red}}}),$$

so without loss of generality from now on we will assume that g is square-free.

Definition 2.2. *Let $C = V(g)$ be a reduced singular plane curve of degree n. We say that C is* free *if the graded R-module $\mathrm{Syz}(J_g)$ of all Jacobian relations for g is a free R-module. If*

$$\mathrm{Syz}(J_g) = R(-n_1) \oplus R(-n_2),$$

with $n_1 + n_2 = n - 1$, the integers (n_1, n_2) are called the exponents *of C. In this case, the Jacobian ideal admits a minimal free resolution of the type*

$$0 \longrightarrow R(-n_1) \oplus R(-n_2) \longrightarrow 3R\big(-(n-1)\big) \longrightarrow J_g \longrightarrow 0.$$

Remark 2.3. The freeness condition for a reduced singular curve $C = V(g)$ in $\mathbb{P}^2$ that is not a set of concurrent lines is equivalent to the Jacobian ideal J_g being arithmetically Cohen–Macaulay of codimension two; such ideals are completely described by the Hilbert–Burch Theorem (see, for instance, [17]): if $I = \langle g_1, \cdots, g_m \rangle \subset R$ is a Cohen-Macaulay ideal of codimension two, then I is defined by the maximal minors of the $(m-1) \times m$ matrix of the first syzygies of the ideal I. Combining this with Euler's formula for a homogeneous polynomial, we get that a free curve $C = V(g)$ in $\mathbb{P}^2$ has a very constrained structure: there exists a 3×3 matrix M, with one column consisting of the 3 variables, and the remaining 2 rows a minimal set of generators of $\mathrm{Syz}(J_g)$, such that $\det(M) \equiv 0 \mod (g)$.

We now explore the Hilbert–Burch matrices of free curves. We start by characterizing triples of polynomials that are the gradient of a homogeneous form (see [4, Lemma 4.2]):

Lemma 2.4. *A triple of polynomials (G_1, G_2, G_3) satisfies the relations*

$$\partial_x G_2 - \partial_y G_1 = 0, \quad \partial_x G_3 - \partial_z G_1 = 0, \quad \partial_y G_3 - \partial_z G_2 = 0,$$

if and only if there exists an element $H \in \Omega^0_{\mathbb{A}^3} \cong R$ such that

$$\nabla H = (G_1, G_2, G_3).$$

If, moreover, the polynomials G_1, G_2, G_3 are homogeneous of degree $n-1$, then by Euler's formula, the polynomial H is homogeneous of degree n.

As a consequence, given a generic matrix $M = \begin{pmatrix} A & D \\ B & E \\ C & F \end{pmatrix}$, if we set

$$G_1 := BF - CE, \quad G_2 := CD - AF, \quad G_3 := AE - BD,$$

we have that M is a first syzygy matrix for some free plane curve $V(g)$ if and only if the triple $(G_1 : G_2 : G_3)$ is proportional to $(\partial_x g : \partial_y g : \partial_z g)$. Moreover, by Lemma 2.4, this holds if and only if the curl $(K_1 : K_2 : K_3)$ of $(G_1 : G_2 : G_3)$ is identically zero, where:

$$K_1 := \partial_x G_2 - \partial_y G_1, \quad K_2 := \partial_x G_3 - \partial_z G_1, \quad K_3 := \partial_y G_3 - \partial_z G_2. \tag{2.1}$$

Such expressions are homogeneous polynomials of degree $n-2$, and they give the zero polynomials if and only if the coefficients of all monomials in x, y, z are zero. These coefficients are, in turn, bihomogeneous polynomials of bidegree $(1,1)$ in the coefficients of the entries of M.

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3. Free curves with a linear syzygy. We focus now on the locus of Hilbert–Burch matrices of type

$$M = \begin{pmatrix} A & D \\ B & E \\ C & F \end{pmatrix}, \tag{3.1}$$

where $A, B, C \in \mathbb{C}[x, y, z]_1$, i.e., are linear forms.

Remark 3.1. Since, by Assumption 2.1, we are not considering unions of concurrent lines, we can assume that at most one of the linear forms A, B, C is zero, and similarly at most one of the forms D, E, F is zero. Indeed, otherwise, one minor of the matrix (3.1) would be identically zero.

Projective plane transformations induce transformations on linear Jacobian syzygies as described in the following result.

Lemma 3.2. *Let $A, B, C \in \mathbb{C}[x, y, z]_1$ and let $H \in \mathbb{C}^{3\times 3}$ be such that $H \cdot \left(\begin{smallmatrix} x \\ y \\ z \end{smallmatrix}\right) = \left(\begin{smallmatrix} A \\ B \\ C \end{smallmatrix}\right)$. A polynomial $g \in \mathbb{C}[x, y, z]_n$ admits a linear Jacobian syzygy of type*

$$A\partial_x g + B\partial_y g + C\partial_z g = 0,$$

if and only if for any $N \in GL(3, \mathbb{C})$ the triple $N^{-1} \cdot H \cdot N \cdot {}^t(x\ y\ z)$ gives a linear Jacobian syzygy of the polynomial $g(N \cdot {}^t(x\ y\ z))$.

Proof. By the chain rule, one gets

$$(\nabla g)\big(N \cdot {}^t(x\ y\ z)\big) = \nabla\Big(g\big(N \cdot {}^t(x\ y\ z)\big)\Big) \cdot N^{-1}. \tag{3.2}$$

Moreover, we have

$$\begin{pmatrix} A\big(N \cdot {}^t(x\ y\ z)\big) \\ B\big(N \cdot {}^t(x\ y\ z)\big) \\ C\big(N \cdot {}^t(x\ y\ z)\big) \end{pmatrix} = H \cdot N \cdot {}^t(x\ y\ z),$$

and the relation $\nabla g \cdot \begin{pmatrix} A \\ B \\ C \end{pmatrix} = 0$ gives

$$(\nabla g)\big(N \cdot {}^t(x\ y\ z)\big) \cdot \begin{pmatrix} A\big(N \cdot {}^t(x\ y\ z)\big) \\ B\big(N \cdot {}^t(x\ y\ z)\big) \\ C\big(N \cdot {}^t(x\ y\ z)\big) \end{pmatrix} = 0.$$

By taking into account (3.2), we finally get

$$\nabla\Big(g\big(N \cdot {}^t(x\ y\ z)\big)\Big) \cdot N^{-1} \cdot H \cdot N \cdot {}^t(x\ y\ z) = 0.$$

□

As a consequence, up to a linear change of coordinates, we can assume that N is in Jordan block form. Thus, any free curve with a linear Jacobian syzygy is projectively equivalent to one with linear Jacobian syzygy of one of the following three forms:

$$A = ax, \qquad B = by, \qquad C = cz, \tag{3.3}$$

$$A = ax + y, \qquad B = ay, \qquad C = cz, \tag{3.4}$$

$$A = ax + y, \qquad B = ay + z, \qquad C = az. \tag{3.5}$$

Remark 3.3. Observe that in case (3.3), we can assume that the three coefficients a, b, c are not all equal; otherwise, we would get the Euler relation.

LEMMA 3.4.

1. *If a polynomial g admits the linear Jacobian syzygy $(ax+y, ay, cz)$, i.e., we are in case (3.4), then either g is a monomial, or its zero locus is a set of concurrent lines, or it is the zero polynomial.*
2. *If a polynomial g admits the linear Jacobian syzygy $(ax+y, ay+z, az)$, i.e., we are in case (3.5), and $a \neq 0$, then $g = 0$.*

Proof. We express a generic degree n polynomial g as follows

$$g = \sum_{(i,j,k)\in I} g_{ijk} x^i y^j z^k, \quad I = \big\{(i,j,k) \mid 0 \le i,j,k \le n,\, i+j+k = n\big\}. \tag{3.6}$$

Case (1). From the relation $(ax+y)\,\partial_x g + ay\,\partial_y g + cz\,\partial_z g = 0$ and the Euler identity, we get

$$a\,n\,g + y\,\partial_x g + (c-a)z\,\partial_z g = 0.$$

By (3.6), we get

$$a\,n \sum_{(i,j,k)\in I} g_{ijk} x^i y^j z^k + y \sum_{(i,j,k)\in I,\, i\ge 1} i\,g_{ijk} x^{i-1} y^j z^k + (c-a)z \sum_{(i,j,k)\in I,\, k\ge 1} k\,g_{ijk} x^i y^j z^{k-1} = 0,$$

that is

$$a\,n \sum_{(i,j,k)\in I} g_{ijk} x^i y^j z^k + \sum_{(i,j,k)\in I,\, i\ge 1} i\,g_{ijk} x^{i-1} y^{j+1} z^k + \sum_{(i,j,k)\in I} (c-a)k\,g_{ijk} x^i y^j z^k = 0.$$

Let v be the left-hand side of the previous equality. Fix $k \le n$ and consider the monomials:

$$x^{n-k} z^k, \quad x^{n-k-1} y z^k, \quad x^{n-k-2} y^2 z^k, \ \dots\ , \ y^{n-k} z^k,$$

(i.e., all the degree n monomials with z of fixed degree k). Let $\alpha_k := a(n-k) + ck$. The coefficients in v of the above monomials are, respectively:

$$\begin{aligned}
\alpha_k g_{n-k\,0\,k},&\\
\alpha_k g_{n-k-1\,1\,k} + (n-k) g_{n-k\,0\,k},&\\
\alpha_k g_{n-k-2\,2\,k} + (n-k-1) g_{n-k-1\,1\,k},&\\
\vdots&\\
\alpha_k g_{0\,n-k\,k} + g_{1\,n-k-1\,k}.&
\end{aligned}$$

Since $v = 0$, all these linear forms in g_{ijk} must be zero. If $\alpha_k \neq 0$, then we see that $g_{ijk} = 0$ for all i, j (where $i+j+k = n$). If, conversely, $\alpha_k = 0$, then we get that $g_{ijk} = 0$ for i, j such that $i+j+k = n$ and $i > 0$, while $g_{0\,n-k\,k}$ can take any value. Therefore, we distinguish three subcases:

Subcase A If $a(n-k) + ck \neq 0$ for all k, then g is the zero polynomial.

