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ON MULTIPOINT BOUNDARY VALUE PROBLEMS FOR INDEX-2 LINEAR SINGULAR DIFFERENCE EQUATIONS*

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Abstract. On the background of a careful analysis of index-2 linear singular difference equations with both constant and varying coefficients cases, multipoint boundary value problems for these equations are considered. Necessary and sufficient conditions for the solvability of multipoint boundary value problems are established. Further, general solution formulae are explicitly constructed.

Key words. Index, Matrix pencil, Linear singular difference equations, Multipoint boundary value problems.

AMS subject classifications. 15A06, 39A05, 39A10.

1. Introduction. In recent years, there has been considerable interest in studying linear singular difference equations (LSDEs) of the form

$$A_n x_{n+1} = B_n x_n + q_n, \quad n \geq 0, \tag{1.1}$$

where $A_n,\ B_n \in \mathbb{R}^{m\times m}$, $q_n \in \mathbb{R}^m$ are given and $\text{rank}A_n = r$ $(1 \leq r \leq m-1)$ for all $n \geq 0$ (see [2]–[8] and references therein). The index notion of a matrix pencil was introduced to investigate Eq. (1.1) with constant coefficients. Further, the solvability of initial value problems (IVPs) has been studied thoroughly [4]–[6]. However, as far as we know the qualitative questions such as the existence, uniqueness, etc. of multipoint boundary value problems (MPBVPs) for (1.1) with constant coefficients have not been discussed. In the varying coefficients case, the index-1 concept of Eq. (1.1) was also introduced in [2, 8] and the solvability of IVPs as well as MPBVPs for index-1 LSDEs has been considered in [2, 3, 8]. Later on, the index-2 concept of Eq. (1.1) has been proposed, and basing on this index-2 notion, the condition of solvability as well as the solution formula of IVPs for index-2 LSDE (1.1) have been established in [7]. As discussed in [7], many valid results for index-1 case can be extended to index-2 case, however, the extension meets with some difficulties.

The main goal of this paper is studying MPBVPs for index-2 LSDE (1.1) in both constant and varying coefficients cases. The index-2 of a matrix pencil and index-2

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of Eq. (1.1) turn to be the keystone in the analysis of MPBVPs. For index-2 LSDEs with constant coefficients, similarly as in [4]–[6], one can solve Eq. (1.1) by means of index of a matrix pencil and Drazin inverse. It is well known that many results for constant coefficients LSDEs cannot be directly generalized to varying coefficients LSDEs (ref. [2, 3, 7, 8]). Thus, in the varying coefficients case, our approach to LSDEs is based on index-2 notion of Eq. (1.1) and projections. We shall develop some techniques of index-1 LSDEs in [3, 8] for index-2 LSDEs.

The paper is organized as follows. In Section 2 we recall some definitions and preliminary results, as well as give some simple results concerning index-2 LSDE (1.1). Necessary and sufficient conditions for the solvability and a general formula solution of MPBVPs for index-2 LSDE (1.1) will be established in Section 3.

2. Preliminaries. We start this section by recalling the Drazin inverse of a matrix and the index notion of a matrix pencil, which have been studied in [4, 6]. Firstly, if $M \in \mathbb{R}^{m\times m}$, the index of M, denoted by $\mathrm{ind}(M)$, is the least non-negative integer ν such that $\ker M^{\nu} = \ker M^{\nu+1}$. It is worth noting that the following theorem plays an important role to study autonomous LSDEs.

Theorem 2.1. [4] *Suppose that $M \in \mathbb{R}^{m\times m}$, $\mathrm{ind}(M) = \nu$ and $\mathrm{rank} M^{\nu} = t$. Then there exists a nonsingular matrix $S \in \mathbb{R}^{m\times m}$ such that*

$$M = S \begin{bmatrix} W & 0 \\ 0 & N \end{bmatrix} S^{-1}, \tag{2.1}$$

where W is a nonsingular $t \times t$ matrix and N is a nilpotent $(m-t)\times(m-t)$ matrix with $\nu=\mathrm{ind}(N)$.

If $M \in \mathbb{R}^{m\times m}$ is given in the form (2.1), then the Drazin inverse of M, denoted by M^D, is defined by

$$M^D = S \begin{bmatrix} W^{-1} & 0 \\ 0 & 0 \end{bmatrix} S^{-1}.$$

It is easy to verify that

$$MM^D = M^D M, \quad M^D M M^D = M^D, \quad M^{k+1}M^D = M^k \text{ for } k \geq \ \mathrm{Ind}(M)$$

and the Drazin inverse is unique.

In what follows, we consider $A, B \in \mathbb{R}^{m\times m}$ and always assume that the matrix pencil (A, B) is regular (i.e., there exists a scalar $\lambda \in \mathbb{C}$ such that $\lambda A + B$ is nonsingular) and let $\widehat{A}_\lambda := (\lambda A + B)^{-1}A$, $\widehat{B}_\lambda := (\lambda A + B)^{-1}B$, $\widehat{f}_\lambda := (\lambda A + B)^{-1} f$ for $f \in \mathbb{R}^m$. Observe that $\widehat{B}_\lambda = I - \lambda \widehat{A}_\lambda$, hence, $\widehat{A}_\lambda$ and $\widehat{B}_\lambda$ commute.

Theorem 2.2. [4] *Suppose that the matrix pencil (A, B) is regular and $f \in \mathbb{R}^m$.*

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Then for all α, $\beta \in \mathbb{C}$ for which $(\alpha A+B)^{-1}$ and $(\beta A+B)^{-1}$ exist, the following statements hold:

(i) $\operatorname{ind}(\widehat{A}_\alpha) = \operatorname{ind}(\widehat{A}_\beta)$,
(ii) $\widehat{A}_\alpha \widehat{A}_\alpha^D = \widehat{A}_\beta \widehat{A}_\beta^D$,
(iii) $\widehat{A}_\alpha^D \widehat{B}_\alpha = \widehat{A}_\beta^D \widehat{B}_\beta$ *and* $\widehat{B}_\alpha^D \widehat{A}_\alpha = \widehat{B}_\beta^D \widehat{A}_\beta$,
(iv) $\widehat{A}_\alpha^D \widehat{f}_\alpha = \widehat{A}_\beta^D \widehat{f}_\beta$ *and* $\widehat{B}_\alpha^D \widehat{f}_\alpha = \widehat{B}_\beta^D \widehat{f}_\beta$.

If (A,B) is regular and $\det(\lambda A+B)\neq 0$, then $\operatorname{ind}(\widehat{A}_\lambda)$ is called the index of the pencil (A,B), denoted by $\operatorname{ind}(A,B)$, i.e., $\operatorname{ind}(A,B) := \operatorname{ind}(\widehat{A}_\lambda)$. Theorem 2.2 guarantees that the definition of the index of the matrix pencil does not depend on the chosen value λ.

Next, to study the index-2 LSDE (1.1) with variable coefficients, we start with some basic definitions for non-autonomous LSDEs (see [2, 3, 8, 7]). Let Q_n be any projection onto $\ker A_n$ and $T_n \in \mathrm{GL}(\mathbb{R}^m)$ for all $n\geq 0$ such that $T_n|_{\ker A_n}$ is an isomorphism from $\ker A_n$ onto $\ker A_{n-1}$, here we put $A_{-1} := A_0$. Denote again by T_n the matrix induced by the operator T_n.

Lemma 2.3. [7] *The matrix $G_n := A_n + B_n T_n Q_n$ is nonsingular if and only if*

$$\ker A_{n-1}\cap S_n = \{0\},$$

where $S_n := \{z\in\mathbb{R}^m : B_n z \in \operatorname{im} A_n\}$.

Definition 2.4. [7] The LSDE (1.1) is said to be of index-1 if

(i) $\operatorname{rank} A_n \equiv r$,
(ii) $\ker A_{n-1}\cap S_n = \{0\}$.

Now we suppose that the matrices G_n are singular for all $n\geq 0$, i.e., Eq. (1.1) is of higher index. Put $P_n := I - Q_n$ for all $n\geq 0$ and let A_n^+ denote the Moore-Penrose generalized inverse of A_n.

Lemma 2.5. [7] *The following relation*

$$(T_n + T_n P_n A_n^+ B_n T_n Q_n)\ker G_n = \ker A_{n-1}\cap S_n$$

is valid.

It is worth noting that the matrices $(T_n + T_n P_n A_n^+ B_n T_n Q_n)$ are nonsingular for all $n\geq 0$, consequently, we come to the following corollary.

Corollary 2.6. [7] $\dim(\ker G_n) = \dim(\ker A_{n-1}\cap S_n)$, $\forall n\geq 0$.

Lemma 2.7. [7] *Let Q_n, $\widetilde{Q}_n$ be two projections onto* $\ker A_n$ *and* T_n, $\widetilde{T}_n \in \mathrm{GL}(\mathbb{R}^m)$ *such that* $T_n|_{\ker A_n}$, $\widetilde{T}_n|_{\ker A_n}$ *are two isomorphisms between* $\ker A_n$ *and* $\ker A_{n-1}$. *Put*

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$G_n := A_n + B_n T_n Q_n$, $\widetilde{G}_n := A_n + B_n \widetilde{T}_n \widetilde{Q}_n$ *and*

$$S_{1,n} := \{z \in \mathbb{R}^m : B_n P_{n-1} z \in \mathrm{im} G_n\},\ \widetilde{S}_{1,n} := \{z \in \mathbb{R}^m : B_n \widetilde{P}_{n-1} z \in \mathrm{im}\widetilde{G}_n\}.$$

Then, the following relations hold:

$$\widetilde{G}_n = G_n(P_n + T_n^{-1}\widetilde{T}_n\widetilde{Q}_n),\ \forall n \ge 0, \tag{2.2}$$

$$\widetilde{S}_{1,n} = (\widetilde{P}_{n-1} + \widetilde{T}_{n-1}^{-1}T_{n-1}Q_{n-1})S_{1,n},\ \forall n \ge 0, \tag{2.3}$$

$$\ker\widetilde{G}_n \cap \widetilde{S}_{1,n+1} = (\widetilde{P}_n + \widetilde{T}_n^{-1}T_nQ_n)(\ker G_n \cap S_{1,n+1}),\ \forall n \ge 0. \tag{2.4}$$

Remark that the identity (2.4) ensures that the following definition does not depend on the choice of the projections onto $\ker A_n$ and the isomorphisms between $\ker A_n$ and $\ker A_{n-1}$. For well-definedness, we put $G_{-1} := G_0$.

Definition 2.8. [7] The LSDE (1.1) is said to be of index-2 if the following conditions

(i) $\mathrm{rank} A_n \equiv r, \quad 1 \le r \le m-1$,
(ii) $\dim(\ker A_{n-1} \cap S_n) \equiv m - s, \quad 1 \le s \le m-1$,
(iii) $\ker G_{n-1} \cap S_{1,n} = \{0\}$

hold for all $n \ge 0$.

From Corollary 2.6, we get that $\mathrm{rank} G_n$ does not depend on the choice of the projections onto $\ker A_n$ and the isomorphisms between $\ker A_n$ and $\ker A_{n-1}$, hence we can suppose that $\mathrm{rank} G_n \equiv s$, $1 \le s \le m-1$. Here, $Q_{1,n}$ denotes a projection onto $\ker G_n$ and let $T_{1,n}$ be a nonsingular operator with the restriction $T_{1,n}|_{\ker G_n}$ is an isomorphism between $\ker G_n$ and $\ker G_{n-1}$. We also denote again by $T_{1,n}$ the matrix induced by the operator $T_{1,n}$.

Lemma 2.9. [7] *The matrix* $G_{1,n} := G_n + B_n P_{n-1} T_{1,n} Q_{1,n}$ *is nonsingular if and only if*

$$\ker G_{n-1} \cap S_{1,n} = \{0\}.$$

Moreover, if $G_{1,n}$ *is nonsingular then*

$$\widehat{Q}_{1,n-1} := T_{1,n} Q_{1,n} G_{1,n}^{-1} B_n P_{n-1}$$

is a projection from $\mathbb{R}^m$ *onto* $\ker G_{n-1}$ *along* $S_{1,n}$.

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Recall that notations $\ker\widetilde{G}_n$ and $\widetilde{S}_{1,n+1}$ have been introduced in Lemma 2.7. We now come to the following lemma which states the relationship between projections $\widehat{Q}_{1,n}$ and $\widehat{\widetilde{Q}}_{1,n}$.

Lemma 2.10. *Suppose that* LSDE (1.1) *is of index-2 and let* $\widehat{\widetilde{Q}}_{1,n}$ *be a projection from* $\mathbb{R}^m$ *onto* $\ker\widetilde{G}_n$ *along* $\widetilde{S}_{1,n+1}$. *Then the following relation holds:*

$$\widehat{\widetilde{Q}}_{1,n} = (\widetilde{P}_n + \widetilde{T}_n^{-1}T_nQ_n)\widehat{Q}_{1,n}. \tag{2.5}$$

Proof. Putting

$$\bar{Q}_{1,n} := (\widetilde{P}_n + \widetilde{T}_n^{-1}T_nQ_n)\widehat{Q}_{1,n}(P_n + T_n^{-1}\widetilde{T}_n\widetilde{Q}_n)$$

and noting that $(P_n + T_n^{-1}\widetilde{T}_n\widetilde{Q}_n)(\widetilde{P}_n + \widetilde{T}_n^{-1}T_nQ_n) = I$, $\widehat{Q}_{1,n}^2 = \widehat{Q}_{1,n}$, we obtain $\bar{Q}_{1,n}^2 = \bar{Q}_{1,n}$, i.e., $\bar{Q}_{1,n}$ is a projection.

