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COMPLETENESS IN THE SENSE OF CAUCHY PRINCIPAL VALUE OF EIGENSYSTEMS FOR INFINITE DIMENSIONAL HAMILTONIAN OPERATORS*

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Abstract. This paper studies the completeness properties of eigensystems associated with a class of infinite dimensional Hamiltonian operators (IDHOs). We establish necessary and sufficient conditions for the completeness in the sense of the Cauchy Principal Value (CCPV) for the eigensystems of these operators. Some examples are provided to illustrate the validity of the criteria. Additionally, we provide sufficient conditions for the CCPV of the eigensystems for specific classes of 4×4 IDHOs.

Key words. Infinite dimensional Hamiltonian operator, Generalized eigenvector, Completeness, Cauchy principal value.

AMS subject classifications. 15A45, 14M15, 53C22.

1. Introduction. In Hamiltonian mechanics, the time evolution of a system is uniquely defined by the Hamiltonian equations [13]:

$$\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q}, \tag{1.1}$$

or equivalently, we can express the Hamiltonian system in matrix form as

$$\frac{dZ}{dt} = J_{2n} \cdot \nabla H, \tag{1.2}$$

where

$$Z = \begin{bmatrix} q \\ p \end{bmatrix}, \quad J_{2n} = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix},$$

and ∇ denotes the gradient of the Hamiltonian $H = H(q(t), p(t))$ often represents the total energy of the system.

In (1.1), let

$$H = \frac{1}{2} Z^\top S Z,$$

where $Z = Z(t) \in \mathbb{R}^{2n},\ t \in \mathbb{R}$, and

$$S = S(t) = \begin{bmatrix} -C(t) & A^\top(t) \\ A(t) & B(t) \end{bmatrix},$$

is a bounded measurable real $2n \times 2n$ matrix-valued function satisfying $S^\top(t) = S(t)$. By calculating the Gâteaux direction derivative, we obtain

$$\frac{\partial H}{\partial p} = \dot{q} = A(t)q + B(t)p,$$

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$$\frac{\partial H}{\partial q} = -\dot{p} = -C(t)q + A^\top(t)p,$$

and the corresponding Hamiltonian system becomes

$$\dot{Z} = \begin{bmatrix} A(t) & B(t) \\ C(t) & -A^\top(t) \end{bmatrix} Z, \tag{1.3}$$

where $\dot{Z}$ denotes the derivative of Z. This is known as a linear Hamiltonian system, and the matrix

$$H(t) = \begin{bmatrix} A(t) & B(t) \\ C(t) & -A^\top(t) \end{bmatrix},$$

is called a Hamiltonian matrix, and it satisfies

$$(J_{2n}H)^\top(t) = J_{2n}H(t).$$

Hamiltonian matrices have numerous applications in the theory of mechanical systems, the calculus of variations, control theory, and other areas [15]. For instance, one can set $Z = [x\ y]^\top$, where $x(t), y(t) \in \mathbb{R}^n$, and

$$H = \begin{bmatrix} 0_n & I_n \\ G(t) & 0_n \end{bmatrix},$$

where $G^\top(t) = G(t)$ is a real $n \times n$ symmetric matrix-valued function. Then the linear Hamiltonian system (1.3) is equivalent to the second-order system

$$x'' = G(t)x,$$

which often arises in the study of mechanical systems near an equilibrium.

In (1.3), by replacing the Euclidean space $\mathbb{R}^n$, the single-variable function Z, the matrix-valued functions, and the matrix transpose operation with an infinite dimensional complex Hilbert space X, a multivariate function Z, linear operators, and the adjoint operation, respectively, we obtain the infinite dimensional linear Hamiltonian system [8, 10, 22]

$$\dot{Z} = \begin{bmatrix} A & B \\ C & -A^* \end{bmatrix} Z, \tag{1.4}$$

where A, B, C are densely defined linear operators on the Hilbert space X, A^* is the adjoint of A, and B and C satisfy $B^* = B$ and $C^* = C$. The densely defined block operator matrix

$$H = \begin{bmatrix} A & B \\ C & -A^* \end{bmatrix},$$

is called an infinite dimensional Hamiltonian operator (abbreviated IDHO). Furthermore, if B and C are nonnegative (denoted by $B \geq 0$ and $C \geq 0$), i.e., $(Bx, x) \geq 0$ for all $x \in \mathcal{D}(B)$ and $(Cx, x) \geq 0$ for all $x \in \mathcal{D}(C)$, then H is called a *nonnegative* Hamiltonian operator. Similarly, we define nonpositive, positive, and negative operators, respectively. IDHOs play a crucial role in various applications, including elasticity, celestial mechanics, and dynamic systems [1, 2, 16, 17]. For example, the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

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is equivalent to the infinite dimensional linear Hamiltonian system

$$\frac{\partial}{\partial y}\begin{bmatrix} u \\ \sigma \end{bmatrix} = \begin{bmatrix} 0 & I \\ -\frac{\partial^2}{\partial x^2} & 0 \end{bmatrix}\begin{bmatrix} u \\ \sigma \end{bmatrix}.$$

The corresponding IDHO is

$$H = \begin{bmatrix} A & B \\ C & -A^* \end{bmatrix} = \begin{bmatrix} 0 & I \\ -\frac{\partial^2}{\partial x^2} & 0 \end{bmatrix}.$$

To apply the Fourier method, i.e., method of separation of variables, to the system (1.4), it is necessary to establish completeness theorem of the IDHO. Consequently, results on the spectral Riesz basis of generalized eigenvectors for IDHOs were developed using the algebraic multiplicities of the eigenvalues [24]. In [19], the plane elasticity problems of two-dimensional octagonal quasicrystals were analyzed using the symplectic approach. In [6], sufficient conditions for the completeness of the eigenfunction system of IDHOs, in the sense of the Cauchy principal value (CCPV), were provided.

In this paper, we study a completeness theory for a class of IDHOs by presenting both sufficient and necessary conditions for the CCPV of their eigensystem. Additionally, we provide sufficient conditions for the CCPV of the eigensystem of a special class of 4×4 IDHOs.

Throughout this paper, we denote by X an infinite dimensional complex Hilbert space with inner produce $\langle \cdot, \cdot \rangle$, by $\mathcal{B}(X)$ the Banach algebra of bounded linear operators on X, and by $\mathcal{C}(X)$ the set of densely defined, closed linear operators on X. In Section 3, we restrict our discussion to separable X.