Subcase B If for a single value $\bar{k}$ of k, we have $a(n-\bar{k}) + \bar{k}c = 0$, while for all $k \neq \bar{k}$, we have $a(n-k)+ck \neq 0$, then g is the monomial $g_{0\,n-\bar{k}\,\bar{k}} y^{n-\bar{k}} z^{\bar{k}}$.

Subcase C If $a(n-k) + kc = 0$ is zero for at least two different values $\bar{k}_1$ and $\bar{k}_2$, then $a = c = 0$. Thus, we get the Jacobian relation $y\partial_x g = 0$, which implies $\partial_x g = 0$ and $V(g)$ is a set of concurrent lines.

Case (2). We argue in a similar way. From the relation $(ax+y)\,\partial_x g + (ay+z)\,\partial_y g + az\,\partial_z g = 0$ and the Euler identity, we get

$$a\,n\,g + y\,\partial_x g + z\,\partial_y g = 0.$$

As in Case (1), by (3.6), we get the following square homogeneous linear system in the variables $\{g_{ijk}\}$:

$$\begin{cases} a\,n\,g_{n00} = 0 \\ a\,n\,g_{0n0} + g_{1n-10} = 0 \\ a\,n\,g_{0jk} + g_{1jk} + j\,g_{0\,j+1\,k-1} = 0, \quad k \geq 1 \\ a\,n\,g_{i0k} + i\,g_{i\,1\,k-1} = 0, \quad k \geq 1 \\ a\,n\,g_{ij0} + (i+1)\,g_{i+1\,j-1\,0} = 0, \\ a\,n\,g_{ijk} + i\,g_{i+1\,j-1\,k} + j\,g_{i\,j+1\,k-1} = 0, \quad j \geq 1,\, k \geq 1. \end{cases}$$

Order the indeterminates g_{ijk} lexicographically, i.e., $g_{i_1,j_1,k_1} \leq g_{i_2,j_2,k_2}$ if and only if $i_1 < i_2$ or ($i_1 = i_2$ and $j_1 < j_2$) or ($i_1 = i_2$, $j_1 = j_2$, and $k_1 < k_2$). In this way, the associated matrix is upper triangular, with $a\,n$ on the diagonal. Hence, if $a \neq 0$, the matrix is of maximal rank; therefore, the system admits only the trivial solution. □

We sum up the findings of this section in the following proposition.

PROPOSITION 3.5. *Let $g \in \mathbb{C}[x,y,z]_n$ be a nonzero polynomial defining a reduced curve that is not a set of concurrent lines, with a linear Jacobian syzygy (A,B,C). Up to linear changes of variables, we have the following two possibilities:*

1. *$A = ax$, $B = by$, $C = cz$ for some a, b, c not all equal and $a \neq 0$;*
2. *$A = y$, $B = z$, $C = 0$.*

Sections 4 to 6 concern the study of the first of the two cases above, while Section 7 briefly analyzes the second case.

4. The case $A = ax$, $B = by$, $C = cz$.

This section is devoted to an accurate analysis of the general case. The final point is to obtain the possible expressions of squarefree polynomials g given by:

$$g = \frac{1}{n}\det\begin{pmatrix} x & ax & D \\ y & by & E \\ z & cz & F \end{pmatrix}, \tag{4.1}$$

and annihilating the curl from (2.1). The final result is summarized in Section 6, Table 1.

To clarify our strategy, we start by illustrating the main ideas and the results of the next three sections.

Observe that the expressions in (2.1) are bihomogeneous in the variables a, b, c and in the coefficients of D, E, F; see Proposition 4.2 for the explicit form. With suitable elementary operations, the system of equations can be transformed in an equivalent, simpler one, composed of four families: three of them split into a product of a linear form in a, b, c and a coefficient of D, or E, or F, while the fourth one is given by linear polynomials in the coefficients of E and F; see Proposition 4.3.

If the linear forms in a, b, c never vanish, the system admits only the trivial solution. Therefore, we focus on the other situation.

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The coefficients of the linear forms determine lattice points in a triangle in $\mathbb{R}^3$, see Definition 4.1 and Fig. 1. The existence and dimension of the space of solutions can be translated to the geometry of segments in the lattice, see Section 4.2 and Fig. 3. Finally, we show in Corollary 4.5 that particular lattice points and suitable maximal segments correspond to free polynomials g.

Moreover, we shall prove that it is possible to explicitly express the general g satisfying the linear Jacobian syzygy and to exclude the situations of concurrent lines (see Section 5) and of nonreduced polynomials, see Section 6.

4.1. The equations. The first task is to codify freeness of a form with exponents $(1, n-2)$ into a system of polynomial equations. Consider the matrix M given in (3.1) with $A = ax$, $B = by$, $C = cz$. As a, b, c are not all zero, we will assume from now on that

$$a \neq 0.$$

Moreover, we observe that, by adding to the second column of M a suitable multiple of the first column, we can assume that D is a generic form of degree $n-2$ in y, z without changing the 2×2 minors of M. Let E, F be generic forms in x, y, z of degree $n-2$. Hence, the forms E, F, and D can be written as:

$$\begin{aligned}
E &= \sum_{(i,j,k)\in I} e_{ijk} x^i y^j z^k, && I = \{(i,j,k) \mid 0 \le i,j,k \le n-2,\ i+j+k = n-2\},\\
F &= \sum_{(i,j,k)\in I} f_{ijk} x^i y^j z^k, && I = \{(i,j,k) \mid 0 \le i,j,k \le n-2,\ i+j+k = n-2\},\\
D &= \sum_{(j,k)\in I} d_{0jk} y^j z^k, && I = \{(j,k) \mid 0 \le j,k \le n-2,\ j+k = n-2\}.
\end{aligned}$$

We consider now the equations $K_1 = 0, K_2 = 0, K_3 = 0$, where the K_i are as in (2.1). We introduce a geometric object that helps us to keep track of the exponents of monomials appearing in such equations, to simplify their handling.

Definition 4.1. *Let $n \in \mathbb{N}$. The* n-triangle grid *$\mathcal{T}_n$ is the set of integer points of the n-th dilation of the standard 2-simplex in $\mathbb{R}^3$, i.e., the integer points of the convex hull of the three points $(n,0,0)$, $(0,n,0)$, and $(0,0,n)$. A* maximal segment *in $\mathcal{T}_n$ is the intersection of $\mathcal{T}_n$ with a line connecting two points in the grid different from $(n,0,0)$, $(0,n,0)$, and $(0,0,n)$; these sets are indeed maximal with respect to inclusion.*

Proposition 4.2. *The polynomial K_1 is identically zero if and only if for every $i,j,k \geq 0$ such that $i+j+k = n-2$, the following equations are satisfied:*

$$\big((i+1)a + (j+1)b\big) f_{ijk} = 0 \quad (\textit{if } k = 0), \tag{4.2}$$

$$\big((i+1)a + (j+1)b\big) f_{ijk} - (j+1)c\, e_{ij+1\,k-1} = 0 \quad (\textit{if } k > 0). \tag{4.3}$$

The polynomial K_2 is identically zero if and only if for every $i,j,k \geq 0$ such that $i+j+k = n-2$ the following equations are satisfied:

$$\big((i+1)a + (k+1)c\big) e_{ijk} = 0 \quad (\textit{if } j = 0), \tag{4.4}$$

$$\big((i+1)a + (k+1)c\big) e_{ijk} - (k+1)b f_{ij-1\,k+1} = 0 \quad (\textit{if } j > 0). \tag{4.5}$$

The polynomial K_3 is identically zero if and only if for every $i,j,k \geq 0$ such that $i+j+k = n-2$ the following equations are satisfied:

$$\big((j+1)b+(k+1)c\big)d_{ijk}=0 \quad (\textit{if } i=0), \tag{4.6}$$

$$(j+1)e_{i-1\,j+1\,k}+(k+1)f_{i-1\,j\,k+1}=0 \quad (\textit{if } i>0). \tag{4.7}$$

Proof. A direct computation gives

$$K_1=-(a+b)F+cz\partial_y E-ax\partial_x F-by\partial_y F.$$

The monomials of the polynomial $\partial_x F$ (a form of degree $n-3$) are represented by the set $\mathcal{T}_{n-3}$; specifically, to the point $(i,j,k)\in\mathcal{T}_{n-3}$ we associate the coefficient $(i+1)f_{i+1\,jk}$. Hence, the monomials of $x\,\partial_x F$ are represented by the points of $\mathcal{T}_{n-2}$; more precisely, to the point $(i,j,k)\in\mathcal{T}_{n-2}$ (which comes, if $i>0$, from the point $(i-1,j,k)\in\mathcal{T}_{n-3}$) it is associated the coefficient if_{ijk}. Note that this formula holds also for the case $i=0$. In a similar way, to the point $(i,j,k)\in\mathcal{T}_{n-2}$, we associate the coefficient jf_{ijk} of the polynomial $y\,\partial_y F$. For the polynomial $z\,\partial_y E$, we distinguish two cases: for the points $(i,j,k)\in\mathcal{T}_{n-2}$ with $k=0$, the coefficient of $z\,\partial_y E$ is 0, while, if $k>0$, the coefficient is $(j+1)e_{ij+1\,k-1}$. The polynomial K_1 is a form of degree $n-2$; hence, its monomials are encoded in $\mathcal{T}_{n-2}$. Let $(i,j,k)\in\mathcal{T}_{n-2}$ with $k=0$. Then, the coefficient in K_1 of the monomial $x^iy^jz^k$ (which is x^iy^j with $i+j=n-2$) is

$$-(a+b)f_{ijk}-aif_{ijk}-bjf_{ijk}=-\big((i+1)a+(j+1)b\big)f_{ijk},$$

and this gives (4.2). If $k>0$, the coefficient of the monomial $x^iy^jz^k$ with $i+j+k=n-2$ and $k>0$ in K_1 is $-\left((i+1)a+(j+1)b\right)f_{ijk}+(j+1)ce_{ij+1\,k-1}$ and this gives (4.3). The other cases are proved in a similar way. ∎

The next proposition provides useful information to solve (4.2)–(4.7).