Applying the relation (2.2) and observing that $G_n\widehat{Q}_{1,n} = 0$, we have

$$\widetilde{G}_n\bar{Q}_{1,n} = G_n\widehat{Q}_{1,n}(P_n + T_n^{-1}\widetilde{T}_n\widetilde{Q}_n) = 0.$$

On the other hand, let $x \in \mathbb{R}^m$ such that $\bar{Q}_{1,n}x = 0$, or equivalently, $\widehat{Q}_{1,n}(P_n + T_n^{-1}\widetilde{T}_n\widetilde{Q}_n)x = 0$. Since $\widehat{Q}_{1,n}$ is the projection onto $\ker G_n$ along $S_{1,n+1}$, it follows $(P_n + T_n^{-1}\widetilde{T}_n\widetilde{Q}_n)x \in S_{1,n+1}$. This leads to $x \in (\widetilde{P}_n + \widetilde{T}_n^{-1}T_nQ_n)S_{1,n+1}$. Hence, using the relation (2.3), we get $x \in \widetilde{S}_{1,n+1}$. Thus, $\bar{Q}_{1,n}$ is a projection onto $\ker\widetilde{G}_n$ along $\widetilde{S}_{1,n+1}$ meaning that

$$\widehat{\widetilde{Q}}_{1,n} = (\widetilde{P}_n + \widetilde{T}_n^{-1}T_nQ_n)\widehat{Q}_{1,n}(P_n + T_n^{-1}\widetilde{T}_n\widetilde{Q}_n).$$

Furthermore, observing that $\widehat{Q}_{1,n} = T_{1,n+1}Q_{1,n+1}G_{1,n+1}^{-1}B_{n+1}P_n$ and $T_n^{-1}\widetilde{T}_n\widetilde{Q}_n = Q_nT_n^{-1}\widetilde{T}_n\widetilde{Q}_n$ yields

$$\widehat{Q}_{1,n}(P_n + T_n^{-1}\widetilde{T}_n\widetilde{Q}_n) = \widehat{Q}_{1,n}.$$

Thus, we obtain Eq. (2.5). ◻

From now on, we put $P_{1,n} := I - Q_{1,n}$ and $\widehat{P}_{1,n} := I - \widehat{Q}_{1,n}$.

Lemma 2.11. [7] *Suppose that the* LSDE (1.1) *is of index-2 and* $\widehat{G}_{1,n} := G_n + B_nP_{n-1}T_{1,n}\widehat{Q}_{1,n}$. *Then the following relations hold:*

$$\widehat{G}_{1,n}^{-1}G_n = \widehat{P}_{1,n}, \quad \widehat{G}_{1,n}^{-1}A_n = \widehat{P}_{1,n}P_n, \tag{2.6}$$

$$\widehat{G}_{1,n}^{-1}B_n = \widehat{G}_{1,n}^{-1}B_nP_{n-1}\widehat{P}_{1,n-1} + T_{1,n}^{-1}\widehat{Q}_{1,n-1} + \widehat{P}_{1,n}T_n^{-1}Q_{n-1}. \tag{2.7}$$

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Suppose that the LSDE (1.1) is of index-2. We also introduce an operator $\widetilde{T}_{1,n} \in \mathrm{GL}(\mathbb{R}^m)$ whose restriction $\widetilde{T}_{1,n}|_{\ker \widetilde{G}_n}$ is an isomorphism between $\ker\widetilde{G}_n$ and $\ker\widetilde{G}_{n-1}$. Put $\widehat{\widetilde{G}}_{1,n} := \widetilde{G}_n + B_n\widetilde{P}_{n-1}\widetilde{T}_{1,n}\widehat{\widetilde{Q}}_{1,n}$. A similar result of the relation (2.2) can be established for index-2 LSDEs, namely, we obtain the following lemma.

LEMMA 2.12. *Let the* LSDE (1.1) *be of index-2. Then the identity*

$$\widehat{\widetilde{G}}_{1,n} = \widehat{G}_{1,n}\Big(P_n + T_n^{-1}\widetilde{T}_n\widetilde{Q}_n + T_n^{-1}Q_{n-1}\widetilde{P}_{n-1}\widetilde{T}_{1,n}\widehat{\widetilde{Q}}_{1,n} + T_{1,n}^{-1}(P_{n-1} + T_{n-1}^{-1}\widetilde{T}_{n-1}\widetilde{Q}_{n-1})\widetilde{T}_{1,n}\widehat{\widetilde{Q}}_{1,n} - \widehat{Q}_{1,n}\Big) \tag{2.8}$$

is valid for each $n \geq 0$.

Proof. Since $T_n^{-1}Q_{n-1} = Q_nT_n^{-1}Q_{n-1}$ and $P_{n-1}\widetilde{P}_{n-1} = P_{n-1}$, we have that

$$G_nT_n^{-1}Q_{n-1}\widetilde{P}_{n-1}\widetilde{T}_{1,n}\widehat{\widetilde{Q}}_{1,n} = B_n\widetilde{P}_{n-1}\widetilde{T}_{1,n}\widehat{\widetilde{Q}}_{1,n} - B_nP_{n-1}\widetilde{T}_{1,n}\widehat{\widetilde{Q}}_{1,n}. \tag{2.9}$$

Observing that $\widehat{Q}_{1,n}Q_n = 0$, we come to the following identity

$$\widehat{Q}_{1,n}(P_n + T_n^{-1}\widetilde{T}_n\widetilde{Q}_n) = \widehat{Q}_{1,n}.$$

This gives

$$P_n + T_n^{-1}\widetilde{T}_n\widetilde{Q}_n - \widehat{Q}_{1,n} = \widehat{P}_{1,n}(P_n + T_n^{-1}\widetilde{T}_n\widetilde{Q}_n).$$

Thus, we obtain

$$B_nP_{n-1}T_{1,n}\widehat{Q}_{1,n}(P_n + T_n^{-1}\widetilde{T}_n\widetilde{Q}_n - \widehat{Q}_{1,n}) = 0. \tag{2.10}$$

Using the relation (2.2), we can easily see that

$$T_{1,n}^{-1}(P_{n-1} + T_{n-1}^{-1}\widetilde{T}_{n-1}\widetilde{Q}_{n-1})\widetilde{T}_{1,n}\widehat{\widetilde{Q}}_{1,n} = \widehat{Q}_{1,n}T_{1,n}^{-1}(P_{n-1} + T_{n-1}^{-1}\widetilde{T}_{n-1}\widetilde{Q}_{n-1})\widetilde{T}_{1,n}\widehat{\widetilde{Q}}_{1,n}.$$

Therefore, we have

$$G_nT_{1,n}^{-1}(P_{n-1} + T_{n-1}^{-1}\widetilde{T}_{n-1}\widetilde{Q}_{n-1})\widetilde{T}_{1,n}\widehat{\widetilde{Q}}_{1,n} = 0. \tag{2.11}$$

Further, since $T_{n-1}^{-1}\widetilde{T}_{n-1}\widetilde{Q}_{n-1} = Q_{n-1}T_{n-1}^{-1}\widetilde{T}_{n-1}\widetilde{Q}_{n-1}$, we get

$$B_nP_{n-1}T_{1,n}\widehat{Q}_{1,n}T_{1,n}^{-1}(P_{n-1} + T_{n-1}^{-1}\widetilde{T}_{n-1}\widetilde{Q}_{n-1})\widetilde{T}_{1,n}\widehat{\widetilde{Q}}_{1,n} = B_nP_{n-1}\widetilde{T}_{1,n}\widehat{\widetilde{Q}}_{1,n}. \tag{2.12}$$

Finally, combining the relation (2.2) with Eqs. (2.9)–(2.12), and observing that $G_n\widehat{Q}_{1,n} = 0$ and $\widehat{Q}_{1,n}Q_n = 0$ implies that

$$\widehat{G}_{1,n}\Big(P_n + T_n^{-1}\widetilde{T}_n\widetilde{Q}_n + T_n^{-1}Q_{n-1}\widetilde{P}_{n-1}\widetilde{T}_{1,n}\widehat{\widetilde{Q}}_{1,n}$$

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$$
\begin{aligned}
&+T_{1,n}^{-1}\big(P_{n-1}+T_{n-1}^{-1}\widetilde{T}_{n-1}\widetilde{Q}_{n-1}\big)\widetilde{T}_{1,n}\widehat{\widetilde{Q}}_{1,n}-\widehat{Q}_{1,n}\Big)\\
&=(G_n+B_nP_{n-1}T_{1,n}\widehat{Q}_{1,n})\Big(P_n+T_n^{-1}\widetilde{T}_n\widetilde{Q}_n+T_n^{-1}Q_{n-1}\widetilde{P}_{n-1}\widetilde{T}_{1,n}\widehat{\widetilde{Q}}_{1,n}\\
&\quad+T_{1,n}^{-1}\big(P_{n-1}+T_{n-1}^{-1}\widetilde{T}_{n-1}\widetilde{Q}_{n-1}\big)\widetilde{T}_{1,n}\widehat{\widetilde{Q}}_{1,n}-\widehat{Q}_{1,n}\Big)\\
&=\widetilde{G}_n+B_n\widetilde{P}_{n-1}\widetilde{T}_{1,n}\widehat{\widetilde{Q}}_{1,n}-B_nP_{n-1}\widetilde{T}_{1,n}\widehat{\widetilde{Q}}_{1,n}+B_nP_{n-1}\widetilde{T}_{1,n}\widehat{\widetilde{Q}}_{1,n}\\
&=\widehat{\widetilde{G}}_{1,n},
\end{aligned}
$$

which is Eq. (2.8) as to be proved. $\square$

The following fact easily follows from Lemma 2.12.

COROLLARY 2.13. *Suppose that the* LSDE (1.1) *is of index-2. Then the matrix* $P_n+T_n^{-1}\widetilde{T}_n\widetilde{Q}_n+T_n^{-1}Q_{n-1}\widetilde{P}_{n-1}\widetilde{T}_{1,n}\widehat{\widetilde{Q}}_{1,n}+T_{1,n}^{-1}\big(P_{n-1}+T_{n-1}^{-1}\widetilde{T}_{n-1}\widetilde{Q}_{n-1}\big)\widetilde{T}_{1,n}\widehat{\widetilde{Q}}_{1,n}-\widehat{Q}_{1,n}$ *is nonsingular. Moreover,*

$$
\begin{aligned}
&\Big(P_n+T_n^{-1}\widetilde{T}_n\widetilde{Q}_n+T_n^{-1}Q_{n-1}\widetilde{P}_{n-1}\widetilde{T}_{1,n}\widehat{\widetilde{Q}}_{1,n}\\
&\qquad+T_{1,n}^{-1}\big(P_{n-1}+T_{n-1}^{-1}\widetilde{T}_{n-1}\widetilde{Q}_{n-1}\big)\widetilde{T}_{1,n}\widehat{\widetilde{Q}}_{1,n}-\widehat{Q}_{1,n}\Big)^{-1}\\
&\qquad=\widetilde{P}_n+\widetilde{T}_n^{-1}T_nQ_n+\widetilde{T}_n^{-1}\widetilde{Q}_{n-1}P_{n-1}T_{1,n}\widehat{Q}_{1,n}\\
&\qquad\quad+\widetilde{T}_{1,n}^{-1}\big(\widetilde{P}_{n-1}+\widetilde{T}_{n-1}^{-1}T_{n-1}Q_{n-1}\big)T_{1,n}\widehat{Q}_{1,n}-\widehat{\widetilde{Q}}_{1,n}.
\end{aligned}
\tag{2.13}
$$

3. Multipoint boundary value problems.

3.1. Constant coefficients case. We shall consider the LSDEs with constant coefficients

$$Ax_{i+1}=Bx_i+q_i,\quad i=\overline{0,N-1}, \tag{3.1}$$

together with the boundary conditions

$$\sum_{i=0}^{N}C_ix_i=\gamma, \tag{3.2}$$

where $A,B,C_i\in\mathbb{R}^{m\times m}$, $q_i,\gamma\in\mathbb{R}^m$ are given and suppose that $\nu:=\text{ind}(A,B)$ is greater than one.