2. Eigenvectors and generalized eigenvectors of an IDHO. Recall that for a finite dimensional linear operator A on a vector space V, a generalized eigenvector x corresponding to λ generates a Jordan chain of linearly independent generalized eigenvectors [20]. For example, if A is the $n \times n$ Jordan block with eigenvalue λ:

$$J_n(\lambda) = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ 0 & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 1 \\ 0 & 0 & 0 & \cdots & \lambda \end{bmatrix},$$

the generalized eigenvectors are

$$e_n,\ e_{n-1} = (A - \lambda I_n)e_n,\ e_{n-2} = (A - \lambda I_n)e_{n-1}, \ldots, e_2 = (A - \lambda I_n)e_3,\ e_1 = (A - \lambda I_n)e_2,$$

and e_1 is the ordinary eigenvector. In general, let x_k be a generalized eigenvector of order k corresponding to the matrix A and the eigenvalue λ. The chain generated by $x = x_k$ is the set of vectors $\{x = x_k, x_{k-1}, \ldots, x_0\}$, given by $x_j = (A-\lambda I)^{k-j}x$, $j = k, k-1, \ldots, 0$, with $(T-\lambda I)^{k+1}x = 0$, $(T-\lambda I)^k x \neq 0$, where $x_0 = (A-\lambda I)^k x$ is always an ordinary eigenvector. The Jordan decomposition theorem [11, 23] ensures that generalized eigenvectors, which are not ordinary eigenvectors, exist for finite dimensional defective linear operators.

Analogously, for a linear operator T on an infinite dimensional Hilbert space X, the set $\sigma_p(T)$ of all eigenvalues of T is called the *point spectrum* of T. A vector $x \in X$ is called a *kth order generalized eigenvector* of T corresponding to an eigenvalue λ if, for some $k \geq 1$, the following conditions hold:

$$(T - \lambda I)^{k+1}x = 0, \quad (T - \lambda I)^k x \neq 0.$$

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Clearly, $x_0 := (T-\lambda I)^k x$ is an ordinary eigenvector corresponding to the eigenvalue λ. In other words, eigenvectors are precisely the generalized eigenvectors of order 0.

Recall the null space of $T-\lambda I$:

$$N(T-\lambda I) := \{x \in \mathcal{D}(T) : (T-\lambda I)x = 0\},$$

which is the eigenspace $E_\lambda(T)$ of T corresponding to λ. Here, $\mathcal{D}(T)$ denotes the domain of T.

The subspace

$$N_\lambda^k := N((T-\lambda I)^{k+1}) = \{x \in \mathcal{D}(T^{k+1}) : (T-\lambda I)^{k+1}x = 0\}, \quad k = 0, 1, 2, \dots,$$

is called the generalized eigenspace of order (or rank) k corresponding to λ. It is invariant under T Clearly, $N_\lambda^0 := N(T-\lambda I) = E_\lambda(T)$, and the inclusion relation $N_\lambda^0 \subset N_\lambda^1 \subset N_\lambda^2 \subset \cdots$ holds.

The *generalized eigenspace* of T corresponding to λ is defined as the linear span of all generalized eigenvectors corresponding to λ, i.e.,

$$L_\lambda(T) := \bigcup_{k=0}^{\infty} N_\lambda^k.$$

Unlike the finite dimensional case, the existence of eigenvectors or generalized eigenvectors is not guaranteed for infinite dimensional operators. For example, the unilateral (forward) shift operator $S : \ell^2 \to \ell^2$, defined by $S(x_1, x_2, \dots) = (0, x_1, x_2, \dots)$, has no eigenvalues [14, p. 49] and thus no generalized eigenvectors, though the spectrum of S is the closed unit disc [12, p. 227].

Lemma 2.1. Let $T \in \mathcal{C}(X)$ and k be a positive integer. Suppose λ is an eigenvalue of T. If for any $f \in N(T-\lambda I)$, there exists $g \in N(T^*-\overline{\lambda}I)^k$ such that $\langle f, g\rangle \neq 0$, then the order of generalized eigenvectors of T corresponding to λ is less than k. In particular, if $k = 1$, then all generalized eigenvectors of T corresponding to λ are ordinary eigenvectors.

Proof. Assume, for the sake of contradiction, that the order of generalized eigenvectors of T corresponding to λ is at least k. Then, there exists $u \in X$ such that

$$(T-\lambda I)^{k+1}u = 0 \quad \text{and} \quad (T-\lambda I)^k u \neq 0,$$

which implies that $(T-\lambda I)^k u \in N(T-\lambda I)$. By assumption, there exists $g \in N(T^*-\overline{\lambda}I)^k$ such that $\langle (T-\lambda I)^k u, g\rangle \neq 0$. So we have

$$0 = \langle u, (T^*-\overline{\lambda}I)^k g\rangle = \langle (T-\lambda I)^k u, g\rangle \neq 0, \qquad \square$$

a contradiction. Hence, the order of generalized eigenvectors of T corresponding to λ must be less than k.

If $k = 1$, the only generalized vectors corresponding to λ are those of rank 0, which are ordinary eigenvectors.

A basic property of the eigenvectors and generalized eigenvectors of the IDHOs is their symplectic orthogonality, due to the symplectic structure J given in (2.5). The following lemma, a specific part of [3, Theorem 2.1.13], is provided with an elementary proof for the sake of completeness.

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LEMMA 2.2. Let $H = \begin{bmatrix} A & B \\ C & -A^* \end{bmatrix} : \mathcal{D}(H) \subset X \times X \to X \times X$ be an IDHO. Suppose $\lambda_1, \lambda_2 \in \sigma_p(H)$ are distinct with $\lambda_1 \neq -\overline{\lambda}_2$. Let $u_i \in N(H - \lambda_i I)$, and let $v_i \in X$ be a generalized eigenvector such that

$$(H - \lambda_i I)^{k+1} v_i = 0, \quad (H - \lambda_i I)^k v_i \neq 0, \quad i = 1, 2.$$

Then the following symplectic orthogonality relations hold:

$$\langle u_1, J u_2 \rangle = 0, \quad \langle u_1, J v_2 \rangle = 0, \quad \langle v_1, J u_2 \rangle = 0, \quad \langle v_1, J v_2 \rangle = 0,$$

where

$$J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}, \tag{2.5}$$

is the canonical symplectic structure.

Proof. Step 1. We prove the symplectic orthogonality of eigenvectors. Let

$$H u_1 = \lambda_1 u_1, \tag{2.6}$$

$$H u_2 = \lambda_2 u_2. \tag{2.7}$$

Taking the inner product of (2.6) with $J u_2$ and $J u_1$ with (2.7) and using that facts that $J = -J^*$ and $JH \subset (JH)^*$, we obtain

$$-\langle J H u_1, u_2 \rangle = \langle J^* H u_1, u_2 \rangle = \langle H u_1, J u_2 \rangle = \lambda_1 \langle u_1, J u_2 \rangle, \tag{2.8}$$

$$\langle J H u_1, u_2 \rangle = \langle (JH)^* u_1, u_2 \rangle = \langle u_1, J H u_2 \rangle = \langle u_1, J \lambda u_2 \rangle = \overline{\lambda_2} \langle u_1, J u_2 \rangle. \tag{2.9}$$

Since $\lambda_1 + \overline{\lambda}_2 \neq 0$, by adding these two equations, we conclude that $\langle u_1, J u_2 \rangle = 0$.