Proposition 4.3. *If $a\neq 0$, then (4.2)–(4.7) are equivalent to the following four sets of equations:*

$$e_{i-1\,j\,k-1}(ia+jb+kc)=0 \quad (i>0,\ k>0,\ i+j+k=n), \tag{4.8}$$

$$f_{i-1\,j-1\,k}(ia+jb+kc)=0 \quad (i>0,\ j>0,\ i+j+k=n), \tag{4.9}$$

$$d_{i\,j-1\,k-1}(ia+jb+kc)=0 \quad (i=0,\ j>0,\ k>0,\ i+j+k=n), \tag{4.10}$$

$$je_{i-1\,j\,k-1}+kf_{i-1\,j-1\,k}=0 \quad (\textit{if } i,j,k>0,\ i+j+k=n). \tag{4.11}$$

Proof. Let us suppose that (4.2)–(4.7) hold true. To simplify the notations, we denote by $\mathcal{E}_1(i,j,k),\dots,\mathcal{E}_6(i,j,k)$ the left-hand side of (4.2)–(4.7) (it is understood that i,j,k are subject to the required restrictions given in Proposition 4.2). The equations of the third line of (4.8) are already obtained in (4.6), while (4.11) is obtained from (4.7) substituting $i+1$ in place of i and $j-1$ in place of j.

If in (4.3) we eliminate f_{ijk} using (4.7), we get:

$$\big((i+1)a+(j+1)b\big)\mathcal{E}_6(i+1,j,k-1)-k\mathcal{E}_2(i,j,k)=e_{i\,j+1\,k-1}(j+1)\big((i+1)a+(j+1)b+kc\big), \tag{4.12}$$

(for $k>0$ and $i+j+k=n-2$). Similarly, from (4.3)–(4.7), we have:

$$\big((i+1)a+(k+1)c\big)\mathcal{E}_6(i+1,j-1,k)-j\mathcal{E}_4(i,j,k)=f_{i\,j-1\,k+1}(k+1)\big((i+1)a+jb+(k+1)c\big), \tag{4.13}$$

(for $j>0$, $i+j+k=n-2$). Suppose that (4.3) and (4.7) are satisfied. Then, from (4.12), we get:

$$e_{i\,j+1\,k-1}\big((i+1)a+(j+1)b+kc\big)=0,$$

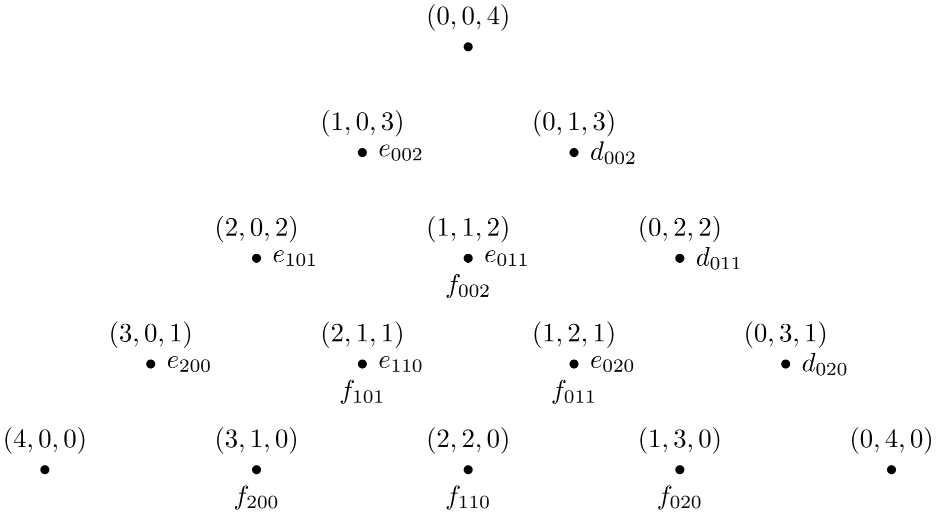

FIG. 1. *The grid for $n = 4$ with the corresponding monomials.*

and, if we write $i-1$ in place of i and $j-1$ in place of j, we get that $e_{i-1\,j,k-1}(ia+jb+kc)=0$ for $i>0, j>0, k>0$ and $i+j+k=n$. But this formula, thanks to (4.4), also holds for $j=0$, hence (4.3) and (4.7) imply that the first set of (4.8) are satisfied. Similarly, from (4.13) follows the second set of (4.8). Now, suppose that, conversely, (4.8) are satisfied. Then, the right-hand side of (4.12) is zero and since also (4.11) holds, (4.3) is satisfied. With similar computations, we can prove (4.5). ☐

4.2. Solving (4.8), . . . , (4.11). Now we determine the explicit solutions of the four sets of equations of Proposition 4.3. To begin with, we determine the solutions of (4.8). One trivial solution is given by $a=b=c=0$, but we discard this case, since we assume $a \neq 0$. Another trivial solution is given by all the coefficients of D, E, and F equal to zero. Let $L(i,j,k)$ be the linear form $ia+jb+kc$ in a,b,c and we associate to it the vector $(i,j,k) \in \mathbb{R}^3$. When $L(i,j,k)$ varies among all the possible cases, the vectors (i,j,k) describe the set $\mathcal{T}'$, which is the n-triangle grid $\mathcal{T}_n$ without the three points $(n,0,0)$, $(0,n,0)$, and $(0,0,n)$. Let $\mathcal{U}$ be the set of all the coefficients of D, E, F together with the element 0, i.e.:

$$\mathcal{U} = \{d_{0jk} \mid (0,j,k) \in \mathcal{T}_{n-2}\} \cup \{e_{ijk} \mid (i,j,k) \in \mathcal{T}_{n-2}\} \cup \{f_{ijk} \mid (i,j,k) \in \mathcal{T}_{n-2}\} \cup \{0\},$$

and consider the maps $\phi_D, \phi_E, \phi_F \colon \mathcal{T}' \longrightarrow \mathcal{U}$ given by:

$$\phi_D(i,j,k) = \begin{cases} d_{0\,j-1\,k-1} & \text{if } i=0,\ j,k>0 \\ 0 & \text{otherwise} \end{cases}, \tag{4.14}$$

$$\phi_E(i,j,k) = \begin{cases} e_{i-1\,j\,k-1} & \text{if } i,k>0 \\ 0 & \text{otherwise} \end{cases}, \tag{4.15}$$

$$\phi_F(i,j,k) = \begin{cases} f_{i-1\,j-1\,k} & \text{if } i,j>0 \\ 0 & \text{otherwise} \end{cases}. \tag{4.16}$$

If H is a subset of $\mathcal{T}'$, we say that a coefficient $\gamma \in \mathcal{U} \setminus \{0\}$ is *tied* to H if $\gamma \notin \phi_D(H)$, $\gamma \notin \phi_E(H)$, and $\gamma \notin \phi_F(H)$.

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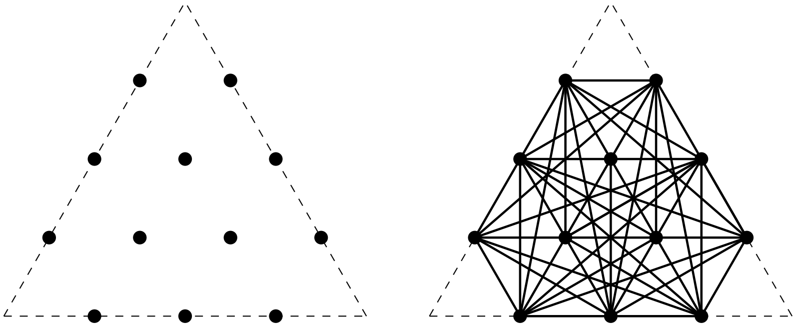

Fig. 2. *Case $n = 4$. The set $\mathcal{T}'$ and its maximal segments.*

Similarly as in Definition 4.1, a subset H of $\mathcal{T}'$ is called a *maximal segment* (of $\mathcal{T}'$) if H has at least two points and is a maximal subset of collinear points of $\mathcal{T}'$.

If H is a maximal segment of $\mathcal{T}'$ or H is given by a single point $(i,j,k) \in \mathcal{T}'$, then we associate to H the following two sets:

$$m(H) = \{\gamma \in \mathcal{U} \setminus \{0\} \mid \gamma \text{ is tied to } H\}, \tag{4.17}$$

$$r(H) = \{ai + bj + ck \mid (i,j,k) \in H\}. \tag{4.18}$$

The first set consists of some coefficients of the polynomials D, E, F, while the second one consists of linear forms in a, b, c.