We suppose that $\lambda\in\mathbb{C}$ such that $\det(\lambda A+B)\neq 0$. Multiply Eq. (3.1) by $(\lambda A+B)^{-1}$ from the left to obtain

$$\widehat{A}_\lambda x_{i+1}=\widehat{B}_\lambda x_i+\widehat{q}_i,\quad i=\overline{0,N-1}, \tag{3.3}$$

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where $\widehat{q}_i := (\lambda A + B)^{-1} q_i$ for all $i = \overline{0, N-1}$. According to Theorem 2.1, there exists a nonsingular matrix $T \in \mathbb{R}^{m\times m}$ such that

$$T^{-1}\widehat{A}_\lambda T = \begin{bmatrix} C & 0 \\ 0 & U \end{bmatrix}, \quad T^{-1}\widehat{B}_\lambda T = \begin{bmatrix} I-\lambda C & 0 \\ 0 & I-\lambda U \end{bmatrix}, \tag{3.4}$$

where $C \in \mathbb{R}^{r\times r}$ is nonsingular with $r := \mathrm{rank}\widehat{A}_\lambda^\nu$ and $U \in \mathbb{R}^{(m-r)\times(m-r)}$ is nilpotent of the order ν. Letting $x_i = Ty_i$ and $f_i = T^{-1}\widehat{q}_i$, then we can rewrite Eq. (3.3) as

$$\begin{bmatrix} C & 0 \\ 0 & U \end{bmatrix} \begin{bmatrix} y_{i+1}^{(1)} \\ y_{i+1}^{(2)} \end{bmatrix} = \begin{bmatrix} I-\lambda C & 0 \\ 0 & I-\lambda U \end{bmatrix} \begin{bmatrix} y_i^{(1)} \\ y_i^{(2)} \end{bmatrix} + \begin{bmatrix} f_i^{(1)} \\ f_i^{(2)} \end{bmatrix}, \quad i = \overline{0, N-1},$$

where $y_i^{(1)},\ f_i^{(1)} \in \mathbb{R}^r$, $y_i^{(2)},\ f_i^{(2)} \in \mathbb{R}^{m-r}$. Note that when $\nu = 1$ then $U = 0$, and in this case, we easily obtain solutions of the above difference equation. The problem of solving (3.1), (3.2) is not difficult, hence, it is omitted here due to lack of space. In this paper, we consider the case $\nu \geq 2$, i.e., $U \neq 0$. However, it is easy to see that these results are still valid for the case $\nu = 1$. Since U has only the eigenvalue 0, it yields that $I - \lambda U$ is nonsingular. Besides, noting that C is a nonsingular matrix, we find that all solutions of Eq. (3.1) are given by

$$\begin{aligned} x_i = {} & \big(\widehat{A}_\lambda^D \widehat{B}_\lambda\big)^i \widehat{A}_\lambda^D \widehat{A}_\lambda \bar{x}_0 + \big(\widehat{B}_\lambda^D \widehat{A}_\lambda\big)^{N-i} (I - \widehat{A}_\lambda^D \widehat{A}_\lambda)\bar{x}_N \\ & + \sum_{l=0}^{i-1} \big(\widehat{A}_\lambda^D \widehat{B}_\lambda\big)^l \widehat{A}_\lambda^D \widehat{q}_{i-l-1} - (I - \widehat{A}_\lambda^D \widehat{A}_\lambda) \sum_{l=0}^{N-i-1} \big(\widehat{B}_\lambda^D \widehat{A}_\lambda\big)^l \widehat{B}_\lambda^D \widehat{q}_{i+l},\ i = \overline{0, N}, \end{aligned} \tag{3.5}$$

where $\bar{x}_0,\ \bar{x}_N \in \mathbb{R}^m$ are arbitrary vectors. Here it is assumed that $\sum\limits_{l=0}^{-1} = 0$.

Notice that the formula (3.5) has also been established in [4]. Further, applying Theorem 2.2 we see that the solution formula (3.5) is independent of the chosen value λ.

REMARK 3.1. An important special case is when A is nonsingular. To study MPBVP (3.1), (3.2), instead of (3.5), we usually use the following solution formula

$$x_i = (A^{-1}B)^i \bar{x}_0 + \sum_{l=0}^{i-1} (A^{-1}B)^l A^{-1} q_{i-l-1}, \quad i = \overline{0, N},$$

where $\bar{x}_0 \in \mathbb{R}^m$ is an arbitrary vector. In another important special case, when B is invertible, the solution to (3.1) is given by

$$x_i = (B^{-1}A)^{N-i} \bar{x}_N - \sum_{l=0}^{N-i-1} (B^{-1}A)^l B^{-1} q_{i+l}, \quad i = \overline{0, N},$$

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where $\bar{x}_N \in \mathbb{R}^m$ is an arbitrary vector. These results were discussed in the theory of boundary value problems for ordinary difference equations, we refer the reader to [1] for more details. The purpose of this paper is to study the MPBVP (3.1), (3.2) in the case, where A and B are both singular.

Let X_i $(i = \overline{0,N})$ be the "fundamental solution" of Eq. (3.1), i.e.,

$$AX_{i+1} = BX_i, \quad i = \overline{0, N-1}.$$

It is clear that

$$X_i = \big(\widehat{A}_\lambda^D \widehat{B}_\lambda\big)^i \widehat{A}_\lambda^D \widehat{A}_\lambda + \big(\widehat{B}_\lambda^D \widehat{A}_\lambda\big)^{N-i}(I - \widehat{A}_\lambda^D \widehat{A}_\lambda), \quad i = \overline{0,N}.$$

Put $X_i^{(1)} := \big(\widehat{A}_\lambda^D \widehat{B}_\lambda\big)^i \widehat{A}_\lambda^D \widehat{A}_\lambda$, $X_i^{(2)} := \big(\widehat{B}_\lambda^D \widehat{A}_\lambda\big)^{N-i}(I - \widehat{A}_\lambda^D \widehat{A}_\lambda)$ $(i = \overline{0,N})$, $D_1 := \sum_{i=0}^{N} C_i X_i^{(1)}$, $D_2 := \sum_{i=0}^{N} C_i X_i^{(2)}$ and $\gamma^* := \gamma - \sum_{i=0}^{N} C_i z_i$, where

$$z_i := \sum_{l=0}^{i-1} \big(\widehat{A}_\lambda^D \widehat{B}_\lambda\big)^l \widehat{A}_\lambda^D \widehat{q}_{i-l-1} - (I - \widehat{A}_\lambda^D \widehat{A}_\lambda) \sum_{l=0}^{N-i-1} \big(\widehat{B}_\lambda^D \widehat{A}_\lambda\big)^l \widehat{B}_\lambda^D \widehat{q}_{i+l}, \quad i = \overline{0,N}.$$

In what follows, we shall deal with the $(m \times 2m)$ matrix (D_1, D_2) with columns of D_1 and D_2 and the $(2m \times 2m)$ matrix

$$R := \begin{bmatrix} \widehat{A}_\lambda^D \widehat{A}_\lambda & 0 \\ 0 & I - \widehat{A}_\lambda^D \widehat{A}_\lambda \end{bmatrix}.$$

From Theorem 2.2 it follows that the matrices (D_1, D_2) and R do not depend on the chosen value λ.

THEOREM 3.2. *Suppose that the matrix pencil* (A, B) *is regular and* $\mathrm{ind}(A, B) \geq 2$. *Then the* MPBVP (3.1) *and* (3.2) *has a unique solution for every* $q_i \in \mathbb{R}^m$ $(i = \overline{0, N-1})$ *and every* $\gamma \in \mathbb{R}^m$ *if and only if*

$$\ker(D_1, D_2) = \ker R \tag{3.6}$$

and it can be represented as

$$x_i = X_i^{(1)}\xi + X_i^{(2)}\zeta + z_i, \quad i = \overline{0,N}, \tag{3.7}$$

where $(\xi^T, \zeta^T)^T = (D_1, D_2)^+\gamma^*$ *with* $(D_1, D_2)^+$ *the generalized inverse in Moore-Penrose's sense of* (D_1, D_2).

Proof. Due to our construction, the relation

$$\ker R \subseteq \ker(D_1, D_2)$$

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is valid.

Assume that the MPBVP (3.1), (3.2) is uniquely solvable, and let $(\bar{x}_0^T, \bar{x}_N^T)^T \in \ker(D_1, D_2)$. Then

$$D_1\bar{x}_0 + D_2\bar{x}_N = 0.$$

Putting $x_i^* := X_i^{(1)}\bar{x}_0 + X_i^{(2)}\bar{x}_N$ $(i = \overline{0, N})$, we find that $\{x_i^*\}_{i=0}^N$ is a solution of the homogeneous MPBVP (3.1), (3.2) with $q_i = 0$, $(i = \overline{0, N-1})$ and $\gamma = 0$. Since the homogeneous MPBVP (3.1) and (3.2) has only a trivial solution, it follows $x_i^* = 0$ for all $i = \overline{0, N}$. In particular, we have $x_0^* = 0$ and $x_N^* = 0$, hence,

$$\widehat{A}_\lambda^D \widehat{A}_\lambda \bar{x}_0 + (\widehat{B}_\lambda^D \widehat{A}_\lambda)^N (I - \widehat{A}_\lambda^D \widehat{A}_\lambda)\bar{x}_N = 0 \tag{3.8}$$

and

$$(\widehat{A}_\lambda^D \widehat{B}_\lambda)^N \widehat{A}_\lambda^D \widehat{A}_\lambda \bar{x}_0 + (I - \widehat{A}_\lambda^D \widehat{A}_\lambda)\bar{x}_N = 0. \tag{3.9}$$

From Eq. (3.4) and the facts that

$$\widehat{A}_\lambda^D = T\begin{bmatrix} C^{-1} & 0 \\ 0 & 0 \end{bmatrix} T^{-1}, \quad \widehat{B}_\lambda^D = T\begin{bmatrix} (I-\lambda C)^D & 0 \\ 0 & (I-\lambda U)^{-1} \end{bmatrix} T^{-1},$$

it follows that

$$\widehat{A}_\lambda^D \widehat{A}_\lambda = T\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T^{-1}, \quad I - \widehat{A}_\lambda^D \widehat{A}_\lambda = T\begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} T^{-1}, \tag{3.10}$$

and

$$\widehat{A}_\lambda^D \widehat{B}_\lambda = T\begin{bmatrix} C^{-1}(I-\lambda C) & 0 \\ 0 & 0 \end{bmatrix} T^{-1}, \tag{3.11}$$

$$\widehat{B}_\lambda^D \widehat{A}_\lambda = T\begin{bmatrix} (I-\lambda C)^D C & 0 \\ 0 & (I-\lambda U)^{-1}U \end{bmatrix} T^{-1}. \tag{3.12}$$

Next, applying formulae (3.10)–(3.12) and putting

$$(\bar{y}_0^{(1)^T}, \bar{y}_0^{(2)^T})^T := T^{-1}\bar{x}_0, \quad (\bar{y}_N^{(1)^T}, \bar{y}_N^{(2)^T})^T := T^{-1}\bar{x}_N$$

with $\bar{y}_0^{(1)}$, $\bar{y}_N^{(1)} \in \mathbb{R}^r$, we can reduce the equalities (3.8), (3.9) to

$$\begin{cases} \bar{y}_0^{(1)} = 0, \\ \big((I-\lambda U)^{-1}U\big)^N \bar{y}_N^{(2)} = 0 \end{cases} \quad \text{and} \quad \begin{cases} \big(C^{-1}(I-\lambda C)\big)^N \bar{y}_0^{(1)} = 0, \\ \bar{y}_N^{(2)} = 0, \end{cases}$$

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respectively. Thus, we obtain

$$\bar{x}_0 = T \begin{bmatrix} 0 \\ \xi \end{bmatrix}, \quad \bar{x}_N = T \begin{bmatrix} \eta \\ 0 \end{bmatrix},$$

where $\xi \in \mathbb{R}^{m-r}$ and $\eta \in \mathbb{R}^r$ are arbitrary vectors, or $\bar{x}_0 \in \ker(\widehat{A}_\lambda^D \widehat{A}_\lambda)$ and $\bar{x}_N \in \ker(I - \widehat{A}_\lambda^D \widehat{A}_\lambda)$, hence $(\bar{x}_0^T, \bar{x}_N^T)^T \in \ker R$. This means that the inclusion

$$\ker(D_1, D_2) \subseteq \ker R$$

must be true, and consequently, (3.6) holds.

Conversely, let (3.6) be valid. Then for each $q_i \in \mathbb{R}^m$ $(i = \overline{0, N-1})$ and $\gamma \in \mathbb{R}^m$ a solution of the MPBVP (3.1), (3.2) is determined by (3.5) and

$$D_1 \bar{x}_0 + D_2 \bar{x}_N = \gamma^*.$$

Let $q_i = 0$ for all $i = \overline{0, N-1}$ and $\gamma = 0$. Then $\bar{x}_0$ and $\bar{x}_N$ satisfy the following equality

$$D_1 \bar{x}_0 + D_2 \bar{x}_N = 0.$$

Therefore, we have $(\bar{x}_0^T, \bar{x}_N^T)^T \in \ker(D_1, D_2) = \ker R$. Now (3.5) ensures that the homogeneous MPBVP (3.1), (3.2) has only a trivial solution.

According to the formula (3.5), any solution of (3.1) can be expressed as (3.7) where $\xi,\ \zeta \in \mathbb{R}^m$ are constant vectors. This solution satisfies the boundary condition (3.2) if and only if

$$D_1 \xi + D_2 \zeta = \gamma^*$$

which means that $(\xi^T, \zeta^T)^T = (D_1, D_2)^+ \gamma^*$. Thus, the unique solution of (3.1), (3.2) has the representation (3.7). $\square$

It is easy to see that $\dim(\ker R) = m$. Denote $p := \dim\big(\ker(D_1, D_2)\big)$. We now consider a case, when (3.6) does not hold, i.e., $p > m$ and the problem (3.1), (3.2) has either no solution or an infinite number of solutions. We denote by $\{w_i^0\}_{i=1}^m$ certain base of $\ker R$. Using the fact that $\ker R \subset \ker(D_1, D_2)$, we can extend $\{w_i^0\}_{i=1}^m$ to a basis $\{w_i^0\}_{i=1}^p$ of $\ker(D_1, D_2)$. Let $u_i^0,\ v_i^0 \in \mathbb{R}^m$ be the first and the second groups of components of w_i^0, i.e., $w_i^0 = ({u_i^0}^T, {v_i^0}^T)^T$, $(i = \overline{1, p})$. We construct the column matrices $\Phi_i := X_i^{(1)} \mathcal{U} + X_i^{(2)} \mathcal{V}$ $(i = \overline{0, N})$, where $\mathcal{U} := (u_{m+1}^0, \ldots, u_p^0)$, $\mathcal{V} := (v_{m+1}^0, \ldots, v_p^0) \in \mathbb{R}^{m \times (p-m)}$. To represent solutions of the MPVBP (3.1), (3.2) we introduce a linear operator $\mathcal{L}$ acting in $\mathbb{R}^{m(N+1)}$, defined by

$$\mathcal{L}(x_0^T, \ldots, x_N^T)^T := \Big((Ax_1 - Bx_0)^T, \ldots, (Ax_N - Bx_{N-1})^T, \big(\sum_{i=0}^N C_i x_i\big)^T\Big)^T.$$

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LEMMA 3.3. $\ker\mathcal{L} = \{((\Phi_0 a)^T, \dots, (\Phi_N a)^T)^T : a \in \mathbb{R}^{p-m}\}$.