Step 2. We prove the symplectic orthogonality of eigenvectors and generalized eigenvectors. Let v_2 be a generalized eigenvector corresponding to λ_2. By definition, for some positive integer,

$$(H - \lambda_2 I)^{k+1} v_2 = 0, \quad (H - \lambda_2 I)^k v_2 \neq 0.$$

Define $v_2^{(j)} = (H - \lambda_2 I)^j v_2$ for $j = 0, 1, \ldots, k$. Then, $v_2^{(0)} \in N(H - \lambda_2 I)$, and $v_2^{(j)} \in N((H - \lambda_2 I)^{k-j+1})$. We proceed by induction to show that $\langle u_1, J v_2^{(j)} \rangle = 0$ for all j. For the base case, $j = 0$, $v_2^{(0)}$ is an eigenvector corresponding to λ_2, so $\langle u_1, J v_2^{(0)} \rangle = \langle u_1, J u_2 \rangle = 0$ from Step 1. Assume $\langle u_1, J v_2^{(j)} \rangle = 0$ for some j. Then, $(H - \lambda_2 I) v_2^{(j)} = v_2^{(j-1)}$. Using $H u_1 = \lambda_1 u_1$ and the induction assumption, we have

$$\langle H u_1, J v_2^{(j)} \rangle = \lambda_1 \langle u_1, J v_2^{(j)} \rangle = 0.$$

Using the property $JH \subset (JH)^*$,

$$\begin{aligned} \langle H u_1, J v_2^{(j)} \rangle &= \langle u_1, -H^* J^* v_2^{(j)} \rangle = \langle u_1, -(JH)^* v_2^{(j)} \rangle = -\langle u_1, J H v_2^{(j)} \rangle \\ &= -\langle u_1, J(v_2^{(j-1)} + \lambda_2 v_2^{(j)}) \rangle = -\langle u_1, J v_2^{(j-1)} \rangle. \end{aligned}$$

Thus, $\langle u_1, J v_2^{(j-1)} \rangle = 0$, completing the induction.

Step 3. Suppose v_1 and v_2 are generalized eigenvectors satisfying $(H - \lambda_1 I)^{k+1} v_1 = 0$, $(H - \lambda_1 I)^k v_1 \neq 0$ and $(H - \lambda_2 I)^{k+1} v_2 = 0$, $(H - \lambda_2 I)^k v_2 \neq 0$. Using a similar inductive argument as in Step 2, we show that $\langle v_1, J v_2 \rangle = 0$. ∎

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We observe that the finite dimensional Hamiltonian matrix

(2.10) $$H = \begin{bmatrix} A & B \\ C & -A^* \end{bmatrix}, \quad \text{where } B = B^*, C = C^*, \quad A, B, C \in \mathbb{C}_{n\times n},$$

satisfies

$$(J_{2n}H)^* = J_{2n}H = \begin{bmatrix} C & -A^* \\ -A & -B \end{bmatrix}, \quad (HJ_{2n})^* = HJ_{2n} = \begin{bmatrix} -B & A \\ A^* & C \end{bmatrix}, \quad J_{2n}HJ_{2n}^{-1} = -H^*,$$

where

(2.11) $$J_{2n} = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix},$$

is the canonical symplectic structure. These relationships demonstrate that $J_{2n}H$ and HJ_{2n} are Hermitian matrices. Moreover, if λ is an eigenvalue of H with eigenvector $v \in \mathbb{C}^n \times \mathbb{C}^n$, i.e., $Hv = \lambda v$, then $H^*J_{2n}v = -J_{2n}Hv = -\lambda J_{2n}v$, i.e., $-\lambda$ is an eigenvalue of H^* with eigenvector $J_{2n}v \in \mathbb{C}^n \times \mathbb{C}^n$. Consequently, the eigenvalues of the finite dimensional Hamiltonian matrix H appear in $(\lambda, -\bar{\lambda})$, and in $(\lambda, -\lambda, \bar{\lambda}, -\bar{\lambda})$ if H is real. This spectrum symmetry is known as the Hamiltonian symmetry [18].

However, such symmetry does not necessarily extend to IDHOs. For example, let

$$X = L^2[0,\infty), \quad A = \frac{d}{dt} : X \to X \text{ defined by } Ax = \frac{dx}{dt}.$$

The domain of A

$$\mathcal{D}(A) := \{x \in X : x \text{ is locally absolutely continuous}, \ x' \in X, \ x(0) = 0\},$$

is dense in X and $A^* = \frac{d}{dt}$ with $\mathcal{D}(A^*) = \{x \in X : x \text{ is absolutely continuous}, \ x' \in X\}$:

$$H = \begin{bmatrix} A & 0 \\ 0 & -A^* \end{bmatrix} = \begin{bmatrix} \frac{d}{dt} & 0 \\ 0 & \frac{d}{dt} \end{bmatrix}.$$

The point spectrum is $\sigma_p(H) = \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda < 0\}$, which is not symmetric about the imaginary axis [7, Example 2.2.3]. This highlights the greater complexity of the infinite dimensional case compared to the finite dimensional one. For example, the adjoint of the finite dimensional Hamiltonian matrix H in (2.10) exists and is given by

$$H^* = \begin{bmatrix} A & B \\ C & -A^* \end{bmatrix}^* = \begin{bmatrix} A^* & C \\ B & -A \end{bmatrix}.$$

Now consider the IDHO

$$H_1 = \begin{bmatrix} A_1 & -A_1 \\ A_1 & -A_1 \end{bmatrix},$$

where A_1 is an unbounded self-adjoint linear operator on X. Then,

$$\begin{bmatrix} A_1 & -A_1 \\ A_1 & -A_1 \end{bmatrix}^* \neq \begin{bmatrix} A_1 & A_1 \\ -A_1 & -A_1 \end{bmatrix}.$$

In fact, $\begin{bmatrix} A_1 & -A_1 \\ A_1 & -A_1 \end{bmatrix}^*$ is a closed operator but $\begin{bmatrix} A_1 & A_1 \\ -A_1 & -A_1 \end{bmatrix}$ is not [7, Example 4.1.2].

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The notation $A \subset B$, where A and B are operators on X, means A is a restriction of B, i.e., $\mathcal{D}(A) \subset \mathcal{D}(B)$ and $Ax = Bx$ for all $x \in \mathcal{D}(A)$.

We recall that a linear operator T on X with dense domain $\mathcal{D}(T)$ is symmetric if $\langle Tx, y\rangle = \langle x, Ty\rangle$ for all $x, y \in \mathcal{D}(T)$. If $\mathcal{D}(T)$ is the entire X, then T is a bounded operator and thus T is self-adjoint. In general, a symmetric operator is self-adjoint if and only if $\mathcal{D}(T^*) \subset \mathcal{D}(T)$.

As

$$H = \begin{bmatrix} A & B \\ C & -A^* \end{bmatrix},$$

is densely defined, we have

$$JHJ = \begin{bmatrix} A^* & C \\ B & -A \end{bmatrix} \subset H^*.$$

The following result establishes that, under certain conditions, the eigenvalues of IDHOs are symmetric about the origin.