Proposition 4.4. *The primary decomposition in $\mathbb{C}[a,b,c,\{d_{ijk}\},\{e_{ijk}\},\{f_{ijk}\}]$ of the radical of the ideal generated by the left-hand side of (4.8), with $a \neq 0$, is given by the prime ideals*

- $\big(\{d_{ijk}\} \cup \{e_{ijk}\} \cup \{f_{ijk}\}\big)$;
- $\big(m(H) \cup r(H)\big)$ *where* $H = \{(i,j,k)\}$ *with* $(i,j,k) \in \mathcal{T}'$;
- $\big(m(H) \cup r(H)\big)$ *where H is a maximal segment of $\mathcal{T}'$ whose points are not aligned with* $(n,0,0)$.

Proof. If the system (4.8) has a nontrivial solution, then there exist points (i_1,j_1,k_1), $\ldots$, (i_r,j_r,k_r) in $\mathcal{T}'$ such that

$$\begin{cases} a\, i_1 + b\, j_1 + c\, k_1 = 0 \\ \qquad\qquad \vdots \\ a\, i_r + b\, j_r + c\, k_r = 0. \end{cases} \tag{4.19}$$

This linear system must have a nontrivial solution in a, b, c, so the rank of the associated matrix must be either 1 or 2.

If the rank is 1, then $r = 1$ because any two points of $\mathcal{T}'$ are linearly independent. In this case, if we set $H = \{(i_1,j_1,k_1)\}$, the solution to equations (4.8) is given by $ai_1 + bj_1 + ck_1 = 0$ and by $\gamma = 0$, where γ are the coefficients of D or of E or of F tied to H; hence, this solution gives the prime ideal generated by $m(H) \cup r(H)$.

If the rank of the linear system (4.19) is 2, the points $(i_1,j_1,k_1),\ldots,(i_r,j_r,k_r)$ are collinear. Moreover, if (i_s,j_s,k_s) is another point of $\mathcal{T}'$ collinear with the r previous points, then the linear system (4.19) plus the equation $ai_s + bj_s + ck_s = 0$ has the same solutions of (4.19). Therefore, if we have some collinear points in $\mathcal{T}'$, we can complete them to a maximal set H of collinear points contained in $\mathcal{T}'$. As soon as the linear equations corresponding to the points of H are satisfied, to find the solutions of the desired equations we set to zero all the coefficients of D, E, and F that are tied to H; hence, we get again the prime ideal generated by $m(H) \cup r(H)$.

Finally, if some collinear points of $\mathcal{T}'$ give (4.19) of rank 2 and these points are collinear with $(n, 0, 0)$, then $na = 0$, which is not possible. ∎

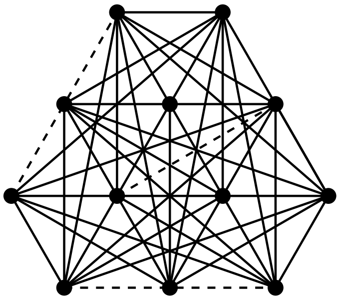

Fig. 3. *The primary decomposition given in Proposition 4.4 (for the case $n = 4$). The black circles are the points of $\mathcal{T}'$, the lines are the maximal segments of $\mathcal{T}'$, the three dashed lines are three maximal segments that have to be excluded since their points are aligned with $(n, 0, 0)$.*

An immediate consequence of the above proposition is the following:

Corollary 4.5. *The primary decomposition of the radical of the ideal generated by the left-hand side of (4.8) and (4.11) is given by the ideals described in Proposition 4.4, where we make the substitutions $e_{i-1\,j\,k-1} = -(k/j) f_{i-1\,j-1\,k}$ when $i, j, k > 0$.*

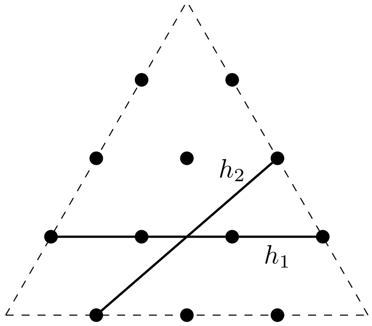

Fig. 4. *Two examples of maximal segments.*

Example 4.6. *Consider the case $n = 4$ and the following two maximal segments (see Fig. 4):*

$$h_1 = \{(3,0,1), (2,1,1), (1,2,1), (0,3,1)\},$$
$$h_2 = \{(3,1,0), (0,2,2)\}.$$

According to (4.14)–(4.16), we have:

$$\phi_D(h_1) = \{0, d_{020}\}, \quad \phi_E(h_1) = \{0, e_{020}, e_{110}, e_{200}\}, \quad \phi_F(h_1) = \{0, f_{011}, f_{101}\},$$

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and

$$\phi_D(h_2) = \{0, d_{011}\}, \quad \phi_E(h_2) = \{0\}, \quad \phi_F(h_2) = \{0, f_{200}\}.$$

Therefore:

$$m(h_1) = \{d_{002}, d_{011}, e_{002}, e_{011}, e_{101}, f_{002}, f_{020}, f_{110}, f_{200}\},$$

and

$$m(h_2) = \{d_{002}, d_{020}, e_{002}, e_{011}, e_{020}, e_{101}, e_{110}, e_{200}, f_{002}, f_{011}, f_{020}, f_{101}, f_{110}\}.$$

Finally, we can compute the polynomials D, E, F under the conditions $m(h_1)$ and $m(h_2)$:

$$D_{h_1} = d_{020}y^2, \quad E_{h_1} = e_{020}y^2 + e_{110}xy + e_{200}x^2, \quad F_{h_1} = f_{011}yz + f_{101}xz.$$

It is possible to count the number of prime ideals in the primary decomposition given by Proposition 4.4.

Lemma 4.7. *The number of elements in the primary decomposition given by Proposition 4.4 is*

$$\frac{1}{2}(n-1)(n+4) + \alpha(n+1) - \sum_{i=2}^{n} \phi(i) - 2 \sum_{i=\lfloor n/2 \rfloor + 1}^{n} \phi(i) - 1,$$

where ϕ is the Euler totient function.

Proof. The number of elements of $\mathcal{T}'$ is

$$p_1 = \frac{(n-1)(n+4)}{2}.$$

The number of lines passing through at least two points of a triangular grid of side n is a known number, given by

$$\alpha(n) = 3 \sum_{j=1}^{n-1} \phi(j) \left(\binom{n-j+1}{2} - \binom{n-2j+1}{2} \right),$$

see https://oeis.org/A244504 (here, we assume $\binom{i}{2}$ is zero if $i \leq 0$). The points of the lines passing through $N = (n, 0, 0)$ and a point $Q = (u, v, w) \in \mathcal{T}'$ (hence with $u + v + w = n$ and $u, v, w \geq 0$) are given by $N + \lambda(Q - N)$. If these points have integer coordinates, they are of the form $N + m/d\,Q$, where $d = \gcd(n-u, v, w) = \gcd(v+w, v, w) = \gcd(v, w)$ and $m \in \mathbb{Z}$. From this, we see that the lines of the grid of side $n+1$ passing through N and Q, and no other points of $\mathcal{T}'$ are in one-to-one correspondence with the points $Q \in \mathcal{T}'$ such that v, w are positive, coprime and are such that $u + v = i$ for $i = 2, 3, \ldots, n$. Hence, their number is

$$p_2 = \sum_{i=2}^{n} \phi(i).$$

These lines have to be subtracted from the computation of the maximal segments of $\mathcal{T}'$. Similarly, other lines of the grid of side $n+1$ that do not give maximal segments of $\mathcal{T}'$ are the lines passing through the point $(0, n, 0)$ and a point $Q = (u, v, w) \in \mathcal{T}'$ such that u, w are coprime and $v \geq \lfloor n/2 \rfloor + 1$, since these lines contain only one point of $\mathcal{T}'$. The number of these lines is given by:

$$p_3 = \sum_{i=\lfloor n/2 \rfloor + 1}^{n} \phi(i).$$

A similar consideration can be given for the lines through $(0, 0, n)$. Finally, also the two maximal segments of $\mathcal{T}'$ given by the line through the two points $(n, 0, 0), (0, n, 0)$ and the maximal segment given by the line through the points $(n,0,0),(0,0,n)$ must be excluded. From these computations, we see that the number of prime ideals of the primary decomposition of Proposition 4.4 is $1+p_1+\alpha(n+1)-p_2-2p_3-2$, i.e.:

$$\frac{1}{2}(n-1)(n+4)+\alpha(n+1)-\sum_{i=2}^{n}\phi(i)-2\sum_{i=\lfloor n/2\rfloor+1}^{n}\phi(i)-1. \qquad \square$$

5. Restriction to solutions corresponding to non-concurrent lines. As observed in Remark 3.1, we want to consider those H given by Proposition 4.4 (or Corollary 4.5) for which at most one among D, E and F is zero. If a point $(i,j,k)\in\mathcal{T}'$ is such that $i=0$, then all coefficients of E and all coefficients of F are tied to $H=\{(i,j,k)\}$, hence in this case E and F evaluate to zero. Similarly, if $j=0$, then D and F evaluate to zero and if $k=0$, then D and E evaluate to zero. Therefore, with our hypothesis, if H is a singleton, it must be an interior point of $\mathcal{T}'$. For the same reason, the three sets of collinear points of $\mathcal{T}'$ given by:

$$\begin{aligned}H_1&=\{(0,j,k)\mid j+k=n,\ j,k>0\},\\ H_2&=\{(i,0,k)\mid i+k=n,\ i,k>0\},\\ H_3&=\{(i,j,0)\mid i+j=n,\ i,j>0\}.\end{aligned}$$

are not acceptable maximal segments of $\mathcal{T}'$ (the last two have already been excluded, since collinear with the point $(n,0,0)$). Let therefore

$$\mathcal{H}_1:=\{\{(i,j,k)\}\mid (i,j,k)\in\mathcal{T}' \text{ and } i,j,k>0\},$$

and

$$\mathcal{H}_2:=\{H\subseteq\mathcal{T}'\mid H \text{ is a maximal segment}\}\setminus\{H_1,H_2,H_3\}.$$

We are interested in the values of D,E,F under the conditions $m(H)$ given by (4.17) with $H\in\mathcal{H}_1\cup\mathcal{H}_2$. Hence, we have to compute the reduced form of D,E,F w.r.t. the monomial ideal $\big(m(H)\big)$. We denote these values by D_H,E_H,F_H. The discussion above immediately implies the following result.