Proof. Suppose that $x = (x_0^T, \dots, x_N^T)^T \in \{((\Phi_0 a)^T, \dots, (\Phi_N a)^T)^T : a \in \mathbb{R}^{p-m}\}$, i.e., there exists a vector $a \in \mathbb{R}^{p-m}$ such that $x_i = \Phi_i a$, $i = \overline{0, N}$. This leads to $x_i = X_i^{(1)}\mathcal{U}a + X_i^{(2)}\mathcal{V}a$. From Eqs. (3.10)–(3.12), by simple computations we find

$$AX_{i+1}^{(1)} = BX_i^{(1)} \quad \text{and} \quad AX_{i+1}^{(2)} = BX_i^{(2)}, \quad i = \overline{0, N-1}.$$

Using the above equations, we have

$$\begin{aligned} Lx &:= \big((Ax_1 - Bx_0)^T, \dots, (Ax_N - Bx_{N-1})^T\big)^T \\ &= \Big(\big((AX_1^{(1)} - BX_0^{(1)})\mathcal{U}a\big)^T, \dots, \big((AX_N^{(1)} - BX_{N-1}^{(1)})\mathcal{U}a\big)^T\Big)^T \\ &\quad + \Big(\big((AX_1^{(2)} - BX_0^{(2)})\mathcal{V}a\big)^T, \dots, \big((AX_N^{(2)} - BX_{N-1}^{(2)})\mathcal{V}a\big)^T\Big)^T \\ &= 0. \end{aligned}$$

Denote $\Gamma x := \sum_{i=0}^{N} C_i x_i = D_1\mathcal{U}a + D_2\mathcal{V}a$. Since $\mathcal{U}$ and $\mathcal{V}$ are column matrices whose columns are u_i^0, v_i^0 and $(u_i^{0^T}, v_i^{0^T})^T \in \ker(D_1, D_2)$ $(i = \overline{m+1, p})$, it gives that $D_1\mathcal{U} + D_2\mathcal{V} = 0$, which immediately implies $\Gamma x = 0$. Thus, we obtain $\mathcal{L}x = 0$, which means that

$$\{((\Phi_0 a)^T, \dots, (\Phi_N a)^T)^T : a \in \mathbb{R}^{p-m}\} \subseteq \ker\mathcal{L}.$$

Conversely, assume that $x = (x_0^T, \dots, x_N^T)^T \in \ker\mathcal{L}$, i.e,

$$\begin{cases} Ax_{i+1} = Bx_i, \quad i = \overline{0, N-1}, \\ \sum_{i=0}^{N} C_i x_i = 0. \end{cases}$$

Due to the formula (3.5), $x_i = X_i^{(1)}\xi + X_i^{(2)}\zeta$ $(i = \overline{0, N})$, where vectors $\xi, \zeta \in \mathbb{R}^m$ satisfy the relation $D_1\xi + D_2\zeta = 0$, hence we have $(\xi^T, \zeta^T)^T \in \ker(D_1, D_2)$. Since $(u_k^{0^T}, v_k^{0^T})^T$ $(k = \overline{1, p})$ is the basis of $\ker(D_1, D_2)$, there exists a sequence $\{\alpha_k\}_{k=1}^p$ such that $(\xi^T, \zeta^T)^T = \sum_{k=1}^{p} \alpha_k (u_k^{0^T}, v_k^{0^T})^T$, hence $\xi = \sum_{k=1}^{p} \alpha_k u_k^0$ and $\zeta = \sum_{k=1}^{p} \alpha_k v_k^0$. Thus,

$$x_i = \sum_{k=1}^{m} \alpha_k \big(X_i^{(1)} u_k^0 + X_i^{(2)} v_k^0\big) + \sum_{k=m+1}^{p} \alpha_k \big(X_i^{(1)} u_k^0 + X_i^{(2)} v_k^0\big), \quad i = \overline{0, N}.$$

Observing that $(u_k^{0^T}, v_k^{0^T})^T \in \ker R$, i.e., $\widehat{A}_\lambda^D \widehat{A}_\lambda u_k^0 = 0$ and $(I - \widehat{A}_\lambda^D \widehat{A}_\lambda) v_k^0 = 0$ for all $k = \overline{1, m}$, we find $X_i^{(1)} u_k^0 = 0$ and $X_i^{(2)} v_k^0 = 0$ for all $k = \overline{1, m}$, $i = \overline{0, N}$. Thus,

$$x_i = X_i^{(1)} \sum_{k=m+1}^{p} \alpha_k u_k^0 + X_i^{(2)} \sum_{k=m+1}^{p} \alpha_k v_k^0, \quad i = \overline{0, N}.$$

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Taking $a := (\alpha_{m+1}, \dots, \alpha_p)^T \in \mathbb{R}^{p-m}$, we get $x_i = X_i^{(1)}\mathcal{U}a + X_i^{(2)}\mathcal{V}a$ $(i = \overline{0,N})$, i.e., $x_i = \Phi_i a$ for all $i = \overline{0,N}$ where $a \in \mathbb{R}^{p-m}$. Thus, we obtain

$$x \in \Big\{ \big((\Phi_0 a)^T, \dots, (\Phi_N a)^T \big)^T : a \in \mathbb{R}^{p-m} \Big\},$$

or

$$\ker\mathcal{L} \subseteq \Big\{ \big((\Phi_0 a)^T, \dots, (\Phi_N a)^T \big)^T : a \in \mathbb{R}^{p-m} \Big\}. \quad \square$$

Next, we let $q := \dim\big(\ker(D_1, D_2)^T\big)$ and denote by $\{w_i\}_{i=1}^q$ certain base of $\ker(D_1, D_2)^T$. Letting $W \in \mathbb{R}^{q\times m}$ be a row matrix whose rows are vectors w_i $(i = \overline{1,q})$, we come to the following theorem.

THEOREM 3.4. *Let the matrix pencil* (A, B) *be regular and* $\operatorname{ind}(A, B) \geq 2$. *Then, the problem* (3.1), (3.2) *is solvable if and only if*

$$W\gamma^* = 0. \tag{3.13}$$

Moreover, a general solution of (3.1), (3.2) *has the following form*

$$x_i = X_i^{(1)}\xi + X_i^{(2)}\zeta + z_i + \Phi_i a, \quad i = \overline{0,N}, \tag{3.14}$$

where $a \in \mathbb{R}^{p-m}$ *is an arbitrary vector and* $(\xi^T, \zeta^T)^T = (D_1, D_2)^+\gamma^*$ *with* $(D_1, D_2)^+$ *the generalized inverse in Moore-Penrose's sense of* (D_1, D_2).

Proof. The problem (3.1), (3.2) is solvable if and only if

$$(q_0^T, \dots, q_{N-1}^T, \gamma^T)^T \in \operatorname{im}\mathcal{L},$$

i.e., there exists $x = (x_0^T, \dots, x_N^T)^T \in \mathbb{R}^{m(N+1)}$ satisfying $\mathcal{L}x = (q_o^T, \dots, q_{N-1}^T, \gamma^T)^T$. Equivalently, there exist vectors $\xi, \zeta \in \mathbb{R}^m$ such that $x_i = X_i^{(1)}\xi + X_i^{(2)}\zeta + z_i$ $(i = \overline{0,N})$ and $\sum_{i=0}^N C_i x_i = \gamma$. Thus, the system (3.1), (3.2) possesses a solution if and only if there exist vectors $\xi, \zeta \in \mathbb{R}^m$ such that $D_1\xi + D_2\zeta = \gamma^*$. Using the fact that $\operatorname{im}(D_1, D_2) = \big(\ker(D_1, D_2)^T\big)^\perp$ we come to the conclusion that the MPBVP (3.1), (3.2) is solvable if and only if $\gamma^* \in \big(\ker(D_1, D_2)^T\big)^\perp$. Thus, the problem (3.1), (3.2) possesses a solution if and only if (3.13) is valid.

Finally, thanks to Lemma 3.3 and the formula (3.5), to show that (3.14) is a general solution formula of the problem (3.1), (3.2) we only need to prove that $\bar{x}_i$ is given by $\bar{x}_i = X_i^{(1)}\xi + X_i^{(2)}\zeta + z_i$ $(i = \overline{0,N})$ with $(\xi^T, \zeta^T)^T = (D_1, D_2)^+\gamma^*$, is a particular solution of the above mentioned problem. $\square$

Theorem 3.2 and Theorem 3.4 imply the following corollary.

COROLLARY 3.5. (Fredholm alternative) *Suppose that the matrix pencil* (A, B) *is regular,* $\operatorname{ind}(A, B) \geq 2$ *and let* $p := \dim\big(\ker(D_1, D_2)\big)$. *Then*

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(i) *either* $p = m$ *and the* MPBVP (3.1), (3.2) *is uniquely solvable for any data* q_i $(i = \overline{0, N-1})$ *and* γ;
(ii) *or* $p > m$ *and the* MPBVP (3.1, (3.2) *is solvable if and only if the condition* (3.13) *is valid.*

Moreover, the solution formula (3.14) *holds.*

3.2. Varying coefficients case. In this subsection, we shall deal with the MP-BVPs for the non-autonomous LSDEs as follows

$$A_i x_{i+1} = B_i x_i + q_i, \quad i = \overline{0, N-1}, \tag{3.15}$$

$$\sum_{i=0}^{N} C_i x_i = \gamma, \tag{3.16}$$

where $A_i, B_i, C_i \in \mathbb{R}^{m\times m}$, $q_i, \gamma \in \mathbb{R}^m$ are given and suppose that the LSDE (3.15) is of index-2 in the sense that the following relations hold:

(i) $\mathrm{rank} A_i \equiv r, \quad 1 \le r \le m-1,$
(ii) $\dim(\ker A_{i-1} \cap S_i) \equiv m - s, \quad 1 \le s \le m-1,$
(iii) $\ker G_{i-1} \cap S_{1,i} = \{0\}$

for all $i = \overline{0, N-1}$. Further, here it is assumed that $A_{-1} := A_0$, $G_{-1} := G_0$ and $\widehat{Q}_{1,N-1}$ is projection onto $\ker G_{N-1}$ such that $\widehat{Q}_{1,N-1} Q_{N-1} = 0$.

Now, we describe shortly the decomposition technique for index-2 LSDEs (see [7] for details). We decompose the index-2 LSDE solution x_i into

$$x_i = Q_{i-1}x_i + P_{i-1}\widehat{P}_{1,i-1}x_i + P_{i-1}\widehat{Q}_{1,i-1}x_i =: w_i + u_i + P_{i-1}v_i.$$

Multiplying Eq. (3.15) by $P_i\widehat{P}_{1,i}\widehat{G}_{1,i}^{-1}$, $Q_i\widehat{P}_{1,i}\widehat{G}_{1,i}^{-1}$, and $\widehat{Q}_{1,i}\widehat{G}_{1,i}^{-1}$, respectively, using the relations (2.6), (2.7) and the fact that $\widehat{Q}_{1,i}Q_i = 0$, and carrying out some technical computations, we decouple the index-2 LSDE (3.15) into the system

$$\begin{cases} u_{i+1} = P_i\widehat{P}_{1,i}\widehat{G}_{1,i}^{-1}B_iu_i + P_i\widehat{P}_{1,i}\widehat{G}_{1,i}^{-1}q_i, \\ -Q_iv_{i+1} = Q_i\widehat{P}_{1,i}\widehat{G}_{1,i}^{-1}B_iu_i + T_i^{-1}w_i + Q_i\widehat{P}_{1,i}\widehat{G}_{1,i}^{-1}q_i, \qquad i = \overline{0, N-1}. \\ 0 = T_{1,i}^{-1}v_i + \widehat{Q}_{1,i}\widehat{G}_{1,i}^{-1}q_i, \end{cases}$$

Thus, we obtain

$$\begin{aligned} v_i &= -T_{1,i}\widehat{Q}_{1,i}\widehat{G}_{1,i}^{-1}q_i, \quad i = \overline{0, N-1}, \\ w_i &= -T_iQ_i\widehat{P}_{1,i}\widehat{G}_{1,i}^{-1}B_iu_i - T_iQ_iv_{i+1} - T_iQ_i\widehat{P}_{1,i}\widehat{G}_{1,i}^{-1}q_i, \quad i = \overline{0, N-1}, \\ u_{i+1} &= P_i\widehat{P}_{1,i}\widehat{G}_{1,i}^{-1}B_iu_i + P_i\widehat{P}_{1,i}\widehat{G}_{1,i}^{-1}q_i, \quad i = \overline{0, N-1}. \end{aligned}$$

We denote

$$\Pi_i := \big(I - T_iQ_i\widehat{P}_{1,i}\widehat{G}_{1,i}^{-1}B_i\big)P_{i-1}\widehat{P}_{1,i-1}, \quad i = \overline{0, N-1}$$

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and

$$M_k^{(i)} := \prod_{l=0}^{k} P_{i-l-1}\widehat{P}_{1,i-l-1}\widehat{G}_{1,i-l-1}^{-1}B_{i-l-1}, \quad i=\overline{1,N},\ k=\overline{-1,i-1},$$

where it is assumed that $\prod_{l=0}^{-1} = I$. Observing that $(P_{i-1}\widehat{P}_{1,i-1})^2 = P_{i-1}\widehat{P}_{1,i-1}$, we get the solution of the LSDE (3.15) as follows

$$\begin{aligned}(3.17)\qquad x_i = {} & \Pi_i\Big(M_{i-1}^{(i)}\bar{x}_0 + \sum_{k=0}^{i-1} M_{i-k-2}^{(i)}P_k\widehat{P}_{1,k}\widehat{G}_{1,k}^{-1}q_k\Big)\\ & + T_iQ_iT_{1,i+1}\widehat{Q}_{1,i+1}\widehat{G}_{1,i+1}^{-1}q_{i+1}\\ & - \big(T_iQ_i\widehat{P}_{1,i}\widehat{G}_{1,i}^{-1} + P_{i-1}T_{1,i}\widehat{Q}_{1,i}\widehat{G}_{1,i}^{-1}\big)q_i, \quad i=\overline{0,N-2},\end{aligned}$$

$$\begin{aligned}(3.18)\quad x_{N-1} = {} & \Pi_{N-1}\Big(M_{N-2}^{(N-1)}\bar{x}_0 + \sum_{k=0}^{N-2} M_{N-k-3}^{(N-1)}P_k\widehat{P}_{1,k}\widehat{G}_{1,k}^{-1}q_k\Big)\\ & - T_{N-1}Q_{N-1}\widehat{Q}_{1,N-1}\bar{x}_N\\ & - \big(T_{N-1}Q_{N-1}\widehat{P}_{1,N-1}\widehat{G}_{1,N-1}^{-1} + P_{N-2}T_{1,N-1}\widehat{Q}_{1,N-1}\widehat{G}_{1,N-1}^{-1}\big)q_{N-1}\end{aligned}$$

and

$$(3.19)\quad x_N = M_{N-1}^{(N)}\bar{x}_0 + \sum_{k=0}^{N-1} M_{N-k-2}^{(N)}P_k\widehat{P}_{1,k}\widehat{G}_{1,k}^{-1}q_k + Q_{N-1}\bar{x}_N + P_{N-1}\widehat{Q}_{1,N-1}\bar{x}_N,$$

where $\bar{x}_0,\ \bar{x}_N \in \mathbb{R}^m$ are arbitrary vectors and it is assumed that $\sum_{k=0}^{-1} = 0$.