Theorem 2.3. Let $H = \begin{bmatrix} A & B \\ C & -A^* \end{bmatrix} : \mathcal{D}(H) \subset X \times X \to X \times X$ be an IDHO, where B is invertible and definite, $\mathcal{D}(B) \subset \mathcal{D}(A^*)$, and $B^{-1}A$ is symmetric, i.e., $B^{-1}A \subset A^*B^{-1}$. Then,

(i) $\sigma_p(H) \subset \mathbb{R} \cup i\mathbb{R}$.
(ii) $\sigma_p(H)$ is symmetric with respect to the origin, i.e., if λ is an eigenvalue of H, then so is $-\lambda$.
(iii) If $\lambda \in \sigma_p(H)\backslash\{0\}$, then all generalized eigenvectors corresponding to λ are ordinary eigenvectors.

Thus, for the finite dimensional Hamiltonian matrix H in (2.10), if B is positive (or negative) definite, and $B^{-1}A$ is Hermitian, then (i), (ii), and (iii) hold for H.

Proof. (i) Let $[u\ v]^\top \in X \times X$ be an eigenvector of H corresponding to λ, i.e.,

$$\begin{bmatrix} A & B \\ C & -A^* \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \lambda \begin{bmatrix} u \\ v \end{bmatrix}, \tag{2.12}$$

or equivalently,

$$Au + Bv = \lambda u, \tag{2.13}$$

$$Cu - A^*v = \lambda v. \tag{2.14}$$

Since B is invertible, from the first equation, we have

$$v = \lambda B^{-1}u - B^{-1}Au, \tag{2.15}$$

implying $u \neq 0$ as $[u\ v]^\top$ is an eigenvector. Multiplying this equation by λ and making use of (2.14), we have

$$\lambda v = \lambda^2 B^{-1}u - \lambda B^{-1}Au = Cu - A^*v.$$

Thus,

$$0 = \lambda^2 B^{-1}u - \lambda B^{-1}Au - Cu + \lambda A^*B^{-1}u - A^*B^{-1}Au = \lambda^2 B^{-1}u - Cu - A^*B^{-1}Au, \tag{2.16}$$

because $B^{-1}A$ is symmetric, i.e., $B^{-1}A \subset A^*B^{-1}$. Taking inner product with u for this equation to have

$$\langle \lambda^2 B^{-1}u - Cu - A^*B^{-1}Au, u\rangle = 0,$$

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and thus

$$\lambda^2 = \frac{\langle A^*B^{-1}Au, u\rangle + \langle Cu, u\rangle}{\langle B^{-1}u, u\rangle} \in \mathbb{R},$$

because both C and $A^*B^{-1}A$ are self-adjoint, and B^{-1} is invertible and definite. In other words, $\sigma_p(H) \subset \mathbb{R} \cup i\mathbb{R}$.

(ii) Let $\lambda \in \sigma_p(H)$ and $U = [u\ v]^\top$ be an eigenvector corresponding to λ. Set $V = \begin{bmatrix} -u \\ \lambda B^{-1}u + B^{-1}Au \end{bmatrix}$. Then, $V \in \mathcal{D}(H)$ and

$$\begin{aligned}(H+\lambda I)V &= \begin{bmatrix} A+\lambda I & B \\ C & -A^* + \lambda I \end{bmatrix} \begin{bmatrix} -u \\ \lambda B^{-1}u + B^{-1}Au \end{bmatrix} \\ &= \begin{bmatrix} -Au - \lambda u + \lambda u + Au \\ -Cu - \lambda A^*B^{-1}u - A^*B^{-1}Au + \lambda^2 B^{-1}u + \lambda B^{-1}Au \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ \lambda^2 B^{-1}u - Cu - A^*B^{-1}Au \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.\end{aligned}$$

The third equality is due to the assumption that $B^{-1}A$ is symmetric, i.e., $B^{-1}A \subset A^*B^{-1}$, so $B^{-1}Au = A^*B^{-1}u$. The last equality follows from (2.16). Thus, $-\lambda \in \sigma_p(H)$ with V as eigenvector.

(iii) Case I: $\lambda \in \mathbb{R} \cap \sigma_p(H)\backslash\{0\}$. Let $U = [u\ v]^\top$ be an eigenvector corresponding to λ, i.e., $HU = \lambda U$. Set $V = \begin{bmatrix} -\lambda B^{-1}u - B^{-1}Au \\ -u \end{bmatrix}$. Using the fact that $JHJ \subset H^*$, $J^2 = -I$, and the calculation in (ii), we have

$$(H^* - \overline{\lambda} I)V = J(H + \lambda I)JV = 0,$$

i.e., $V \in N(H^* - \overline{\lambda} I)$ since $\lambda \in \mathbb{R}$. Moreover, because $B^{-1}A$ is symmetric and B is invertible and definite, and $\lambda \neq 0$, we have

$$\begin{aligned}\langle U, V\rangle &= \left\langle \begin{bmatrix} u \\ \lambda B^{-1}u - B^{-1}Au \end{bmatrix}, \begin{bmatrix} -\lambda B^{-1}u - B^{-1}Au \\ -u \end{bmatrix} \right\rangle \\ &= -\lambda\langle u, B^{-1}u\rangle - \langle u, B^{-1}Au\rangle - \lambda\langle B^{-1}u, u\rangle + \langle B^{-1}Au, u\rangle \\ &= -2\lambda\langle B^{-1}u, u\rangle \neq 0.\end{aligned}$$

By Lemma 2.1, all generalized eigenvectors of H corresponding to λ are eigenvectors.

Case II: $\lambda \in i\mathbb{R} \cap \sigma_p(H)\backslash\{0\}$. Let $U = [u\ v]^\top$ be an eigenvector corresponding to λ. Set $V = \begin{bmatrix} -\lambda B^{-1}u + B^{-1}Au \\ u \end{bmatrix}$. Using the facts that $JHJ \subset H^*$, $J^2 = -I$, and calculation similar to those in case (ii), we have

$$(H^* - \overline{\lambda} I)V = J(H - \lambda I)JV = J(H - \lambda I)U = 0,$$

i.e., $V \in N(H^* - \overline{\lambda} I)$ since $\lambda \in i\mathbb{R}$. Moreover, because $B^{-1}A$ is symmetric and B is invertible and definite, and $\lambda \neq 0$, we have

$$\begin{aligned}\langle U, V\rangle &= \left\langle \begin{bmatrix} u \\ \lambda B^{-1}u - B^{-1}Au \end{bmatrix}, \begin{bmatrix} -\lambda B^{-1}u + B^{-1}Au \\ u \end{bmatrix} \right\rangle \\ &= -\overline{\lambda}\langle u, B^{-1}u\rangle + \langle u, B^{-1}Au\rangle + \lambda\langle B^{-1}u, u\rangle - \langle B^{-1}Au, u\rangle \\ &= 2\lambda\langle B^{-1}u, u\rangle \neq 0.\end{aligned}$$

By Lemma 2.1, all generalized eigenvectors of H corresponding to λ are eigenvectors. □

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3. Completeness of eigensystem in the sense of Cauchy principal value. The completeness theory of eigensystems forms the theoretical basis for the Fourier method. The completeness of the eigensystem for a class of complex Schrödinger operators was studied in [21]. Let us recall the notion of completeness for a set of vectors in a Banach space.