Proposition 5.1. *If $H\in\mathcal{H}_1\cup\mathcal{H}_2$, we have:*

$$\begin{aligned}D_H&=\sum_{(i,j,k)\in H}\phi_D(i,j,k)x^iy^{j-1}z^{k-1},\\ E_H&=\sum_{(i,j,k)\in H}\phi_E(i,j,k)x^{i-1}y^jz^{k-1},\\ F_H&=\sum_{(i,j,k)\in H}\phi_F(i,j,k)x^{i-1}y^{j-1}z^k.\end{aligned}\tag{5.1}$$

Remark 5.2. In particular, if $H=\{(i,j,k)\}\in\mathcal{H}_1$, then

$$\begin{aligned}D_H&=0,\\ E_H&=e_{i-1\,j\,k-1}x^{i-1}y^jz^{k-1},\\ F_H&=f_{i-1\,j-1\,k}x^{i-1}y^{j-1}z^k.\end{aligned}\tag{5.2}$$

If $H\in\mathcal{H}_2$, then D_H is 0, unless there exist j_0,k_0 such that $(0,j_0,k_0)\in H$; in this case, $D_H=d_{0\,j_0\,k_0}y^{j_0-1}z^{k_0-1}$. We see therefore that in case $H\in\mathcal{H}_1$, D_H is zero, while E_H and F_H are nonzero monomials, while if $H\in\mathcal{H}_2$, D_H is either 0 or a monomial.

If we want the values of the polynomials D, E, F for a solution of all the four families of equations of Proposition 4.3, corresponding to a set $H \in \mathcal{H}_1 \cup \mathcal{H}_2$, it is enough to modify the function ϕ_E defined in (4.15) as follows, taking into account (4.11):

$$\phi^*_E(i,j,k) = \begin{cases} e_{i-1\,j\,k-1} & \text{if } i,k>0, j=0 \\ -(k/j)\, f_{i-1,j-1,k} & \text{if } i,j,k>0 \\ 0 & \text{otherwise} \end{cases} . \tag{5.3}$$

Hence, in this case, we define E^*_H by:

$$E^*_H = \sum_{(i,j,k)\in H} \phi^*_E(i,j,k) x^{i-1} y^j z^{k-1}.$$

Example 5.3. *In Example 4.6, we have:*

$$E^*_{h_1} = -\frac{1}{2} f_{011} y^2 - f_{101} xy + e_{200} x^2,$$

and

$$D_{h_2} = d_{011} yz, \quad E_{h_2} = E^*_{h_2} = 0, \quad F_{h_2} = f_{200} x^2.$$

Let us now describe the final shape of the generic polynomial g, not corresponding to concurrent lines, in terms of a, b, c and the coefficients of D, E, and F.

Lemma 5.4. *Let $H \in \mathcal{H}_1 \cup \mathcal{H}_2$ and let g_H be the polynomial determined by (4.1) using the forms D_H, E_H, and F_H given by (5.1). Then, we have*

$$n \cdot g_H = \sum_{(i,j,k)\in H} \big((c-b)\phi_D(i,j,k) + (a-c)\phi_E(i,j,k) + (b-a)\phi_F(i,j,k)\big) x^i y^j z^k. \tag{5.4}$$

Proof. By (4.1), we have

$$g = \frac{1}{n}\big((c-b)yzD + (a-c)xzE + (b-a)xyF\big).$$

By substituting D_H, E_H, F_H from (5.1) in place of D, E, F, we get:

$$\begin{aligned} n \cdot g_H = {} & (c-b) \sum_{(i,j,k)\in H} \phi_D(i,j,k) x^i y^j z^k + (a-c) \sum_{(i,j,k)\in H} \phi_E(i,j,k) x^i y^j z^k + \\ & (b-a) \sum_{(i,j,k)\in H} \phi_F(i,j,k) x^i y^j z^k \\ = {} & \sum_{(i,j,k)\in H} \big((c-b)\phi_D(i,j,k) + (a-c)\phi_E(i,j,k) + (b-a)\phi_F(i,j,k)\big) x^i y^j z^k, \end{aligned}$$

which proves the claim. ∎

Example 5.5. *In the case $n = 9$, an example of a maximal segment is*

$$h = \big\{(0,2,7), (1,3,5), (2,4,3), (3,5,1)\big\}.$$

From (5.4), we have that

$$\begin{aligned} g_h = {} & \frac{1}{9} y^2 z \Big(\big((a-c)e_{250} + (b-a)f_{241}\big)x^3y^3 + \big((a-c)e_{142} + (b-a)f_{133}\big)x^2y^2z^2 \\ & + \big((a-c)e_{034} + (b-a)f_{025}\big)xyz^4 + (c-b)d_{016}z^6\Big) . \end{aligned}$$

Note that here g is not reduced.

Remark 5.6. The polynomial g_H is a homogeneous polynomial in x, y, z of degree n and from the above expression, we see that the only monomials that appear in g_H are those of the form $x^i y^j z^k$ where $(i, j, k) \in H$; therefore, the set H in the triangular grid $\mathcal{T}'$ that expresses a solution of (4.8) can be used here with a different meaning: it represents, in the triangular grid $\mathcal{T}_n$ of degree n monomials, the monomials appearing in g_H.

Definition 5.7. *We set g_H^* to be the polynomial obtained by substituting D_H, E_H^*, F_H in* (3.1)*, and g_H^{**} to be the final value when also a, b, c are evaluated according to the conditions given by the ideal generated by $r(H)$.*

Let us analyze explicitly the two cases.

5.1. Case $H \in \mathcal{H}_1$.

Here, $H = \{(i, j, k)\}$. From (5.2), we see that

$$g_H = \frac{1}{n}\left((a-c)e_{i-1\,j\,k-1} + (b-a)f_{i-1\,j-1\,k}\right)x^i y^j z^k,$$

while

$$g_H^* = \left(\frac{(b-a)}{n} - \frac{k(a-c)}{j}\right) f_{i-1\,j-1\,k}x^i y^j z^k,$$

and

$$g_H^{**} = \left(\frac{(b-a)}{n} - \frac{(k-i)a}{j} - b\right) f_{i-1\,j-1\,k}x^i y^j z^k.$$

5.2. Case $H \in \mathcal{H}_2$.

Taking into account (5.1) and (5.3), if we denote by g_H^* the polynomial determined by (4.1), we get

$$\begin{aligned} n \cdot g_H^* = & \sum_{(0,j,k)\in H} (c-b)d_{0\,j-1\,k-1}y^j z^k + \sum_{(i,0,k)\in H} (a-c)e_{i-1\,0\,k-1}x^i z^k \\ & + \sum_{(i,j,0)\in H} (b-a)f_{i-1\,j-1\,0}x^i y^j + \sum_{\substack{(i,j,k)\in H \\ i,j,k>0}} \left((b-a) - \frac{k}{j}(a-c)\right) f_{i-1\,j-1\,k}x^i y^j z^k. \end{aligned}$$

Notice that the first three sums contribute by at most one monomial in x, y, z each.

The last formula shows that there are no relations among the coefficients of $g_H^\star$, so they can get any possible value. The greatest common divisor of the monomials $x^i y^j z^k$ for $(i, j, k) \in H$ is given by $x^u y^v z^w$, where u is the minimum of the integers i such that $(i, j, k) \in H$, v is the minimum of the integers j such that $(i, j, k) \in H$, and w is the minimum of the integers k such that $(i, j, k) \in H$. The restrictions we have on H imply that this greatest common divisor is never 1, hence g_H (and g_H^* and g_H^{**}) is always a multiple of x or of y or of z (which is trivially true in Section 5.1).

Moreover, we observe that the explicit expressions for g_H^{**} are quite involved, but since we are interested in squarefree polynomials, we shall determine them in the next section.

6. Selecting squarefree solutions.

Since we are interested only in square-free polynomials g, we have other restrictions to consider. If $H = \{(i, j, k)\}$, from (5.2), we see that g_H is never squarefree (unless $n = 3$), so the case in which H is a point (therefore, the whole set $\mathcal{H}_1$) can be set aside.

Hence, let us focus on the case where H is not a singleton. From now on, we assume $n \geq 4$.