Remark 3.6. Similar to Remark 3.1, if A_i (resp., B_i) is nonsingular for each $i=\overline{0,N-1}$ then we will use the corresponding solution formulae for (3.15), (3.16)

$$x_i = \Big(\prod_{l=0}^{i-1} A_{i-l-1}^{-1}B_{i-l-1}\Big)\bar{x}_0 + \sum_{k=0}^{i-1}\Big(\prod_{l=0}^{i-k-2} A_{i-l-1}^{-1}B_{i-l-1}\Big)A_k^{-1}q_k, \quad i=\overline{0,N},$$

or

$$x_i = \Big(\prod_{l=i}^{N-1} B_l^{-1}A_l\Big)\bar{x}_N - \sum_{k=i}^{N-1}\Big(\prod_{l=i}^{k-1} B_l^{-1}A_l\Big)B_k^{-1}q_k, \quad i=\overline{0,N},$$

where $\bar{x}_0, \bar{x}_N \in \mathbb{R}^m$ are arbitrary vectors instead of the formulae (3.17)–(3.19). See [1] for details. The formulae (3.17)–(3.19) are useful in the case, where A_i and B_i are

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both singular, and Eq. (3.15) is of index-2. Further, it is clear that these formulae are an extension of the above mentioned formulae.

LEMMA 3.7. *Let the* LSDE (3.15) *be of index-2. Then the matrices*

$$\Pi_i,\ \Pi_i M_{i-1}^{(i)},\ \Pi_i M_{i-k-2}^{(i)} P_k \widehat{P}_{1,k} \widehat{G}_{1,k}^{-1},\ T_i Q_i T_{1,i+1} \widehat{Q}_{1,i+1} \widehat{G}_{1,i+1}^{-1},$$

$$(T_i Q_i \widehat{P}_{1,i} + P_{i-1} T_{1,i} \widehat{Q}_{1,i}) \widehat{G}_{1,i}^{-1}$$

are independent of the choice of the T_i, Q_i *and* $T_{1,i}$.

Proof. Let $\widetilde{T}_i$ be another transformation, whose restriction $\widetilde{T}_i|_{\ker A_i}$ is an isomorphism from $\ker A_i$ onto $\ker A_{i-1}$ and $\widetilde{Q}_i$ be another projection onto $\ker A_i$, $\widetilde{P}_i := I - \widetilde{Q}_i$. We denote $\widetilde{G}_i := A_i + B_i \widetilde{T}_i \widetilde{Q}_i$. Let $\widehat{\widetilde{Q}}_{1,i}$ be a projection onto $\ker \widetilde{G}_i$ along $S_{1,i+1}$, $\widehat{\widetilde{P}}_{1,i} := I - \widehat{\widetilde{Q}}_{1,i}$, $\widetilde{T}_{1,i}|_{\ker \widetilde{G}_i}$ denote an isomorphism from $\ker \widetilde{G}_i$ onto $\ker \widetilde{G}_{i-1}$ and put

$$\widehat{\widetilde{G}}_{1,i} := \widetilde{G}_i + B_i \widetilde{P}_{i-1} \widetilde{T}_{1,i} \widehat{\widetilde{Q}}_{1,i}, \quad \widetilde{\Pi}_i := \big(I - \widetilde{T}_i \widetilde{Q}_i \widehat{\widetilde{P}}_{1,i} \widehat{\widetilde{G}}_{1,i}^{-1} B_i\big) \widetilde{P}_{i-1} \widehat{\widetilde{P}}_{1,i-1}.$$

First, we put

$$Z_i := \widetilde{P}_i + \widetilde{T}_i^{-1} T_i Q_i + \widetilde{T}_i^{-1} \widetilde{Q}_{i-1} P_{i-1} T_{1,i} \widehat{Q}_{1,i} + \widetilde{T}_{1,i}^{-1} \big(\widetilde{P}_{i-1} + \widetilde{T}_{i-1}^{-1} T_{i-1} Q_{i-1}\big) T_{1,i} \widehat{Q}_{1,i} - \widehat{\widetilde{Q}}_{1,i}.$$

From the identities (2.8) and (2.13), we have

$$\widetilde{T}_i \widetilde{Q}_i \widehat{\widetilde{P}}_{1,i} \widehat{\widetilde{G}}_{1,i}^{-1} = \widetilde{T}_i \widetilde{Q}_i \widehat{\widetilde{P}}_{1,i} Z_i \widehat{G}_{1,i}^{-1}. \tag{3.20}$$

Using the facts that $\widehat{\widetilde{Q}}_{1,i} \widetilde{Q}_i = 0$, $\widetilde{T}_i^{-1} T_i Q_i = \widetilde{Q}_i \widetilde{T}_i^{-1} T_i Q_i$ and

$$\widetilde{T}_{1,i}^{-1} \big(\widetilde{P}_{i-1} + \widetilde{T}_{i-1}^{-1} T_{i-1} Q_{i-1}\big) T_{1,i} \widehat{Q}_{1,i} = \widehat{\widetilde{Q}}_{1,i} \widetilde{T}_{1,i}^{-1} \big(\widetilde{P}_{i-1} + \widetilde{T}_{i-1}^{-1} T_{i-1} Q_{i-1}\big) T_{1,i} \widehat{Q}_{1,i},$$

we see that

$$\widetilde{T}_i \widetilde{Q}_i \widehat{\widetilde{P}}_{1,i} \widetilde{P}_i = -\widetilde{T}_i \widetilde{Q}_i \widehat{\widetilde{Q}}_{1,i} \widetilde{P}_i = -\widetilde{T}_i \widetilde{Q}_i \widehat{\widetilde{Q}}_{1,i},$$

$$\widetilde{T}_i \widetilde{Q}_i \widehat{\widetilde{P}}_{1,i} \widetilde{T}_i^{-1} T_i Q_i = \widetilde{T}_i \widetilde{Q}_i (I - \widehat{\widetilde{Q}}_{1,i}) \widetilde{Q}_i \widetilde{T}_i^{-1} T_i Q_i = T_i Q_i,$$

$$\begin{aligned} \widetilde{T}_i \widetilde{Q}_i \widehat{\widetilde{P}}_{1,i} \widetilde{T}_i^{-1} \widetilde{Q}_{i-1} P_{i-1} T_{1,i} \widehat{Q}_{1,i} &= \widetilde{T}_i \widetilde{Q}_i (I - \widehat{\widetilde{Q}}_{1,i}) \widetilde{Q}_i \widetilde{T}_i^{-1} \widetilde{Q}_{i-1} P_{i-1} T_{1,i} \widehat{Q}_{1,i} \\ &= \widetilde{Q}_{i-1} P_{i-1} T_{1,i} \widehat{Q}_{1,i} \end{aligned}$$

and

$$\widetilde{T}_i \widetilde{Q}_i \widehat{\widetilde{P}}_{1,i} \widetilde{T}_{1,i}^{-1} \big(\widetilde{P}_{i-1} + \widetilde{T}_{i-1}^{-1} T_{i-1} Q_{i-1}\big) T_{1,i} \widehat{Q}_{1,i} = 0.$$

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Combining the above relations with Eq. (2.5), it follows that

$$
\begin{aligned}
\widetilde{T}_i\widetilde{Q}_i\widehat{\widetilde{P}}_{1,i}Z_i &= -\widetilde{T}_i\widetilde{Q}_i\widehat{\widetilde{Q}}_{1,i} + T_iQ_i + \widetilde{Q}_{i-1}P_{i-1}T_{1,i}\widehat{Q}_{1,i}\\
&= -\widetilde{T}_i\widetilde{Q}_i(\widetilde{P}_i + \widetilde{T}_i^{-1}T_iQ_i)\widehat{Q}_{1,i} + T_iQ_i + \widetilde{Q}_{i-1}P_{i-1}T_{1,i}\widehat{Q}_{1,i}\\
&= T_iQ_i\widehat{P}_{1,i} + \widetilde{Q}_{i-1}P_{i-1}T_{1,i}\widehat{Q}_{1,i}.
\end{aligned} \tag{3.21}
$$

Thus,

$$
\widetilde{\Pi}_i = \Big(I - T_iQ_i\widehat{P}_{1,i}\widehat{G}_{1,i}^{-1}B_i - \widetilde{Q}_{i-1}P_{i-1}T_{1,i}\widehat{Q}_{1,i}\widehat{G}_{1,i}^{-1}B_i\Big)\widetilde{P}_{i-1}\widehat{\widetilde{P}}_{1,i-1}.
$$

Observe that

$$
\widetilde{P}_{i-1}\widehat{\widetilde{P}}_{1,i-1} = \widetilde{P}_{i-1} - \widetilde{P}_{i-1}(\widetilde{P}_{i-1} + \widetilde{T}_{i-1}^{-1}T_{i-1}Q_{i-1})\widehat{Q}_{1,i-1} = \widetilde{P}_{i-1}\widehat{P}_{1,i-1}
$$

and

$$
\widehat{Q}_{1,i-1}\widetilde{P}_{i-1}\widehat{P}_{1,i-1} = \widehat{Q}_{1,i-1}(I - \widetilde{Q}_{i-1})\widehat{P}_{1,i-1} = 0.
$$

Further, we note that

$$
\widehat{G}_{1,i}^{-1}B_iQ_{i-1} = \widehat{G}_{1,i}^{-1}(A_i + B_iT_iQ_i)Q_iT_i^{-1}Q_{i-1} = \widehat{P}_{1,i}T_i^{-1}Q_{i-1},
$$

implying that

$$
\begin{aligned}
\widehat{Q}_{1,i-1} &:= T_{1,i}\widehat{Q}_{1,i}\widehat{G}_{1,i}^{-1}B_iP_{i-1}\\
&= T_{1,i}\widehat{Q}_{1,i}\widehat{G}_{1,i}^{-1}B_i - T_{1,i}\widehat{Q}_{1,i}\widehat{G}_{1,i}^{-1}B_iQ_{i-1}\\
&= T_{1,i}\widehat{Q}_{1,i}\widehat{G}_{1,i}^{-1}B_i.
\end{aligned}
$$

This leads to

$$
\widetilde{\Pi}_i = \big(I - T_iQ_i\widehat{P}_{1,i}\widehat{G}_{1,i}^{-1}B_i\big)\widetilde{P}_{i-1}P_{i-1}\widehat{P}_{1,i-1} = \Pi_i - \big(I - T_iQ_i\widehat{P}_{1,i}\widehat{G}_{1,i}^{-1}B_i\big)\widetilde{Q}_{i-1}P_{i-1}\widehat{P}_{1,i-1}.
$$

Since

$$
\begin{aligned}
\big(I - T_iQ_i\widehat{P}_{1,i}\widehat{G}_{1,i}^{-1}B_i\big)\widetilde{Q}_{i-1} &= \widetilde{Q}_{i-1} - T_iQ_i\widehat{P}_{1,i}\widehat{G}_{1,i}^{-1}B_iQ_{i-1}\widetilde{Q}_{i-1}\\
&= \widetilde{Q}_{i-1} - T_iQ_i\widehat{P}_{1,i}\widehat{P}_{1,i}T_i^{-1}Q_{i-1}\widetilde{Q}_{i-1}\\
&= \widetilde{Q}_{i-1} - T_iQ_i(I - \widehat{Q}_{1,i})Q_iT_i^{-1}\widetilde{Q}_{i-1}\\
&= 0,
\end{aligned}
$$

we have $\widetilde{\Pi}_i = \Pi_i$.