Definition 3.1. *[5] A set of vectors $\{e_\alpha\}_{\alpha\in\Lambda}$ is said to be complete in the Banach space $\mathcal{B}$ if the linear span of the set is dense in X, i.e., $\mathcal{B} = \overline{Span_{\alpha\in\Lambda}\{e_\alpha\}}$.*

Completeness is a property weaker than a basis and is often sufficient for ensuring that all elements of the space can be approximated, even if not uniquely.

In the forthcoming discussion, we assume that X is *separable* and study completeness in the sense of Cauchy principal value of the eigensystems of a class of IDHOs.

Definition 3.2. *[6]* Two set of vectors, $\{\alpha_k\}_{k\in\Lambda}$ and $\{\alpha_{-k}\}_{k\in\Lambda}$, in X are said to be *complete in the sense of Cauchy principal value* (abbreviated CCPV), if, for any $x \in X$, there exist $\{C_k\}_{k\in\Lambda}$ and $\{C_{-k}\}_{k\in\Lambda}$ such that

$$x = \sum_{k\in\Lambda}(C_k\alpha_k + C_{-k}\alpha_{-k}).$$

The following example exhibits a CCPV system, which is not complete in X.

Example 3.3. Let $X = L^2(0,1)\times L^2(0,1)$ and define the sequences

$$\left\{u_k = \begin{bmatrix}\sin(k\pi x)\\ k\pi\sin(k\pi x)\end{bmatrix}\right\}_{k=1}^{\infty}, \quad \left\{u_{-k} = \begin{bmatrix}-\sin(k\pi x)\\ k\pi\sin(k\pi x)\end{bmatrix}\right\}_{k=1}^{\infty}.$$

Then, $\{u_k\}_{k=1}^\infty$ and $\{u_{-k}\}_{k=1}^\infty$ are CCPV in X. In fact, for any vector $\begin{bmatrix}f(x)\\ g(x)\end{bmatrix} \in X$, choose

$$C_k = \frac{1}{k\pi}\int_0^1 (k\pi f(x) + g(x))\sin(k\pi x)\,dx, \quad k = 1,2,\ldots,$$

$$C_{-k} = -\frac{1}{k\pi}\int_0^1 (k\pi f(x) - g(x))\sin(k\pi x)\,dx, \quad k = 1,2,\ldots.$$

Then, we have

$$\sum_{k=1}^{\infty}(C_k u_k + C_{-k}u_{-k}) = \sum_{k=1}^{\infty}\begin{bmatrix}2\int_0^1 f(x)\sin(k\pi x)\,dx\sin(k\pi x)\\ 2\int_0^1 g(x)\sin(k\pi x)\,dx\sin(k\pi x)\end{bmatrix} = \begin{bmatrix}f(x)\\ g(x)\end{bmatrix},$$

which implies that $\{u_k\}_{k=1}^\infty$ and $\{u_{-k}\}_{k=1}^\infty$ are CCPV in X.

Suppose on the contrary that $\{u_{\pm k}\}_{k=1}^\infty$ is complete in X. Then, there would exist constants $\{C_{\pm k}\}_{k=1}^\infty$ such that

$$\begin{bmatrix}1\\ 0\end{bmatrix} = \sum_{k=-\infty,\ k\neq 0}^{\infty} C_k u_k = \lim_{m\to\infty}\sum_{k=1}^{m} C_k u_k + \lim_{n\to\infty}\sum_{k=1}^{n} C_{-k}u_{-k}. \tag{3.17}$$

Using the fact that

$$\langle u_m, Ju_n\rangle = \begin{cases} 0, & n \neq -m,\\ m\pi, & n = -m,\end{cases}$$

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where J is given in (2.5), we obtain

$$C_k = -C_{-k} = \frac{1-(-1)^k}{k\pi}.$$

From the second equality in (3.17), we have

$$0 = \lim_{m\to\infty}\sum_{k=1}^{m}(1-(-1)^k)\sin(k\pi x) - \lim_{n\to\infty}\sum_{k=1}^{n}(1-(-1)^k)\sin(k\pi x).$$

However, $\lim_{m\to\infty}\sum_{k=1}^{m}(1-(-1)^k)\sin(k\pi x)$ is divergent, leading to a contradiction. This example shows that $\{u_k\}_{k=-\infty,k\neq 0}^{\infty}$ is CCPV in X but not complete in X.

When the vectors are eigenvectors, CCPV property is particularly relevant in scenarios involving unbounded operators, where the spectral theory often yields eigenvectors that do not form a basis.

Definition 3.4. *A linear operator $T \in \mathcal{C}(X)$ is called simple if the point spectrum $\sigma_p(T)$ of T is at most countable and $\dim N(T-\lambda I) = 1$ for each $\lambda \in \sigma_p(T)$.*

Theorem 3.5. Let X be a separable space, and let $H = \begin{bmatrix} A & B \\ C & -A^* \end{bmatrix} : \mathcal{D}(H) \subset X \times X \to X \times X$ be an IDHO. Assume that B is invertible and definite, $\mathcal{D}(B) \subset \mathcal{D}(A^*)$, $B^{-1}A$ is symmetric, $C + A^*B^{-1}A$ is definite, and $B(C + A^*B^{-1}A)$ is an injective, simple operator. Then,

(i)

$$\sigma_p(H) = \{\lambda \in \mathbb{R} \cup i\mathbb{R} : \lambda \neq 0,\ \lambda^2 \in \sigma_p(B(C + A^*B^{-1}A))\}. \tag{3.18}$$

Thus, both H and $B(C + A^*B^{-1}A)$ have countably infinitely many eigenvalues.

(ii) u is an eigenvector of $B(C + A^*B^{-1}A)$ corresponding to λ^2 if and only if

$$\begin{bmatrix} u \\ \lambda B^{-1}u - B^{-1}Au \end{bmatrix}, \quad \begin{bmatrix} -u \\ \lambda B^{-1}u + B^{-1}Au \end{bmatrix}, \tag{3.19}$$

are eigenvectors of H corresponding to λ and $-\lambda$, respectively. Thus, H is a simple operator.

(iii) The eigenvectors of H are CCPV in $X \times X$ if and only if the eigenvectors of $B(C + A^*B^{-1}A)$ are complete in X.

Proof. (i) It is easy to see that the following decomposition holds:

$$H = \begin{bmatrix} I & 0 \\ -A^*B^{-1} & I \end{bmatrix} \begin{bmatrix} 0 & B \\ C + A^*B^{-1}A & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ \overline{B^{-1}A} & I \end{bmatrix},$$

where $\overline{B^{-1}A}$ denotes the closure of the operator $B^{-1}A$.