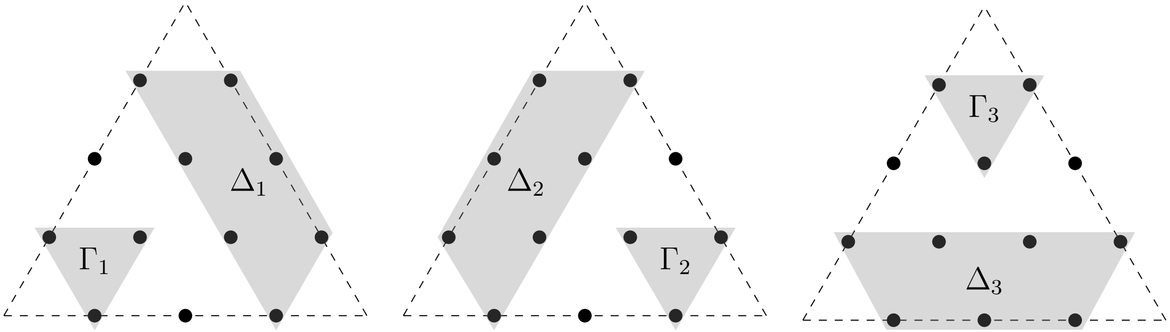

FIG. 5. *The sets Γ_i and Δ_i for $i \in \{1,2,3\}$.*

DEFINITION 6.1. *In $\mathcal{T}'$, we select these subsets:*

$$\begin{aligned}
\Gamma_1 &= \{(n-1,1,0),(n-1,0,1),(n-2,1,1)\},\\
\Gamma_2 &= \{(0,n-1,1),(1,n-1,0),(1,n-2,1)\},\\
\Gamma_3 &= \{(0,1,n-1),(1,0,n-1),(1,1,n-2)\},\\
\Delta_1 &= \{(i,j,k) \in \mathcal{T}' \mid i = 0,1\},\\
\Delta_2 &= \{(i,j,k) \in \mathcal{T}' \mid j = 0,1\},\\
\Delta_3 &= \{(i,j,k) \in \mathcal{T}' \mid k = 0,1\},
\end{aligned}$$

and we consider the set

$$\mathcal{H}_{\rm red} = \big\{H \in \mathcal{H}_2 \mid \text{ there exists } \ell \in \{1,2,3\} \text{ such that } H \cap \Gamma_\ell \neq \emptyset \text{ and } H \cap \Delta_\ell \neq \emptyset\big\}.$$

DEFINITION 6.2. *We denote the three maximal segments parallel to the three edges of the triangle $\mathcal{T}_n$ by:*

$$\begin{aligned}
L_1 &:= \{(1,n-1,0),\dots,(1,0,n-1)\},\\
L_2 &:= \{(n-1,1,0),\dots,(0,1,n-1)\},\\
L_3 &:= \{(n-1,0,1),\dots,(0,n-1,1)\}.
\end{aligned}$$

DEFINITION 6.3. *If P and Q are two points in $\mathcal{T}'$, we denote by $H(P,Q)$ the maximal segment of $\mathcal{T}'$ containing P and Q.*

LEMMA 6.4. *Let $P \in \Gamma_\ell$ and $Q \in \Delta_\ell$ for some $\ell \in \{1,2,3\}$. If $H(P,Q) \neq L_i$ for all $i \in \{1,2,3\}$, then P and Q are the extremes of $H(P,Q)$.*

Let us describe the points of $H(P,Q)$ when P and Q are its extremes. In this case, the elements of $H(P,Q)$ are the points R_λ given by

$$R_\lambda = P + \frac{\lambda}{d}(Q-P), \quad \lambda = 0,\dots,d,$$

where d is the greatest common divisor of the entries of the vector $Q-P$. From this and Lemma 6.4, we can easily describe the elements of $\mathcal{H}_{\rm red}$.

LEMMA 6.5. *If $H \in \mathcal{H}_2$ is such that g_H is squarefree, then $H \in \mathcal{H}_{\rm red}$.*

Proof. If $H \notin \mathcal{H}_{\mathrm{red}}$, then the greatest common divisor of the monomials of g_H contains x, or y, or z to a power greater than 1, so g_H cannot be squarefree. ∎

Hence, to describe the polynomials g that are squarefree, it is enough to consider maximal segments in $\mathcal{H}_{\mathrm{red}}$, which determine the triple $(a : b : c)$. For instance, if a segment H has an extreme in Γ_2 and an extreme in Δ_2, one has six possible cases, see Table 1. The other cases involved in $\mathcal{H}_{\mathrm{red}}$ can be treated similarly, and the resulting polynomials are essentially obtained by permuting the variables x, y, and z.

The following theorem, which provides a classification of reduced free curves with a linear Jacobian syzygy of type (3.3), is an immediate consequence of the inspection of the zero loci of polynomials appearing in Table 1.

Theorem 6.6. *If a polynomial g of degree $n \geq 4$ corresponds to a reduced free curve, not a set of concurrent lines, with a linear Jacobian syzygy of type (3.3), then the zero set of g falls in one of the following four families:*

Table 1

All six possible cases when we take a segment H with a vertex in Γ_2 and a vertex in Δ_2. If $P_2 \in \Gamma_2$ and $Q_2 \in \Delta_2$, let d be the greatest common divisor of the entries of the vector $Q_2 - P_2$. We define the vector $(\alpha, \beta, \gamma) := \frac{1}{d}(Q_2 - P_2)$. The polynomials T appearing in the table are bivariate homogeneous polynomials of degree d; hence, each $T(y^\beta, x^\alpha z^\gamma)$ admits a factorization of the form $T(y^\beta, x^\alpha z^\gamma) = \prod_{s=1}^{d}(\lambda_s y^\beta + \mu_s x^\alpha z^\gamma)$. The cases (2), $i = n-2$; (5), $i = n-1$; (6), $i = n-2$ are omitted, since sub-cases of (1), $i = n-1$. Similarly, the cases (4), (5), (6), $i = 1$ are also omitted, since sub-cases of (3), $i = 1$.

	(P_2, Q_2)	$(a : b : c)$		g_H
(1)	$\left(\begin{smallmatrix} 0 & n-1 & 1 \\ i & 0 & n-i \end{smallmatrix}\right)$, $1 \leq i \leq n-1$	$\begin{pmatrix} (n-i)(n-1) \\ i \\ -i(n-1) \end{pmatrix}$	$i = 1$:	$z\,(\lambda\, y^{n-1} + \mu\, x z^{n-2})$
			$i = n-1$:	$z\,T(x, y)$
			$1 < i < n-1$:	$z\,T(y^\beta, x^\alpha z^\gamma)$
(2)	$\left(\begin{smallmatrix} 0 & n-1 & 1 \\ i & 1 & n-i-1 \end{smallmatrix}\right)$, $1 \leq i \leq n-1$	$\begin{pmatrix} (n-i-1)(n-1)-1 \\ -i \\ -i(n-1) \end{pmatrix}$	$i = 1$:	$yz\,(\lambda\, y^{n-2} + \mu\, x\, z^{n-3})$
			$i = n-1$:	$y\,(\lambda\, y^{n-2} z + \mu\, x^{n-1})$
			$1 < i < n-2$:	$yz\,T(y^\beta, x^\alpha z^\gamma)$
(3)	$\left(\begin{smallmatrix} 1 & n-1 & 0 \\ i & 0 & n-i \end{smallmatrix}\right)$, $1 \leq i \leq n-1$	$\begin{pmatrix} (n-i)(n-1) \\ -(n-i) \\ -i(n-1) \end{pmatrix}$	$i = 1$:	$x\,T(y, z)$
			$i = n-1$:	$x\,T(\lambda y^{n-1} + \mu x^{n-2} z)$
			$1 < i < n-1$:	$x\,T(y^\beta, x^\alpha z^\gamma)$
(4)	$\left(\begin{smallmatrix} 1 & n-1 & 0 \\ i & 1 & n-i-1 \end{smallmatrix}\right)$, $0 \leq i \leq n-2$	$\begin{pmatrix} (n-i-1)(n-1) \\ -(n-i-1) \\ 1-i(n-1) \end{pmatrix}$	$i = 0$:	$y\,(\lambda x y^{n-2} + \mu z^{n-1})$
			$i = n-2$:	$xy\,(\lambda y^{n-2} + \mu x^{n-3} z)$
			$1 < i < n-2$:	$xy\,T(y^\beta, x^\alpha z^\gamma)$
(5)	$\left(\begin{smallmatrix} 1 & n-2 & 1 \\ i & 0 & n-i \end{smallmatrix}\right)$, $1 \leq i \leq n-1$	$\begin{pmatrix} (n-i)(n-2) \\ 2i-n \\ -i(n-2) \end{pmatrix}$	$1 < i < n-1$:	$xz\,T(y^\beta, x^\alpha z^\gamma)$
(6)	$\left(\begin{smallmatrix} 1 & n-2 & 1 \\ i & 1 & n-i-1 \end{smallmatrix}\right)$, $0 \leq i \leq n-1$	$\begin{pmatrix} (n-i-1)(n-2)-1 \\ 2i+1-n \\ 1-i(n-2) \end{pmatrix}$	$i = 0$:	$yz\,(\lambda x y^{n-3} + \mu z^{n-2})$
			$i = n-1$:	$xy\,(\lambda y^{n-3} z + \mu x^{n-2})$
			$1 < i < n-2$:	$xyz\,T(y^\beta, x^\alpha z^\gamma)$

- *union of a line with a set of concurrent lines;*
- *union of bitangent conics with at least one of their common tangent line;*
- *union of unicuspidal curves, belonging to a pencil, all with the same cusp and the same cuspidal tangent, with the common tangent line, and possibly a general line through the cusp;*
- *union of bicuspidal curves belonging to a pencil, all with the same two cusps, with at least one cuspidal tangent, and possibly the other tangent and/or the line passing through the two cusps.*

Note that, besides lines, all the irreducible components of the zero set of g *have the same degree.*

We conclude this section by computing the cardinality of $\mathcal{H}_{\mathrm{red}}$.