Applying the identities (2.8), (2.13) we get

$$
\widetilde{P}_{i-1}\widehat{\widetilde{P}}_{1,i-1}\widehat{\widetilde{G}}_{1,i-1}^{-1} = \widetilde{P}_{i-1}\widehat{\widetilde{P}}_{1,i-1}Z_{i-1}\widehat{G}_{1,i-1}^{-1}.
$$

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On the other hand, since

$$\widetilde{P}_{i-1}\widehat{\widetilde{P}}_{1,i-1}\widetilde{P}_{i-1} = \widetilde{P}_{i-1} - \widetilde{P}_{i-1}\widehat{\widetilde{Q}}_{1,i-1}(I - \widetilde{Q}_{i-1}) = \widetilde{P}_{i-1}\widehat{\widetilde{P}}_{1,i-1} = \widetilde{P}_{i-1}\widehat{P}_{1,i-1},$$

$$\widetilde{P}_{i-1}\widehat{\widetilde{P}}_{1,i-1}\widetilde{T}_{i-1}^{-1}T_{i-1}Q_{i-1} = \widetilde{P}_{i-1}(I - \widehat{\widetilde{Q}}_{1,i-1})\widetilde{Q}_{i-1}\widetilde{T}_{i-1}^{-1}T_{i-1}Q_{i-1} = 0,$$

$$\begin{aligned}&\widetilde{P}_{i-1}\widehat{\widetilde{P}}_{1,i-1}\widetilde{T}_{i-1}^{-1}\widetilde{Q}_{i-2}P_{i-2}T_{1,i-1}\widehat{Q}_{1,i-1}\\ &= \widetilde{P}_{i-1}(I - \widehat{\widetilde{Q}}_{1,i-1})\widetilde{Q}_{i-1}\widetilde{T}_{i-1}^{-1}\widetilde{Q}_{i-2}P_{i-2}T_{1,i-1}\widehat{Q}_{1,i-1} = 0\end{aligned}$$

and

$$\begin{aligned}&\widetilde{P}_{i-1}\widehat{\widetilde{P}}_{1,i-1}\widetilde{T}_{1,i-1}^{-1}(\widetilde{P}_{i-2} + \widetilde{T}_{i-2}^{-1}T_{i-2}Q_{i-2})T_{1,i-1}\widehat{Q}_{1,i-1}\\ &= \widetilde{P}_{i-1}\widehat{\widetilde{P}}_{1,i-1}\widehat{\widetilde{Q}}_{1,i-1}\widetilde{T}_{1,i-1}^{-1}(\widetilde{P}_{i-2} + \widetilde{T}_{i-2}^{-1}T_{i-2}Q_{i-2})T_{1,i-1}\widehat{Q}_{1,i-1} = 0,\end{aligned}$$

we have that

$$\widetilde{P}_{i-1}\widehat{\widetilde{P}}_{1,i-1}\widehat{\widetilde{G}}_{1,i-1}^{-1} = \widetilde{P}_{i-1}\widehat{P}_{1,i-1}\widehat{G}_{1,i-1}^{-1} = P_{i-1}\widehat{P}_{1,i-1}\widehat{G}_{1,i-1}^{-1} - \widetilde{Q}_{i-1}P_{i-1}\widehat{P}_{1,i-1}\widehat{G}_{1,i-1}^{-1}.$$

Observing that $\Pi_i\widetilde{Q}_{i-1} = \big(I - T_iQ_i\widehat{P}_{1,i}\widehat{G}_{1,i}^{-1}B_i\big)P_{i-1}\widehat{P}_{1,i-1}Q_{i-1}\widetilde{Q}_{i-1} = 0$, we obtain

$$\begin{aligned}\widetilde{\Pi}_i\widetilde{P}_{i-1}\widehat{\widetilde{P}}_{1,i-1}\widehat{\widetilde{G}}_{1,i-1}^{-1} &= \Pi_iP_{i-1}\widehat{P}_{1,i-1}\widehat{G}_{1,i-1}^{-1} - \Pi_i\widetilde{Q}_{i-1}P_{i-1}\widehat{P}_{1,i-1}\widehat{G}_{1,i-1}^{-1}\\ &= \Pi_iP_{i-1}\widehat{P}_{1,i-1}\widehat{G}_{1,i-1}^{-1},\end{aligned}$$

hence

$$\widetilde{\Pi}_i\widetilde{P}_{i-1}\widehat{\widetilde{P}}_{1,i-1}\widehat{\widetilde{G}}_{1,i-1}^{-1} = \Pi_iP_{i-1}\widehat{P}_{1,i-1}\widehat{G}_{1,i-1}^{-1}.$$

Since $\widehat{G}_{1,i-1}^{-1}B_{i-1}Q_{i-2} = \widehat{P}_{1,i-1}T_{i-1}^{-1}Q_{i-2}$ and $P_{i-1}\widehat{P}_{1,i-1}Q_{i-1} = 0$, we have

$$\begin{aligned}&\Pi_iP_{i-1}\widehat{P}_{1,i-1}\widehat{G}_{1,i-1}^{-1}B_{i-1}\widetilde{Q}_{i-2}P_{i-2}\widehat{P}_{1,i-2}\widehat{G}_{1,i-2}^{-1}\\ &= \Pi_iP_{i-1}\widehat{P}_{1,i-1}\widehat{G}_{1,i-1}^{-1}B_{i-1}Q_{i-2}\widetilde{Q}_{i-2}P_{i-2}\widehat{P}_{1,i-2}\widehat{G}_{1,i-2}^{-1}\\ &= \Pi_iP_{i-1}\widehat{P}_{1,i-1}\widehat{P}_{1,i-1}T_{i-1}^{-1}Q_{i-2}\widetilde{Q}_{i-2}P_{i-2}\widehat{P}_{1,i-2}\widehat{G}_{1,i-2}^{-1}\\ &= \Pi_iP_{i-1}\widehat{P}_{1,i-1}Q_{i-1}T_{i-1}^{-1}\widetilde{Q}_{i-2}P_{i-2}\widehat{P}_{1,i-2}\widehat{G}_{1,i-2}^{-1}\\ &= 0.\end{aligned}$$

Thus,

$$\begin{aligned}&\widetilde{\Pi}_i\widetilde{P}_{i-1}\widehat{\widetilde{P}}_{1,i-1}\widehat{\widetilde{G}}_{1,i-1}^{-1}B_{i-1}\widetilde{P}_{i-2}\widehat{\widetilde{P}}_{1,i-2}\widehat{\widetilde{G}}_{1,i-2}^{-1}\\ &= \Pi_iP_{i-1}\widehat{P}_{1,i-1}\widehat{G}_{1,i-1}^{-1}B_{i-1}P_{i-2}\widehat{P}_{1,i-2}\widehat{G}_{1,i-2}^{-1}.\end{aligned}$$

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Thus, the matrices $\Pi_i M^{(i)}_{i-1}$, $\Pi_i M^{(i)}_{i-k-2} P_k \widehat{P}_{1,k} \widehat{G}^{-1}_{1,k}$ do not depend on the choice of the T_i, Q_i and $T_{1,i}$, as it was to be proved.

From Eqs. (2.8), (2.13) it follows that

$$\widetilde{T}_i \widetilde{Q}_i \widetilde{T}_{1,i+1} \widehat{\widetilde{Q}}_{1,i+1} \widehat{\widetilde{G}}^{-1}_{1,i+1} = \widetilde{T}_i \widetilde{Q}_i \widehat{T}_{1,i+1} \widehat{\widetilde{Q}}_{1,i+1} Z_{i+1} \widehat{G}^{-1}_{1,i+1}.$$

Besides, $\widehat{\widetilde{Q}}_{1,i+1} = \widehat{\widetilde{Q}}_{1,i+1}(\widetilde{P}_{i+1} + \widetilde{T}^{-1}_{i+1} T_{i+1} Q_{i+1})$, therefore, we get

$$\begin{aligned}\widetilde{T}_i \widetilde{Q}_i \widetilde{T}_{1,i+1} \widehat{\widetilde{Q}}_{1,i+1} Z_{i+1} \widehat{G}^{-1}_{1,i+1} &= \widetilde{T}_i \widetilde{Q}_i \widetilde{T}_{1,i+1} \widehat{\widetilde{Q}}_{1,i+1} \Big(\widehat{\widetilde{P}}_{1,i+1} (\widetilde{P}_{i+1} + \widetilde{T}^{-1}_{i+1} T_{i+1} Q_{i+1}) \\ &\quad + \widetilde{T}^{-1}_{i+1} \widetilde{Q}_i P_i T_{1,i+1} \widehat{Q}_{1,i+1} + \widetilde{T}^{-1}_{1,i+1} (\widetilde{P}_i + \widetilde{T}^{-1}_i T_i Q_i) T_{1,i+1} \widehat{Q}_{1,i+1} \Big) \widehat{G}^{-1}_{1,i+1} \\ &= T_i Q_i T_{1,i+1} \widehat{Q}_{1,i+1} \widehat{G}^{-1}_{1,i+1},\end{aligned}$$

or equivalently,

$$\widetilde{T}_i \widetilde{Q}_i \widetilde{T}_{1,i+1} \widehat{\widetilde{Q}}_{1,i+1} \widehat{\widetilde{G}}^{-1}_{1,i+1} = T_i Q_i T_{1,i+1} \widehat{Q}_{1,i+1} \widehat{G}^{-1}_{1,i+1}.$$

From Eqs. (3.20)–(3.21), it follows that

$$\widetilde{T}_i \widetilde{Q}_i \widehat{\widetilde{P}}_{1,i} \widehat{\widetilde{G}}^{-1}_{1,i} = T_i Q_i \widehat{P}_{1,i} \widehat{G}^{-1}_{1,i} + \widetilde{Q}_{i-1} P_{i-1} T_{1,i} \widehat{Q}_{1,i} \widehat{G}^{-1}_{1,i},$$

further,

$$\begin{aligned}\widetilde{P}_{i-1} \widetilde{T}_{1,i} \widehat{\widetilde{Q}}_{1,i} \widehat{\widetilde{G}}^{-1}_{1,i} &= \widetilde{P}_{i-1} \widetilde{T}_{1,i} \widehat{\widetilde{Q}}_{1,i} \Big(\widehat{\widetilde{P}}_{1,i} (\widetilde{P}_i + \widetilde{T}^{-1}_i T_i Q_i) + \widetilde{Q}_i \widetilde{T}^{-1}_i \widetilde{Q}_{i-1} P_{i-1} T_{1,i} \widehat{Q}_{1,i} \\ &\quad + \widetilde{T}^{-1}_{1,i} (\widetilde{P}_{i-1} + \widetilde{T}^{-1}_{i-1} T_{i-1} Q_{i-1}) T_{1,i} \widehat{Q}_{1,i} \Big) \widehat{G}^{-1}_{1,i} = \widetilde{P}_{i-1} T_{1,i} \widehat{Q}_{1,i} \widehat{G}^{-1}_{1,i},\end{aligned}$$

giving

$$\begin{aligned}\widetilde{T}_i \widetilde{Q}_i \widehat{\widetilde{P}}_{1,i} \widehat{\widetilde{G}}^{-1}_{1,i} + \widetilde{P}_{i-1} \widetilde{T}_{1,i} \widehat{\widetilde{Q}}_{1,i} \widehat{\widetilde{G}}^{-1}_{1,i} &= T_i Q_i \widehat{P}_{1,i} \widehat{G}^{-1}_{1,i} + \widetilde{Q}_{i-1} P_{i-1} T_{1,i} \widehat{Q}_{1,i} \widehat{G}^{-1}_{1,i} \\ &\quad + \widetilde{P}_{i-1} T_{1,i} \widehat{Q}_{1,i} \widehat{G}^{-1}_{1,i} \\ &= T_i Q_i \widehat{P}_{1,i} \widehat{G}^{-1}_{1,i} \\ &\quad + \big((I - \widetilde{P}_{i-1}) P_{i-1} + \widetilde{P}_{i-1}\big) T_{1,i} \widehat{Q}_{1,i} \widehat{G}^{-1}_{1,i} \\ &= T_i Q_i \widehat{P}_{1,i} \widehat{G}^{-1}_{1,i} + P_{i-1} T_{1,i} \widehat{Q}_{1,i} \widehat{G}^{-1}_{1,i}.\end{aligned}$$

Thus, we obtain

$$\big(\widetilde{T}_i \widetilde{Q}_i \widehat{\widetilde{P}}_{1,i} + \widetilde{P}_{i-1} \widetilde{T}_{1,i} \widehat{\widetilde{Q}}_{1,i}\big) \widehat{\widetilde{G}}^{-1}_{1,i} = \big(T_i Q_i \widehat{P}_{1,i} + P_{i-1} T_{1,i} \widehat{Q}_{1,i}\big) \widehat{G}^{-1}_{1,i}. \quad \square$$

From the formulae (3.17)–(3.19), it follows that the "fundamental solution" of equations

$$A_i X_{i+1} = B_i X_i, \quad i = \overline{0, N-1}$$

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can be determined as $X_i := \Pi_i M^{(i)}_{i-1},\ i = \overline{0, N-1}$ and $X_N := M^{(N)}_{N-1}$. We define $R := \operatorname{diag}(P_{-1}\widehat{P}_{1,-1}, I - P_{N-1}\widehat{P}_{1,N-1})$ and the matrix (D_1, D_2), whose columns are the columns of the matrices $D_1 := \sum\limits_{i=0}^{N} C_i X_i$ and

$$D_2 := -C_{N-1}T_{N-1}Q_{N-1}\widehat{Q}_{1,N-1} + C_N Q_{N-1} + C_N P_{N-1}\widehat{Q}_{1,N-1}.$$

LEMMA 3.8. *Suppose that the* LSDE (3.15) *is of index-2. Then the following condition*

$$\ker(D_1, D_2) = \ker R \tag{3.22}$$

does not depend on the chosen T_i, Q_i and $T_{1,i}$.