Note that $0 \notin \sigma_p(B(C + A^*B^{-1}A))$ as $B(C + A^*B^{-1}A)$ is injective. Since $C + A^*B^{-1}A$ is injective and B is invertible, we conclude $0 \notin \sigma_p(H)$ by the above decomposition. Since $B^{-1}A$ is symmetric, i.e., $B^{-1}A \subset A^*B^{-1}$, by Theorem 2.3(i), $\sigma_p(H) \subset \mathbb{R} \cup i\mathbb{R}$ and all generalized eigenvectors corresponding to λ are ordinary eigenvectors, for each $\lambda \in \sigma_p(H)$. To prove (3.30), let $\lambda \in \sigma_p(H)$ with corresponding eigenvector $[u\ v]^\top$. By (2.15),

$$v = \lambda B^{-1}u - B^{-1}Au,$$

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and $u \neq 0$. By (2.16), we have

$$(C + A^*B^{-1}A)u = \lambda^2 B^{-1}u, \tag{3.20}$$

which implies that λ^2 is an eigenvalue of $B(C + A^*B^{-1}A)$ with corresponding eigenvector u.

Conversely, let $\mu \in \sigma_p(B(C + A^*B^{-1}A))$ with corresponding eigenvector u. So $B(C + A^*B^{-1}A)u = \mu u$, i.e.,

$$(C + A^*B^{-1}A)u = \mu B^{-1}u.$$

Hence,

$$\mu = \frac{\langle (C + A^*B^{-1}A)u, u\rangle}{\langle B^{-1}u, u\rangle} \in \mathbb{R}, \tag{3.21}$$

since $C + A^*B^{-1}A$ is definite and B is invertible and definite. Thus, (3.30) is established.

(ii) Suppose that λ^2 is an eigenvalue of $B(C + A^*B^{-1}A)$ with corresponding eigenvector u, i.e., $B(C + A^*B^{-1}A)u = \lambda^2 u$. Applying (2.16) and the fact that $B^{-1}A$ is symmetric, we have

$$\begin{aligned}
&\begin{bmatrix} A - \lambda I & B \\ C & -A^* - \lambda I \end{bmatrix} \begin{bmatrix} u \\ \lambda B^{-1}u - B^{-1}Au \end{bmatrix} \\
&= \begin{bmatrix} Au - \lambda u + \lambda u + Au \\ Cu - \lambda A^*B^{-1}u + A^*B^{-1}Au - \lambda^2 B^{-1}u + \lambda B^{-1}Au \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ -\lambda^2 B^{-1}u + Cu + A^*B^{-1}Au \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},
\end{aligned} \tag{3.22}$$

i.e., $[u \ \lambda B^{-1}u - B^{-1}Au]^\top \in X \times X$ is an eigenvector of H corresponding to the eigenvalue λ. Similarly,

$$\begin{aligned}
&\begin{bmatrix} A + \lambda I & B \\ C & -A^* + \lambda I \end{bmatrix} \begin{bmatrix} -u \\ \lambda B^{-1}u + B^{-1}Au \end{bmatrix} \\
&= \begin{bmatrix} -Au - \lambda u + \lambda u + Au \\ -Cu - \lambda A^*B^{-1}u - A^*B^{-1}Au + \lambda^2 B^{-1}u + \lambda B^{-1}Au \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ \lambda^2 B^{-1}u - Cu - A^*B^{-1}Au \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},
\end{aligned} \tag{3.23}$$

i.e., $[-u \ \lambda B^{-1}u - B^{-1}Au]^\top \in X \times X$ is an eigenvector of H corresponding to the eigenvalue $-\lambda$. Hence, H is a simple operator.

Conversely, if $[u \ v]^\top$ is an eigenvector of H corresponding to the eigenvalue λ, then by (2.15), $v = \lambda B^{-1}u - B^{-1}Au$. By (3.22), we have

$$(C + A^*B^{-1}A)u = \lambda^2 B^{-1}u \in \mathcal{D}(B), \quad B(C + A^*B^{-1}A)u = \lambda^2 u.$$

So u is an eigenvector of $B(C + A^*B^{-1}A)$ corresponding to λ^2. Thus, the proof of (ii) is complete.

(iii) We now proceed to prove that the eigenvectors of H are CCPV in $X \times X$ if and only if the eigenvectors of $B(C + A^*B^{-1}A)$ are complete in X.

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By (i) and (ii), since $B(C+A^*B^{-1}A)$ and H are simple operators, we can assume that $\sigma_p(B(C+A^*B^{-1}A))=\{\mu_n\}_{n=1}^\infty\subset\mathbb{R}^+$ or $\mathbb{R}^-$ with corresponding eigenvectors $\{f_n\}_{n=1}^\infty$ and $\sigma_p(H)=\{\pm\sqrt{\mu_n}\}_{n=1}^\infty\subset\mathbb{R}$ or $i\mathbb{R}$, respectively, and the corresponding eigenvectors of $\sqrt{\mu}_n$ and $-\sqrt{\mu}_n$ are, respectively,

$$V_n=\begin{bmatrix} f_n \\ \sqrt{\mu_n}B^{-1}f_n-B^{-1}Af_n\end{bmatrix}\in N(H-\sqrt{\mu}_nI),$$
$$V_{-n}=\begin{bmatrix} -f_n \\ \sqrt{\mu_n}B^{-1}f_n+B^{-1}Af_n\end{bmatrix}\in N(H+\sqrt{\mu}_nI).$$

Suppose that the eigenvectors of H are CCPV in $X\times X$. We consider the following two cases:

Case I: If both B and $C+A^*B^{-1}A$ are positive or both are negative, then from (3.21), we have

$$\sigma_p(B(C+A^*B^{-1}A))\subset\mathbb{R}^+,\quad \sigma_p(H)\subset\mathbb{R}\setminus\{0\}.$$

Let $\sigma_p(B(C+A^*B^{-1}A))=\{\mu_n\}_{n=1}^\infty\subset\mathbb{R}^+$ with corresponding eigenvectors $\{f_n\}_{n=1}^\infty$. Because the algebraic multiplicity is 1 for each eigenvalue of $B(C+A^*B^{-1}A)$, $\{f_n\}_{n=1}^\infty$ is a linearly independent set. Then, $\sigma_p(H)=\{\pm\sqrt{\mu_n}\}_{n=1}^\infty$ and V_n and V_{-n} are the eigenvectors corresponding to $\sqrt{\mu}_n$ and $-\sqrt{\mu}_n$, respectively. Since the set of eigenvectors $\{V_{\pm k}\}_{k=1}^\infty$ of H is CCPV in $X\times X$, we conclude that for any $f,g\in X$, there exists a sequence $\{C_{\pm k}\}_{k=1}^\infty$ such that

$$\begin{bmatrix} f \\ g\end{bmatrix}=\sum_{k=1}^\infty(C_kV_k+C_{-k}V_{-k}),$$

which implies that

$$f=\sum_{k=1}^\infty(C_k-C_{-k})f_k\in\overline{\operatorname{Span}\{f_n\}_{n=1}^\infty},$$

i.e., the set $\{f_n\}_{n=1}^\infty$ is complete in X.