LEMMA 6.7. *The number of maximal segments* $H(P,Q) \subseteq \mathcal{H}_2$ *with* $P \in \Gamma_\ell$ *and* $Q \in \Delta_\ell$ *is* $6n-11$ *(here* $\ell = 1,2,3$*).*

Proof. We can consider the case $\ell = 1$. The other two cases are symmetric. We have that $\Delta_1 = D_a \cup D_b$, where $D_a = \{(0,j,n-j) \mid j = 1,\dots,n-1\}$ and $D_b = \{(1,j,n-j-1) \mid j = 0,\dots,n-1\}$. Let $P_1 = (n-1,1,0)$, then

$$U_1 = \{H(P_1,Q) \mid Q \in D_a\},$$

has $n-1$ maximal segments all contained in $\mathcal{H}_2$. The set

$$U_2 = \{H(P_1,Q) \mid Q \in D_b\},$$

has n maximal segments but contains $\{(n-1,1,0),\dots,(1,n-1,0)\}$, which is not in $\mathcal{H}_2$, so $U_2 \cap \mathcal{H}_2$ has $n-1$ elements. The sets U_1 and $U_2 \cap \mathcal{H}_2$ have the maximal segment L_2 in common. Overall, starting from the point P_1, we have therefore a set $X_1 = U_1 \cap U_2 \cap \mathcal{H}_2$ with $2n-3$ distinct maximal segments. We can do the same construction starting from $P_2 = (n-1,0,1)$, and we get a similar result, i.e., a set X_2 with $2n-3$ maximal segments contained in $\mathcal{H}_2$ (here, in place of L_2, we find L_3). Finally, if we take the point $P_3 = (n-2,1,1)$, we get that $\{H(P_3,Q) \mid Q \in D_a\}$ and $\{H(P_3,Q) \mid Q \in D_b\}$ again have $n-1$ and n elements, respectively, but now the two sets have the two maximal segments L_2 and L_3 in common. Hence, the corresponding set X_3 of maximal segments has again $2n-3$ elements. Finally, it holds

$$X_1 \cap X_2 = \emptyset, \quad X_1 \cap X_3 = \{L_1\}, \quad X_2 \cap X_3 = \{L_2\}.$$

Therefore, the number of maximal segments contained in $\mathcal{H}_2$ with one point in Γ_1 and another point in Δ_1 is $3(2n-3)-2 = 6n-11$. ☐

PROPOSITION 6.8. *The number of elements of* $\mathcal{H}_{\mathrm{red}}$ *is* $6(3n-8)$.

Proof. Let $\mathcal{M}_\ell$ be the set of maximal segments $H(P,Q)$ with $P \in \Gamma_\ell$ and $Q \in \Delta_\ell$, for $\ell = 1,2,3$. From Lemma 6.7, we know that each $\mathcal{M}_l$ has $6n-11$ elements. To compute the cardinality of $\mathcal{H}_{\mathrm{red}} = \mathcal{M}_1 \cup \mathcal{M}_2 \cup \mathcal{M}_3$, we determine the common elements between $\mathcal{M}_\ell$ and $\mathcal{M}_m$. Take, for instance, $\ell = 1$ and $m = 2$. Then,

$$\mathcal{M}_1 \cap \mathcal{M}_2 = \big\{H(P,Q) \in \mathcal{H}_2 \mid P \in \Gamma_1 \cap \Delta_2 \text{ and } Q \in \Delta_1 \cap \Gamma_2\big\}.$$

Let $P_1 = (n-1,1,0)$, $P_2 = (n-1,0,1)$, and $P_3 = (n-2,1,1)$ be the three points of Γ_1 and let P_1', P_2', P_3' be the corresponding points of Γ_2. Note that P_1, P_3, P_1', P_3' are aligned, so $H(P_1,P_1') = H(P_1,P_3') = \cdots = H(P_3,P_3')$. Hence, from these points, we get a single element of $\mathcal{M}_1 \cap \mathcal{M}_2$. The other elements of this intersection are four and are given by:

$$H(P_1,P_2'), \quad H(P_2,P_1'), \quad H(P_2,P_3'), \quad H(P_3,P_2').$$

Note that $H(P_2,P_2')$ is not considered, since it is not in $\mathcal{H}_2$. From this computation, we see that the number of elements of $\mathcal{H}_{\mathrm{red}}$ is $3(6n-11) - 3 \cdot 5 = 6(3n-8)$. ☐

7. The case $A = y$, $B = z$, $C = 0$. In the case $A = y$, $B = z$, and $C = 0$, the existence of a linear Jacobian syzygy is sufficient to obtain a classification of the corresponding curves, and the only two possibilities for the zero set of the polynomial g are the so-called *Płoski curves* (see [7]), namely:

- a union of conics, belonging to a hyperosculating pencil, if n is even;
- a union of conics, belonging to a hyperosculating pencil, together with their common tangent line, if n is odd.

Such curves are all free by [5, Example 3.2 (2)], and they are precisely the curves attaining both the local and global bounds on the *Milnor number*, see [21] and [22].

The discussion in this case can be carried in the same spirit of Section 4. Since the result is already proved via other techniques in [20, Proposition 4.4], we only give a sketch of the proof for readability reasons.

We focus on the case when n is even; the proof when n is odd goes similarly. Let

$$g = \sum_{i+j+k=n} g_{ijk} x^i y^j z^k,$$

be a homogeneous, degree n polynomial in x, y, z.

To simplify the notations, a point of $\mathcal{T}_n$ (i.e., a monomial of degree n in x, y, z) will be denoted by (i, j) instead of $(i, j, n-i-j)$, and the coefficient $g_{ij\,n-i-j}$ of g will be denoted by g_{ij}. Let

$$v := y\partial_x g + z\partial_y g.$$

The polynomial v is homogeneous of degree n. Let $c(i, j)$ denote the coefficient in v of the monomial $x^i y^j z^{n-i-j}$. Then,

$$c(i,j) = \begin{cases} (j+1)g_{ij+1} + (i+1)g_{i+1\,j-1} & \text{if } j \geq 1 \text{ and } k \geq 1, \\ (i+1)g_{i+1\,j-1} & \text{if } j \geq 1 \text{ and } k = 0, \\ g_{ij+1} & \text{if } j = 0 \text{ and } k \geq 1. \end{cases} \tag{7.1}$$

We determine the conditions on the coefficients of g that make v vanish. Therefore, we solve the linear system in g_{ij}:

$$c(i,j) = 0 \quad \text{for } (i,j) \in \mathcal{T}_n\,. \tag{7.2}$$

The following maximal segments of $\mathcal{T}_n$:

$$\begin{aligned} \mathcal{A}_j &= \bigl\{(0,j) + t(1,-2) \mid t \in \mathbb{N},\ t \leq j/2\bigr\}, && \text{for } j = 0, \dots, n-1 \\ \mathcal{B}_j &= \bigl\{(n-j,j) + t(1,-2) \mid t \in \mathbb{N},\ t \leq j/2\bigr\}, && \text{for } j = 0, \dots, n, \end{aligned}$$

form a partition of $\mathcal{T}_n$, so (7.2) is equivalent to

$$c(i,j) = 0 \quad \text{for } (i,j) \in \mathcal{A}_j \text{ for } j = 0, \dots, n-1, \tag{7.3}$$

$$c(i,j) = 0 \quad \text{for } (i,j) \in \mathcal{B}_j \text{ for } j = 0, \dots, n. \tag{7.4}$$

Moreover, from (7.1), it is possible to see that for every coefficient $g_{h\ell}$ of g, there exists one and only one index j such that $g_{h\ell}$ appears in (7.3) and (7.4), so they can be solved independently.

As an example, consider the linear system $c(i,j)=0$ for $(i,j)\in A_6$ (assuming n is big enough). Since A_6 is given by

$$(0,6),(1,4),(2,2),(3,0)\,,$$

from (7.1), we get the system of equations:

$$\begin{cases} 7g_{07}+\ \ g_{15}=0\\ 5g_{15}+2g_{23}=0\\ 3g_{23}+3g_{31}=0\\ \ g_{31}\qquad\quad\ =0\end{cases}.$$

The only solution of this system is the zero solution.

With a computation very similar to the one of the example above, we see that all the maximal segments $\mathcal{B}_j$ and all the maximal segments $\mathcal{A}_j$ when j is even give that the involved coefficients of g are zero; therefore, the polynomial g becomes:

$$\bar{g}=\sum_{j=1,3,5,\dots} h_j \quad \text{where} \quad h_j=\sum_{(i,j)\in\mathcal{A}_j} g_{ij}x^iy^jz^{n-i-j},$$

and g_{ij} satisfies (7.3) for j odd.

Here are the first cases of (7.3):

$j=1$	$2g_{02}+g_{10}=0$
$j=3$	$4g_{04}+g_{12}=0,\ 2g_{12}+2g_{20}=0$
$j=5$	$6g_{06}+g_{14}=0,\ 4g_{14}+2g_{22}=0,\ 2g_{22}+3g_{30}=0$
$j=7$	$8g_{08}+g_{16}=0,\ 6g_{16}+2g_{24}=0,\ 4g_{24}+3g_{32}=0,\ 2g_{32}+4g_{40}=0$

The solutions of these equations are (up to a scalar factor):

$j=1$	$g_{02}=1,\ g_{10}=-2$
$j=3$	$g_{04}=1,\ g_{12}=-4,\ g_{20}=4$
$j=5$	$g_{06}=1,\ g_{14}=-6,\ g_{22}=12,\ g_{30}=-8$
$j=7$	$g_{08}=1,\ g_{16}=-8,\ g_{24}=24,\ g_{32}=-32, g_{40}=16$

Therefore,

$$\begin{aligned} h_1 &= g_{02}z^{n-2}\left(y^2-2xz\right)\\ h_3 &= g_{04}z^{n-4}\left(y^4-4xy^2z+4x^2z^2\right)\\ &= g_{04}z^{n-4}\left(y^2-2xz\right)^2\\ h_5 &= g_{06}z^{n-6}\left(y^6-6xy^4z+12x^2y^2z^2-8x^3z^3\right)\\ &= g_{06}z^{n-6}\left(y^2-2xz\right)^3\\ h_7 &= g_{08}z^{n-8}\left(y^8-8xy^6z+24x^2y^4z^2-32x^3y^2z^3-16x^4z^4\right)\\ &= g_{08}z^{n-8}\left(y^2-2xz\right)^4, \end{aligned}$$

and these expressions suggest the general form for h_j. In conclusion, since $n = 2m$ is even, if we set $q = y^2 - 2xz$, the polynomial g becomes:

$$\bar{g} = \sum_{\ell=1}^{m} \lambda_\ell q^\ell z^{m-\ell},$$

where $\lambda_\ell = g_{0\,2\ell}$ are free parameters. Hence, $\bar{g}$ is a product of conics.