Proof. Assume that $\widetilde{Q}_i$ is another projection onto $\ker A_i$ and $\widetilde{T}_i$ (resp., $\widetilde{T}_{1,i}$) is another transformation with $\widetilde{T}_i|_{\ker A_i}$ (resp., $\widetilde{T}_{1,i}|_{\ker \widetilde{G}_i}$) being an isomorphism from $\ker A_i$ onto $\ker A_{i-1}$ (resp., $\ker\widetilde{G}_i$ onto $\ker\widetilde{G}_{i-1}$). Here, the matrices $\widetilde{P}_i$, $\widetilde{G}_i$, $\widehat{\widetilde{Q}}_{1,i}$, $\widehat{\widetilde{P}}_{1,i}$, $\widehat{\widetilde{G}}_{1,i}$ and $\widetilde{\Pi}_i$ are defined in the proof of Lemma 3.7. We put

$$\widetilde{D}_1 := \sum_{i=0}^{N} C_i \widetilde{X}_i,\ \widetilde{X}_i := \widetilde{\Pi}_i \widetilde{M}^{(i)}_{i-1},\ i = \overline{0, N-1},\ \widetilde{X}_N := \widetilde{M}^{(N)}_{N-1},$$

$$\widetilde{D}_2 := -C_{N-1}\widetilde{T}_{N-1}\widetilde{Q}_{N-1}\widehat{\widetilde{Q}}_{1,N-1} + C_N\widetilde{Q}_{N-1} + C_N\widetilde{P}_{N-1}\widehat{\widetilde{Q}}_{1,N-1}$$

and

$$\widetilde{R} := \operatorname{diag}(\widetilde{P}_{-1}\widehat{\widetilde{P}}_{1,-1}, I - \widetilde{P}_{N-1}\widehat{\widetilde{P}}_{1,N-1}),$$

where $\widetilde{M}^{(i)}_{i-1} := \prod\limits_{l=0}^{i-1} \widetilde{P}_{i-l-1}\widehat{\widetilde{P}}_{1,i-l-1}\widehat{\widetilde{G}}^{-1}_{1,i-l-1}B_{i-l-1}$ $(i = \overline{0, N})$.

Lemma 3.7 ensures that $X_i = \widetilde{X}_i$ for all $i = \overline{0, N-1}$. Besides, using Eq. (2.7) and the facts that $\widetilde{P}_{N-1}\widehat{\widetilde{P}}_{1,N-1}\widetilde{T}^{-1}_{N-1}\widetilde{Q}_{N-2} = 0$, $\widetilde{P}_{N-2}\widehat{\widetilde{P}}_{1,N-2}\widetilde{\Pi}_{N-1} = \widetilde{P}_{N-2}\widehat{\widetilde{P}}_{1,N-2}$ we have

$$\widetilde{M}^{(N)}_{N-1} = \widetilde{P}_{N-1}\widehat{\widetilde{P}}_{1,N-1}\widehat{\widetilde{G}}^{-1}_{1,N-1}B_{N-1}\widetilde{\Pi}_{N-1}\widetilde{M}^{(N-1)}_{N-2}.$$

Applying Lemma 3.7 again and noting that

$$\widetilde{P}_{N-1}\widehat{\widetilde{P}}_{1,N-1}\widehat{\widetilde{G}}^{-1}_{1,N-1} = P_{N-1}\widehat{P}_{1,N-1}\widehat{G}^{-1}_{1,N-1} - \widetilde{Q}_{N-1}P_{N-1}\widehat{P}_{1,N-1}\widehat{G}^{-1}_{1,N-1},$$

we obtain

$$\widetilde{M}^{(N)}_{N-1} = M^{(N)}_{N-1} - \widetilde{Q}_{N-1}P_{N-1}\widehat{P}_{1,N-1}\widehat{G}^{-1}_{1,N-1}B_{N-1}\Pi_{N-1}M^{(N-1)}_{N-2},$$

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or

$$\widetilde{X}_N = X_N - \widetilde{Q}_{N-1}P_{N-1}\widehat{P}_{1,N-1}\widehat{G}_{1,N-1}^{-1}B_{N-1}X_{N-1}.$$

This implies that

$$\widetilde{D}_1 = D_1 - C_N\widetilde{Q}_{N-1}P_{N-1}\widehat{P}_{1,N-1}\widehat{G}_{1,N-1}^{-1}B_{N-1}X_{N-1}. \tag{3.23}$$

According to Eq. (2.5), we get

$$\widetilde{T}_{N-1}\widetilde{Q}_{N-1}\widehat{\widetilde{Q}}_{1,N-1} = T_{N-1}Q_{N-1}\widehat{Q}_{1,N-1} \text{ and } \widetilde{P}_{N-1}\widehat{\widetilde{Q}}_{1,N-1} = \widetilde{P}_{N-1}\widehat{Q}_{1,N-1}.$$

Therefore,

$$\widetilde{D}_2 = D_2 + C_N(\widetilde{Q}_{N-1} - Q_{N-1}) + C_N(\widetilde{P}_{N-1} - P_{N-1})\widehat{Q}_{1,N-1},$$

or equivalently,

$$\widetilde{D}_2 = D_2 + C_N(\widetilde{Q}_{N-1} - Q_{N-1})\widehat{P}_{1,N-1}. \tag{3.24}$$

Now suppose that Eq. (3.22) is valid. Let $(\bar{x}_0^T, \bar{x}_N^T)^T \in \ker(\widetilde{D}_1, \widetilde{D}_2)$ then

$$\widetilde{D}_1\bar{x}_0 + \widetilde{D}_2\bar{x}_N = 0.$$

From Eqs. (3.23)–(3.24), the above equation can be rewritten as follows

$$\begin{aligned} D_1\bar{x}_0 + D_2\bar{x}_N - C_N\widetilde{Q}_{N-1}P_{N-1}\widehat{P}_{1,N-1}\widehat{G}_{1,N-1}^{-1}B_{N-1}X_{N-1}\bar{x}_0 \\ +C_N(\widetilde{Q}_{N-1} - Q_{N-1})\widehat{P}_{1,N-1}\bar{x}_N = 0. \end{aligned}$$

Put

$$\xi := -\widetilde{Q}_{N-1}P_{N-1}\widehat{P}_{1,N-1}\widehat{G}_{1,N-1}^{-1}B_{N-1}X_{N-1}\bar{x}_0$$

and

$$\zeta := (\widetilde{Q}_{N-1} - Q_{N-1})\widehat{P}_{1,N-1}\bar{x}_N.$$

Using the facts that $\widetilde{Q}_{N-1} = Q_{N-1}\widetilde{Q}_{N-1}$ and $\widehat{Q}_{1,N-1}Q_{N-1} = 0$, we have

$$\begin{aligned} D_2(\xi + \zeta) = -C_N\widetilde{Q}_{N-1}P_{N-1}\widehat{P}_{1,N-1}\widehat{G}_{1,N-1}^{-1}B_{N-1}X_{N-1}\bar{x}_0 \\ +C_N(\widetilde{Q}_{N-1} - Q_{N-1})\widehat{P}_{1,N-1}\bar{x}_N. \end{aligned}$$

This gives

$$D_1\bar{x}_0 + D_2(\bar{x}_N + \xi + \zeta) = 0.$$

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This means that $\left(\bar{x}_0^T, (\bar{x}_N + \xi + \zeta)^T\right)^T \in \ker(D_1, D_2) = \ker R$. It ensures that $P_{-1}\widehat{P}_{1,-1}\bar{x}_0 = 0$ and $(I - P_{N-1}\widehat{P}_{1,N-1})(\bar{x}_N + \xi + \zeta) = 0$. Now applying Eq. (2.5) we come to the conclusion that $\widetilde{P}_{-1}\widehat{\widehat{P}}_{1,-1} = \widetilde{P}_{-1}\widehat{P}_{1,-1}$. Further, $\widetilde{P}_{-1} = \widetilde{P}_{-1}P_{-1}$, hence $\widetilde{P}_{-1}\widehat{\widehat{P}}_{1,-1} = \widetilde{P}_{-1}P_{-1}\widehat{P}_{1,-1}$. This implies that

$$\widetilde{P}_{-1}\widehat{\widehat{P}}_{1,-1}\bar{x}_0 = 0. \tag{3.25}$$

On the other hand, since $P_{-1}\widehat{P}_{1,-1}\bar{x}_0 = 0$ and $X_{N-1} = X_{N-1}P_{-1}\widehat{P}_{1,-1}$, it follows that $X_{N-1}\bar{x}_0 = 0$. Thus, $\xi = 0$ and we obtain $(I - P_{N-1}\widehat{P}_{1,N-1})(\bar{x}_N + \zeta) = 0$. Observing that $\zeta = Q_{N-1}\zeta$ and $P_{N-1}\widehat{P}_{1,N-1}Q_{N-1} = 0$, we get

$$\bar{x}_N + Q_{N-1}\zeta = P_{N-1}\widehat{P}_{1,N-1}\bar{x}_N.$$

This relation leads to that $\widehat{Q}_{1,N-1}\bar{x}_N = 0$, hence $\zeta = (\widetilde{Q}_{N-1} - Q_{N-1})\bar{x}_N$. This implies that

$$Q_{N-1}\bar{x}_N + Q_{N-1}(\widetilde{Q}_{N-1} - Q_{N-1})\bar{x}_N = 0$$

or we have $\widetilde{Q}_{N-1}\bar{x}_N = 0$. Thus,

$$\widetilde{Q}_{N-1}\bar{x}_N + \widetilde{P}_{N-1}\widehat{Q}_{1,N-1}\bar{x}_N = 0.$$

This means that

$$\widetilde{Q}_{N-1}\bar{x}_N + \widetilde{P}_{N-1}\widehat{\widehat{Q}}_{1,N-1}\bar{x}_N = 0.$$

The last equation is equivalent to

$$(I - \widetilde{P}_{N-1}\widehat{\widehat{P}}_{1,N-1})\bar{x}_N = 0. \tag{3.26}$$

Combining Eqs. (3.25)–(3.26), we come to the conclusion that $(\bar{x}_0^T, \bar{x}_N^T)^T \in \ker\widetilde{R}$. Thus, the inclusion $\ker(\widetilde{D}_1, \widetilde{D}_2) \subseteq \ker\widetilde{R}$ is proved. To show the converse inclusion, we observe that for arbitrary $(\bar{x}_0^T, \bar{x}_N^T)^T \in \ker\widetilde{R}$, i.e, $\widetilde{P}_{-1}\widehat{\widehat{P}}_{1,-1}\bar{x}_0 = 0$ and $(I - \widetilde{P}_{N-1}\widehat{\widehat{P}}_{1,N-1})\bar{x}_N = 0$. Due to $\widetilde{X}_i = \widetilde{X}_i\widetilde{P}_{-1}\widehat{\widehat{P}}_{1,-1}$ $(i = \overline{0,N})$, it implies that $\widetilde{D}_1\bar{x}_0 = 0$. Notice that the equality $(I - \widetilde{P}_{N-1}\widehat{\widehat{P}}_{1,N-1})\bar{x}_N = 0$ is equivalent to the following relation

$$\widetilde{Q}_{N-1}\bar{x}_N + \widetilde{P}_{N-1}\widehat{\widehat{Q}}_{1,N-1}\bar{x}_N = 0.$$

Moreover, since $\widehat{\widehat{Q}}_{1,N-1}\widetilde{P}_{N-1}\widehat{\widehat{P}}_{1,N-1} = 0$ it follows $\widehat{\widehat{Q}}_{1,N-1}\bar{x}_N = 0$. Therefore, we obtain $\widetilde{D}_2\bar{x}_N = 0$. It implies that $\widetilde{D}_1\bar{x}_0 + \widetilde{D}_2\bar{x}_N = 0$ or $(\bar{x}_0^T, \bar{x}_N^T)^T \in \ker(\widetilde{D}_1, \widetilde{D}_2)$. $\square$

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We denote by

$$z_i := \Pi_i \sum_{k=0}^{i-1} M^{(i)}_{i-k-2} P_k \widehat{P}_{1,k} \widehat{G}^{-1}_{1,k} q_k + T_i Q_i T_{1,i+1} \widehat{Q}_{1,i+1} \widehat{G}^{-1}_{1,i+1} q_{i+1} - T_i Q_i \widehat{P}_{1,i} \widehat{G}^{-1}_{1,i} q_i$$
$$- P_{i-1} T_{1,i} \widehat{Q}_{1,i} \widehat{G}^{-1}_{1,i} q_i, \quad i = \overline{0, N-2},$$

$$z_{N-1} := \Pi_{N-1} \sum_{k=0}^{N-2} M^{(N-1)}_{N-k-3} P_k \widehat{P}_{1,k} \widehat{G}^{-1}_{1,k} q_k - T_{N-1} Q_{N-1} \widehat{P}_{1,N-1} \widehat{G}^{-1}_{1,N-1} q_{N-1}$$
$$- P_{N-2} T_{1,N-1} \widehat{Q}_{1,N-1} \widehat{G}^{-1}_{1,N-1} q_{N-1},$$

$$z_N := \sum_{k=0}^{N-1} M^{(N)}_{N-k-2} P_k \widehat{P}_{1,k} \widehat{G}^{-1}_{1,k} q_k \quad \text{and} \quad \gamma^* := \gamma - \sum_{i=0}^{N} C_i z_i.$$

Note that Lemma 3.8 guarantees that the following theorem does not depend on the chosen T_i, Q_i and $T_{1,i}$.