Case II: If one of B and $C+A^*B^{-1}A$ is positive and the other is negative, then by (3.20), we have

$$\sigma_p(B(C+A^*B^{-1}A))\subset\mathbb{R}^-,\quad \sigma_p(H)\subset i\mathbb{R}\setminus\{0\}.$$

Let $\sigma_p(B(C+A^*B^{-1}A))=\{\mu_n\}_{n=1}^\infty\subset\mathbb{R}^-$ with corresponding eigenvectors $\{f_n\}_{n=1}^\infty$. Because the algebraic multiplicity is 1 for each eigenvalue of $B(C+A^*B^{-1}A)$, $\{f_n\}_{n=1}^\infty$ is a linearly independent set. Then, $\sigma_p(H)=\{\pm i\sqrt{-\mu_n}\}_{n=1}^\infty$ and V_n and V_{-n} are the eigenvectors corresponding to $i\sqrt{-\mu_n}$ and $-i\sqrt{-\mu_n}$, respectively. Similar to Case I, we conclude that the set $\{f_n\}_{n=1}^\infty$ is complete in X.

Conversely, suppose that the set of eigenvectors $\{f_n\}_{n=1}^\infty$ of $B(C+A^*B^{-1}A)$ is complete in X. We now consider the following two cases.

Case I: Suppose $\sigma_p(B(C+A^*B^{-1}A))\subset\mathbb{R}^+$ and $\sigma_p(H)\subset\mathbb{R}\setminus\{0\}$. Let $\{\widehat{f}_n\}_{n=1}^\infty$ be the orthonormal vectors obtained by applying the Gram–Schmidt orthogonalization process [9, p. 14] on $\{f_n\}_{n=1}^\infty$. Let

$$U_n=\begin{bmatrix} \widehat{f}_n \\ \sqrt{\mu_n}B^{-1}\widehat{f}_n-B^{-1}A\widehat{f}_n\end{bmatrix},\quad U_{-n}=\begin{bmatrix} -\widehat{f}_n \\ \sqrt{\mu_n}B^{-1}\widehat{f}_n+B^{-1}A\widehat{f}_n\end{bmatrix}\in X\times X.$$

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We will see that they form an eigensystem of H and $\{\widehat{f}_n\}_{n=1}^{\infty}$ is an eigensystem of $B(C + A^*B^{-1}A)$. To prove the set of vectors $\{U_{\pm k}\}_{k=1}^{\infty}$ of H is CCPV in $X \times X$, for any $[f\ g]^\top \in X \times X$, set

$$C_k = \frac{\left\langle \begin{bmatrix} f \\ g \end{bmatrix}, JU_{-k} \right\rangle}{\langle U_k, JU_{-k}\rangle}, \quad C_{-k} = \frac{\left\langle \begin{bmatrix} f \\ g \end{bmatrix}, JU_{k} \right\rangle}{\langle U_{-k}, JU_{k}\rangle}, \quad k = 1, 2, \dots$$

Because $\sqrt{\mu}_k \in \mathbb{R} \setminus \{0\}$, B is invertible and definite, and $B^{-1}A$ is symmetric, we have

$$\begin{aligned}
\langle U_k, JU_{-k}\rangle &= \left\langle \begin{bmatrix} \widehat{f}_k \\ \sqrt{\mu_k}B^{-1}\widehat{f}_k - B^{-1}A\widehat{f}_k \end{bmatrix}, \begin{bmatrix} \sqrt{\mu_k}B^{-1}\widehat{f}_k + B^{-1}A\widehat{f}_k \\ \widehat{f}_k \end{bmatrix} \right\rangle \\
&= \langle \widehat{f}_k, \sqrt{\mu_k}\widehat{f}_k + B^{-1}A\widehat{f}_k\rangle + \langle \sqrt{\mu_k}B^{-1}\widehat{f}_k - B^{-1}A\widehat{f}_k, \widehat{f}_k\rangle \\
&= 2\sqrt{\mu_k}\langle B^{-1}\widehat{f}_k, \widehat{f}_k\rangle \neq 0,
\end{aligned}$$

and

$$\begin{aligned}
\langle U_{-k}, JU_{k}\rangle &= \left\langle \begin{bmatrix} -\widehat{f}_k \\ \sqrt{\mu_k}B^{-1}\widehat{f}_k + B^{-1}A\widehat{f}_k \end{bmatrix}, \begin{bmatrix} \sqrt{\mu_k}B^{-1}\widehat{f}_k - B^{-1}A\widehat{f}_k \\ -\widehat{f}_k \end{bmatrix} \right\rangle \\
&= \langle -\widehat{f}_k, \sqrt{\mu_k}\widehat{f}_k - B^{-1}A\widehat{f}_k\rangle + \langle \sqrt{\mu_k}B^{-1}\widehat{f}_k + B^{-1}A\widehat{f}_k, -\widehat{f}_k\rangle \\
&= -2\sqrt{\mu_k}\langle B^{-1}\widehat{f}_k, \widehat{f}_k\rangle \neq 0.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\sum_{k=1}^{\infty}(C_kU_k + C_{-k}U_{-k}) \\
&= \sum_{k=1}^{\infty}\left(\frac{\left\langle \begin{bmatrix} f \\ g \end{bmatrix}, JU_{-k} \right\rangle}{\langle U_k, JU_{-k}\rangle} U_k + \frac{\left\langle \begin{bmatrix} f \\ g \end{bmatrix}, JU_{k} \right\rangle}{\langle U_{-k}, JU_{k}\rangle} U_{-k} \right) \\
&= \sum_{k=1}^{\infty} \frac{\left\langle \begin{bmatrix} f \\ g \end{bmatrix}, \begin{bmatrix} \sqrt{\mu_k}B^{-1}\widehat{f}_k + B^{-1}A\widehat{f}_k \\ \widehat{f}_k \end{bmatrix} \right\rangle}{2\sqrt{\mu_k}\langle B^{-1}\widehat{f}_k, \widehat{f}_k\rangle} \begin{bmatrix} \widehat{f}_k \\ \sqrt{\mu_k}B^{-1}\widehat{f}_k - B^{-1}A\widehat{f}_k \end{bmatrix} \\
&\quad + \frac{\left\langle \begin{bmatrix} f \\ g \end{bmatrix}, \begin{bmatrix} \sqrt{\mu_k}B^{-1}\widehat{f}_k - B^{-1}A\widehat{f}_k \\ -\widehat{f}_k \end{bmatrix} \right\rangle}{-2\sqrt{\mu_k}\langle B^{-1}\widehat{f}_k, \widehat{f}_k\rangle} \begin{bmatrix} -\widehat{f}_k \\ \sqrt{\mu_k}B^{-1}\widehat{f}_k + B^{-1}A\widehat{f}_k \end{bmatrix} \\
&= \sum_{k=1}^{\infty} \frac{\langle f, \sqrt{\mu_k}B^{-1}\widehat{f}_k + B^{-1}A\widehat{f}_k\rangle + \langle g, \widehat{f}_k\rangle}{2\sqrt{\mu_k}\langle B^{-1}\widehat{f}_k, \widehat{f}_k\rangle} \begin{bmatrix} \widehat{f}_k \\ \sqrt{\mu_k}B^{-1}\widehat{f}_k - B^{-1}A\widehat{f}_k \end{bmatrix} \\
&\quad - \sum_{k=1}^{\infty} \frac{\langle f, \sqrt{\mu_k}B^{-1}\widehat{f}_k - B^{-1}A\widehat{f}_k\rangle - \langle g, \widehat{f}_k\rangle}{2\sqrt{\mu_k}\langle B^{-1}\widehat{f}_k, \widehat{f}_k\rangle} \begin{bmatrix} -\widehat{f}_k \\ \sqrt{\mu_k}B^{-1}\widehat{f}_k + B^{-1}A\widehat{f}_k \end{bmatrix}
\end{aligned}$$