8. Applications and final comments.

8.1. Comparison with related results. We compare our results with [6, Theorem 3.5], where a characterization of the syzygy matrices of free curves with a Jacobian syzygy of the type (ax, by, cz) is given. The authors prove that if $abc \neq 0$, then the existence of the syzygy (ax, by, cz) guarantees that the curve is free, and a syzygy matrix is given by

$$\begin{pmatrix} ax & \left(\frac{1}{c} - \frac{1}{b}\right) \ (n+2)^{-1}\partial_{yz} g \\ by & \left(\frac{1}{a} - \frac{1}{c}\right) \ (n+2)^{-1}\partial_{xz} g \\ cz & \left(\frac{1}{b} - \frac{1}{a}\right) \ (n+2)^{-1}\partial_{xy} g \end{pmatrix}.$$

If $c = 0$ and $ab \neq 0$, the authors prove that the curve is free if and only if $\partial_z f \in (x, y)$.

Our results describe precisely the form of the second column of the syzygy matrix, in particular with respect to the support of the polynomials appearing there.

8.2. Free curves with a linear Jacobian syzygy and quasi-homogeneous singularities. As an application of our results, we can characterize the free curves with a linear Jacobian syzygy and only quasi-homogeneous singularities. We recall the following definition.

Definition 8.1. *An isolated singularity of a hypersurface is* quasi-homogeneous *if and only if there exists a holomorphic change of variables so that a local defining equation is weighted homogeneous. Recall that* $f(y_1, \ldots, y_n) = \sum c_{i_1\cdots i_n} y_1^{i_1} \cdots y_n^{i_n}$ *is said to be* weighted homogeneous *if there exist* n *rational numbers* $\alpha_1, \cdots, \alpha_n$ *such that* $\sum c_{i_1\cdots i_n} y_1^{i_1\alpha_1} \cdots y_n^{i_n\alpha_n}$ *is homogeneous.*

It turns out that the quasi-homogeneity can be detected from a first syzygy matrix M by [3, Theorem 3.3], for the case of free and nearly free curves and [2, Theorem 4.8] for hypersurfaces with isolated singularities:

Theorem 8.2. *Let $C = V(g)$ be a reduced plane curve, let M be a first syzygy matrix in a resolution of the Jacobian singular scheme, and let $p \in Sing(C)$. Then, it holds*

$$p \textit{ is a quasi-homogeneous singularity} \iff rk\, M(p) \geq 1.$$

Remark 8.3. Some related results have been given also in [23, Theorem 2.2], where the author proves that a plane curve with only isolated quasi-homogeneous singularities is free if and only if it admits a Jacobian syzygy given by a regular sequence. Moreover, Hilbert–Burch matrices with columns given by nonregular sequences have been investigated in [18].

Let us apply the criterion of Theorem 8.2 to the case of free curves with a linear Jacobian syzygy.

A necessary condition for non-quasi-homogeneity is clearly given by the linear dependence of the three linear entries of M. By the assumption of working with a Jordan normal form and by the discussion in

Lemma 3.4, this may occur only in the general (ax, by, cz) case with either $b = 0$ or $c = 0$, since we always assume $a \neq 0$, or in the last case $(y, z, 0)$. It is well known, see [21], that in the latter case all the curves have only one singular point, which is indeed a non-quasi-homogeneous singularity if $n \geq 4$.

In the general case, we analyze the cases listed in Table 1, and we see that $bc = 0$ if and only if

1. n is even and $i = \frac{n}{2}$ in case (5), or
2. n is odd and $i = \frac{n-1}{2}$ in case (6).

In case (1), we have

$$d = \gcd\left(\frac{n-2}{2}, -(n-2), \frac{n-2}{2}\right) = \frac{n-2}{2}, \quad (\alpha, \beta, \gamma) = (1, 2, 1),$$

$$H = \big\{(1, n-2, 1), (2, n-4, 2), (3, n-6, 3), \dots, (n/2, 0, n/2)\big\},$$

and a first syzygy matrix is given by

$$M = \begin{pmatrix} \frac{n}{2}(n-2)x & 0 \\ 0 & E \\ -\frac{n}{2}(n-2)z & F \end{pmatrix},$$

where

$$E = e_{0\,n-2\,0}y^{n-2} + e_{1\,n-4\,1}xy^{n-4}z + \dots + e_{\frac{n-2}{2}\,0\,\frac{n-2}{2}}x^{\frac{n-2}{2}}z^{\frac{n-2}{2}},$$

and

$$F = -(n-2)e_{0\,n-2\,0}\ y^{n-3}z - \frac{n-4}{2}e_{1\,n-4\,1}xy^{n-5}z^2 - \dots - (n-2)e_{\frac{n-4}{2}\,2\,\frac{n-4}{2}}x^{\frac{n-4}{2}}yz^{\frac{n-2}{2}}.$$

We see that $M(0,1,0) = 0$ if and only if $e_{0\,n-2\,0} = 0$; in this case, we have $E = xz\widetilde{E}$ and $F = xz^2\widetilde{F}$ for suitable polynomials $\widetilde{E}$ and $\widetilde{F}$. The resulting polynomial g would be of the type

$$n\,g = \det\begin{pmatrix} x & \frac{n}{2}(n-2)x & 0 \\ y & 0 & xz\widetilde{E} \\ z & -\frac{n}{2}(n-2)z & xz^2\widetilde{F} \end{pmatrix} = n(n-2)x^2z^2\widetilde{E} - \frac{n}{2}(n-2)x^2yz^2\widetilde{F},$$

which gives a nonreduced curve.

Similarly, in case (2), a first syzygy matrix is given by

$$M = \begin{pmatrix} (\frac{n-1}{2}(n-2)-1)x & 0 \\ 0 & E \\ 1 - \frac{n-1}{2}(n-2)z & F \end{pmatrix}.$$

As before, if $M(0,1,0) = 0$, then the resulting polynomial g is not square-free.

Thus, we conclude that all the singularities of the reduced free curves with a syzygy (ax, by, cz) are always quasi-homogeneous. We summarize our conclusions as follows.

THEOREM 8.4. *If a reduced free curve with a linear Jacobian syzygy has some non quasi-homogeneous singularity, then it is either an even or an odd Płoski curve.*

We point out that the knowledge of the precise form of the Hilbert–Burch matrix is crucial for the previous result.

8.3. Classification of nearly free curves with a linear Jacobian syzygy. By a result of Du Plessis and Wall, see [16, Proposition 1.1], the reduced plane curves admitting a linear Jacobian syzygies are exactly those having a positive dimensional automorphism group. The latter have been classified in [1, Section 1]. Our results allow one to detect precisely the free curves.

Indeed, the minimal degree of a Jacobian syzygy is related to the *total Tjurina number* $\tau(g)$, that is, the degree of the zero-dimensional singular Jacobian scheme, and we have the following result (see [9, Theorem 1.2]):

Lemma 8.5. *Let* $C = V(g)$ *be a reduced singular curve in* $\mathbb{P}^2$ *of degree* $n \geq 3$ *that is not a union of concurrent lines. If* g *admits a linear Jacobian syzygy, then* $n^2 - 3n + 2 \leq \tau(C) \leq n^2 - 3n + 3$.

Moreover, we have $\tau(C) = n^2-3n+3$, *respectively,* $\tau(C) = n^2-3n+2$, *if and only if* g *is free, respectively, is nearly free — that is, the module* $\mathrm{Syz}(J_g)$ *is generated by three syzygies of degrees* $(1, n-1, n-1)$.

As a consequence, our results can be used to give a classification of nearly free curves with a linear Jacobian syzygy, which we now list.

Corollary 8.6. *If a polynomial* g *of degree* $n \geq 4$ *corresponds to a reduced nearly free curve with a linear Jacobian syzygy, then up to a projectivity, such a syzygy is of the type* (ax, by, cz), *and the zero set of* g *falls in one of the following families:*

- *a union of bitangent conics belonging to a bitangent pencil, and possibly the secant line through the two bitangency points;*
- *a union of unicuspidal curves belonging to a pencil, all with the same cusp and the same cuspidal tangent, and possibly a general line through the cusp;*
- *a union of bicuspidal curves belonging to a pencil, all with the same two cusps, and possibly the line passing through the two cusps.*

Remark 8.7. By [6, Theorem 3.5], in the above cases, we have $abc = 0$; this is indeed confirmed by the classification of nearly free conic-line arrangements with a linear Jacobian syzygy, given in [5, Example 3.3].

8.4. Further investigation. While our approach allows a complete description in the case of a linear Jacobian syzygy, it seems hardly extendable to free curves with a higher degree first Jacobian syzygy, due to the lack of a simultaneous normal form. However, our technique could be extended to higher dimensions and it deserves further investigation.

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