Theorem 3.9. *Let the* LSDE (3.15) *be of index-2. Then the* MPBVP (3.15), (3.16) *is uniquely solvable for every* $q_i \in \mathbb{R}^m$ $(i = \overline{0, N-1})$ *and every* $\gamma \in \mathbb{R}^m$ *if and only if the condition* (3.22) *holds. Moreover, the unique solution can be represented as*

$$\left\{ \begin{array}{l} x_i = X_i \xi + z_i, \quad i = \overline{0, N-2}, \\ x_{N-1} = X_{N-1} \xi + z_{N-1} - T_{N-1} Q_{N-1} \widehat{Q}_{1,N-1} \zeta, \\ x_N = X_N \xi + z_N + Q_{N-1} \zeta + P_{N-1} \widehat{Q}_{1,N-1} \zeta, \end{array} \right. \tag{3.27}$$

where $(\xi^T, \zeta^T)^T = (D_1, D_2)^+ \gamma^*$ *with* $(D_1, D_2)^+$ *the generalized inverse in Moore-Penrose's sense of* (D_1, D_2).

Proof. The proof of Theorem 3.9 is quite similar to that of Theorem 3.2, hence it will be outlined only.

First, observe that the equations corresponding to (3.8) and (3.9) are

$$(I - T_0 Q_0 \widehat{P}_{1,0} \widehat{G}_{1,0} B_0) P_{-1} \widehat{P}_{1,-1} \bar{x}_0 = 0 \tag{3.28}$$

and

$$X_N \bar{x}_0 + Q_{N-1} \bar{x}_N + P_{N-1} \widehat{Q}_{1,N-1} \bar{x}_N = 0. \tag{3.29}$$

Using the facts that $P_{-1} \widehat{P}_{1,-1} = P_0 \widehat{P}_{1,0}$, $T_0 = I$ and $P_0 Q_0 = 0$, we obtain that Eq. (3.28) yields

$$P_0 \widehat{P}_{1,0} \bar{x}_0 = 0.$$

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Moreover, $X_N\bar{x}_0 = X_N P_0 \widehat{P}_{1,0}\bar{x}_0 = 0$, hence, Eq. (3.29) implies that

$$Q_{N-1}\bar{x}_N + P_{N-1}\widehat{Q}_{1,N-1}\bar{x}_N = 0,$$

or equivalently,

$$(I - P_{N-1}\widehat{P}_{1,N-1})\bar{x}_N = 0.$$

In the proof of the converse part, we note that the solution formula (3.5) is replaced with the formulae (3.17)–(3.19). Since $X_i = X_i P_{-1}\widehat{P}_{1,-1}$, $\forall i = \overline{0,N}$ and $P_{-1}\widehat{P}_{1,-1}\bar{x}_0 = 0$, it gives $X_i\bar{x}_0 = 0$ $(i = \overline{0,N})$. Using the fact that $P_{N-1}Q_{N-1} = 0$, we can conclude that the equality $Q_{N-1}\bar{x}_N + P_{N-1}\widehat{Q}_{1,N-1}\bar{x}_N = 0$ implies that $Q_{N-1}\bar{x}_N = 0$ and $P_{N-1}\widehat{Q}_{1,N-1}\bar{x}_N = 0$. Besides, since

$$\bar{x}_N = Q_{N-1}\bar{x}_N + P_{N-1}\widehat{Q}_{1,N-1}\bar{x}_N + P_{N-1}\widehat{P}_{1,N-1}\bar{x}_N,$$

we get $\bar{x}_N = P_{N-1}\widehat{P}_{1,N-1}\bar{x}_N$. This leads to

$$\widehat{Q}_{1,N-1}\bar{x}_N = \widehat{Q}_{1,N-1}P_{N-1}\widehat{P}_{1,N-1}\bar{x}_N = \widehat{Q}_{1,N-1}(I - Q_{N-1})\widehat{P}_{1,N-1}\bar{x}_N = 0. \quad \square$$

Next, we have the following useful lemma.

LEMMA 3.10. *The dimensions of* $\ker R$ *and* $\ker(D_1, D_2)$ *are independent of the choice of* T_i*,* Q_i *and* $T_{1,i}$*, moreover,* $\dim(\ker R) = m$ *and* $\dim\big(\ker(D_1, D_2)\big) =: p \geq m$*.*

Proof. Firstly, we observe that

$$\ker(P_i\widehat{P}_{1,i}) = \ker A_i \oplus \ker G_i, \quad i = \overline{-1, N-1}. \tag{3.30}$$

Indeed, let $\xi \in \ker(P_i\widehat{P}_{1,i})$ means that $P_i\widehat{P}_{1,i}\xi = 0$. We write ξ as $\xi = \widehat{P}_{1,i}\xi + \widehat{Q}_{1,i}\xi$. Clearly, $\widehat{Q}_{1,i}\xi \in \ker G_i$. Furthermore, since $P_i\widehat{P}_{1,i}\xi = 0$, it implies that $\widehat{P}_{1,i}\xi = (P_i + Q_i)\widehat{P}_{1,i}\xi = Q_i\widehat{P}_{1,i}\xi \in \ker A_i$. Thus, we get

$$\ker(P_i\widehat{P}_{1,i}) \subseteq \ker A_i + \ker G_i, \quad i = \overline{-1, N-1}.$$

Conversely, for arbitrary $\xi = y + z \in \ker A_i + \ker G_i$, we see that $P_i\widehat{P}_{1,i}\xi = P_i\widehat{P}_{1,i}Q_i y + P_i\widehat{P}_{1,i}\widehat{Q}_{1,i}z = 0$. Therefore,

$$\ker A_i + \ker G_i \subseteq \ker(P_i\widehat{P}_{1,i}), \quad i = \overline{-1, N-1},$$

which yields

$$\ker(P_i\widehat{P}_{1,i}) = \ker A_i + \ker G_i, \quad i = \overline{-1, N-1}.$$

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On the other hand, since $\widehat{Q}_{1,i}Q_i = 0$, it is easy to verify that $\ker A_i \cap \ker G_i = \{0\}$. This leads to the identity (3.30), as it was to be proved.

Noting that $\mathrm{rank} A_i = r$ and $\mathrm{rank} G_i = s$ for all $i = \overline{-1, N-1}$ and applying Eq. (3.30), we obtain that

$$\mathrm{rank}(P_i\widehat{P}_{1,i}) = r + s - m, \quad i = \overline{-1, N-1}.$$

In particular, we have $\mathrm{rank}(P_{-1}\widehat{P}_{1,-1}) = r+s-m$ and $\mathrm{rank}(P_{N-1}\widehat{P}_{1,N-1}) = r+s-m$. It follows that $\dim(\ker R) = m$.

Now using notations and the arguments in the proof of Lemma 3.8, we consider a linear operator $\mathcal{F}$ from $\ker(\widetilde{D}_1, \widetilde{D}_2)$ to $\ker(D_1, D_2)$, defined by

$$\begin{aligned}\mathcal{F}(\xi^T, \zeta^T)^T := \Big(\xi^T, (\zeta + (\widetilde{Q}_{N-1} - Q_{N-1})\widehat{P}_{1,N-1}\zeta \\ -\widetilde{Q}_{N-1}P_{N-1}\widehat{P}_{1,N-1}\widehat{G}^{-1}_{1,N-1}B_{N-1}X_{N-1}\xi)^T\Big)^T.\end{aligned}$$

Let $(y^T, z^T)^T \in \ker(D_1, D_2)$ be arbitrary, i.e., $D_1y + D_2z = 0$. Then we determine two vectors ξ and ζ by $\xi = y$ and

$$\zeta = z + (Q_{N-1} - \widetilde{Q}_{N-1})\widehat{\widehat{P}}_{1,N-1}z + \widetilde{Q}_{N-1}P_{N-1}\widehat{P}_{1,N-1}\widehat{G}^{-1}_{1,N-1}B_{N-1}X_{N-1}y.$$

From Eq. (2.5) and the facts that $\widetilde{Q}_{N-1}Q_{N-1} = Q_{N-1}$ and $\widehat{Q}_{1,N-1}Q_{N-1} = 0$, we obtain

$$\mathcal{F}(\xi^T, \zeta^T)^T = (y^T, z^T)^T.$$

Moreover, it is easy to see that

$$\widetilde{D}_1\xi + \widetilde{D}_2\zeta = 0, \text{ i.e., } (\xi^T, \zeta^T)^T \in \ker(\widetilde{D}_1, \widetilde{D}_2).$$

This implies that $\mathcal{F}$ is sujective, hence $\mathcal{F}\big(\ker(\widetilde{D}_1, \widetilde{D}_2)\big) = \ker(D_1, D_2)$. According to the property of the linear operator, we get $\dim\mathcal{F}\big(\ker(\widetilde{D}_1, \widetilde{D}_2)\big) \le \dim\big(\ker(\widetilde{D}_1, \widetilde{D}_2)\big)$, hence

$$\dim\big(\ker(D_1, D_2)\big) \le \dim\big(\ker(\widetilde{D}_1, \widetilde{D}_2)\big).$$

Similarly, we also have the following inequality

$$\dim\big(\ker(\widetilde{D}_1, \widetilde{D}_2)\big) \le \dim\big(\ker(D_1, D_2)\big),$$

which implies that

$$\dim\big(\ker(\widetilde{D}_1, \widetilde{D}_2)\big) = \dim\big(\ker(D_1, D_2)\big).$$

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Thus, $\dim\bigl(\ker(D_1, D_2)\bigr)$ does not depend on the choice of T_i, Q_i and $T_{1,i}$. On the other hand, the following inclusion

$$\ker R \subseteq \ker(D_1, D_2)$$

is always valid. Therefore, $\dim\bigl(\ker(D_1, D_2)\bigr) := p \geq m$. □

As a direct consequence of Theorem 3.9 and Lemma 3.10 we come to the following corollary.

Corollary 3.11. *Suppose that the* LSDE (3.15) *is of index-2. Then the* MPBVP (3.15), (3.16) *is uniquely solvable if and only if* $\dim\bigl(\ker(D_1, D_2)\bigr) = m$.

Now we turn to the case when (3.22) is not valid, i.e., $p > m$ and the MPVBP (3.15), (3.16) has either no solution or an infinite number of solutions. Using the same notations as in the constant coefficients case, we define column matrices $\Phi_i := X_i\mathcal{U}$ $(i = \overline{0, N-2})$, $\Phi_{N-1} := X_{N-1}\mathcal{U} - T_{N-1}Q_{N-1}\widehat{Q}_{1,N-1}\mathcal{V}$ and $\Phi_N := X_N\mathcal{U} + Q_{N-1}\mathcal{V} + P_{N-1}\widehat{Q}_{1,N-1}\mathcal{V}$ and a linear operator $\mathcal{L}$ acting in $\mathbb{R}^{m(N+1)}$,

$$\mathcal{L}(x_0^T, \ldots, x_N^T)^T := \Bigl((A_0x_1 - B_0x_0)^T, \ldots, (A_{N-1}x_N - B_{N-1}x_{N-1})^T, \bigl(\sum_{i=0}^{N} C_ix_i\bigr)^T\Bigr)^T.$$

The counterpart of Lemma 3.3 is now as follows.

Lemma 3.12. $\ker\mathcal{L} = \bigl\{\bigl((\Phi_0a)^T, \ldots, (\Phi_Na)^T\bigr)^T : a \in \mathbb{R}^{p-m}\bigr\}$.

Proof. The proof is similar to that for Lemma 3.3 except that now we note that $T_{N-1}Q_{N-1} = Q_{N-2}T_{N-1}Q_{N-1}$, $A_iQ_i = 0$ $(i = N-2,\ N-1)$, $G_{N-1}\widehat{Q}_{1,N-1} = 0$, $X_i = X_iP_{-1}\widehat{P}_{1,-1}$ $(i = \overline{0, N})$, and the equality $(I - P_{N-1}\widehat{P}_{1,N-1})v_k^0 = 0$ is equivalent to the relation $Q_{N-1}v_k^0 + P_{N-1}\widehat{Q}_{1,N-1}v_k^0 = 0$ and implies that $\widehat{Q}_{1,N-1}v_k^0 = 0$ for each $k = \overline{1, m}$. □

Theorem 3.13. *Suppose that the* LSDE (3.15) *is of index-2 and* $p > m$. *Then, the problem* (3.15), (3.16) *possesses a solution if and only if*

$$W\gamma^* = 0. \tag{3.31}$$

Moreover, a general solution of (3.15), (3.16) *can be given by*

$$\begin{cases} x_i = X_i\xi + z_i + \Phi_ia, \quad i = \overline{0, N-2}, \\ x_{N-1} = X_{N-1}\xi + z_{N-1} - T_{N-1}Q_{N-1}\widehat{Q}_{N-1}\zeta + \Phi_{N-1}a, \\ x_N = X_N\xi + z_N + Q_{N-1}\zeta + P_{N-1}\widehat{Q}_{N-1}\zeta + \Phi_Na, \end{cases} \tag{3.32}$$

where $a \in \mathbb{R}^{p-m}$ *is an arbitrary vector and* $(\xi^T, \zeta^T)^T = (D_1, D_2)^+\gamma^*$ *with* $(D_1, D_2)^+$ *the generalized inverse in Moore-Penrose's sense of* (D_1, D_2).

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Proof. The proof of Theorem 3.13 is similar to that of Theorem 3.4 and will be omitted. □

Combining Theorem 3.9 and Theorem 3.13 we come to the following statement.

COROLLARY 3.14. (Fredholm alternative) *Assume that the* LSDE (3.15) *is of index-2 and let* $p := \dim\big(\ker(D_1, D_2)\big)$. *Then*

(i) *either* $p = m$ *and the problem* (3.15), (3.16) *is uniquely solvable for any data* q_i $(i = \overline{0, N-1})$ *and* γ*;*
(ii) *or* $p > m$ *and the problem* (3.15), (3.16) *is solvable if and only if the condition* (3.31) *is valid.*

Moreover, the solution formula (3.32) *holds.*

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