(3.24)
$$= \sum_{k=1}^{\infty} \begin{bmatrix} \dfrac{\langle f, B^{-1}\widehat{f}_k\rangle}{\langle B^{-1}\widehat{f}_k, \widehat{f}_k\rangle}\widehat{f}_k \\ \dfrac{\langle f, B^{-1}A\widehat{f}_k\rangle + \langle g, \widehat{f}_k\rangle}{\langle B^{-1}\widehat{f}_k, \widehat{f}_k\rangle}B^{-1}\widehat{f}_k - \dfrac{\langle f, B^{-1}\widehat{f}_k\rangle}{\langle B^{-1}\widehat{f}_k, \widehat{f}_k\rangle}B^{-1}A\widehat{f}_k \end{bmatrix}.$$

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Because the eigenvalues of $B(C + A^*B^{-1}A)$ are distinct, i.e., $\sqrt{\mu_n} \neq \pm\sqrt{\mu_m}$ for $n \neq m$, $n, m = 1, 2, \dots$, by the first symplectic orthogonality relation in Lemma 2.2, we have

$$\langle U_n, JU_m\rangle = (\sqrt{\mu_n} + \sqrt{\mu_m})\langle B^{-1}\widehat{f}_n, \widehat{f}_m\rangle = 0, \quad n \neq m, \quad n, m = 1, 2, \dots.$$

Thus,

$$\langle B^{-1}\widehat{f}_n, \widehat{f}_m\rangle = 0, \quad n \neq m, \quad n, m = 1, 2, \dots.$$

As $\{\widehat{f}_n\}_{n=1}^{\infty}$ is the orthonormal set obtained from $\{f_n\}_{n=1}^{\infty}$ via the Gram–Schmidt process, $\operatorname{Span}\{\widehat{f}_n\}_{n=1}^{\ell} = \operatorname{Span}\{f_n\}_{n=1}^{\ell}$, for all $\ell = 1, 2, \dots$ So, $\{\widehat{f}_n\}_{n=1}^{\infty}$ is complete in X and is indeed an orthonormal basis of X. Thus, we have

$$B^{-1}\widehat{f}_k = \sum_{j=1}^{\infty}\langle B^{-1}\widehat{f}_k, \widehat{f}_j\rangle\widehat{f}_j = \langle B^{-1}\widehat{f}_k, \widehat{f}_k\rangle\widehat{f}_k, \quad k = 1, 2, \dots. \tag{3.25}$$

i.e., $\{\widehat{f}_n\}_{n=1}^{\infty}$ is an orthonormal eigensystem of B^{-1}. Hence, the first coordinate of (3.24) becomes

$$\sum_{k=1}^{\infty}\frac{\langle f, B^{-1}\widehat{f}_k\rangle}{\langle B^{-1}\widehat{f}_k, \widehat{f}_k\rangle}f_k = \sum_{k=1}^{\infty}\langle f, \widehat{f}_k\rangle f_k = f. \tag{3.26}$$

Since $B^{-1}A$ is symmetric, using (3.25), we have

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{\langle f, B^{-1}\widehat{f}_k\rangle}{\langle B^{-1}\widehat{f}_k, \widehat{f}_k\rangle}B^{-1}A\widehat{f}_k &= \sum_{k=1}^{\infty}\frac{\langle f, B^{-1}\widehat{f}_k\rangle}{\langle B^{-1}\widehat{f}_k, \widehat{f}_k\rangle}A^*B^{-1}\widehat{f}_k = A^*\left(\sum_{k=1}^{\infty}\frac{\langle f, B^{-1}\widehat{f}_k\rangle}{\langle B^{-1}\widehat{f}_k, \widehat{f}_k\rangle}B^{-1}\widehat{f}_k\right)\\ &= A^*B^{-1}f = B^{-1}Af.\end{aligned} \tag{3.27}$$

By (3.25) and (3.27), the second coordinate of (3.24) becomes

$$\begin{aligned}&\sum_{k=1}^{\infty}\left(\frac{\langle f, B^{-1}A\widehat{f}_k\rangle + \langle g, \widehat{f}_k\rangle}{\langle B^{-1}\widehat{f}_k, \widehat{f}_k\rangle}B^{-1}\widehat{f}_k - \frac{\langle f, B^{-1}\widehat{f}_k\rangle}{\langle B^{-1}\widehat{f}_k, \widehat{f}_k\rangle}B^{-1}A\widehat{f}_k\right)\\ &= \sum_{k=1}^{\infty}\left(\langle f, B^{-1}A\widehat{f}_k\rangle\widehat{f}_k + \langle g, \widehat{f}_k\rangle\widehat{f}_k - \frac{\langle f, B^{-1}\widehat{f}_k\rangle}{\langle B^{-1}\widehat{f}_k, \widehat{f}_k\rangle}B^{-1}A\widehat{f}_k\right)\\ &= \sum_{k=1}^{\infty}\frac{\langle g, \widehat{f}_k\rangle}{\langle B^{-1}\widehat{f}_k, \widehat{f}_k\rangle}B^{-1}\widehat{f}_k\\ &= g.\end{aligned} \tag{3.28}$$

By (3.26) and (3.28), we have the relationship

$$\sum_{k=1}^{\infty}\langle C_kU_k + C_{-k}U_{-k}\rangle = \begin{bmatrix} f\\ g\end{bmatrix},$$

i.e., the set of vectors $\{U_{\pm n}\}_{n=1}^{\infty}$ is CCPV in $X \times X$.

Case II: Suppose $\sigma_p(B(C + A^*B^{-1}A)) \subset \mathbb{R}^-$ and $\sigma_p(H) \subset i\mathbb{R}\setminus\{0\}$. Let $\{\widehat{f}_n\}_{n=1}^{\infty}$ be the orthonormal vectors obtained by applying the Gram–Schmidt orthogonalization process [9, p.14] on $\{f_n\}_{n=1}^{\infty}$. Let

$$W_n = \begin{bmatrix}\widehat{f}_n\\ i\sqrt{-\mu_n}B^{-1}\widehat{f}_n - B^{-1}A\widehat{f}_n\end{bmatrix}, \quad W_{-n} = \begin{bmatrix}-\widehat{f}_n\\ i\sqrt{-\mu_n}B^{-1}\widehat{f}_n + B^{-1}A\widehat{f}_n\end{bmatrix} \in X \times X.$$

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We will see that these vectors form an eigensystem of H, though $\{\widehat{f}_n\}_{n=1}^{\infty}$ is no longer an eigensystem of $B(C + A^*B^{-1}A)$. To prove the set of vectors $\{W_{\pm k}\}_{k=1}^{\infty}$ of H is CCPV in $X \times X$, for any $[f\ g]^\top \in X \times X$, set

$$C_k = \frac{\left\langle \begin{bmatrix} f \\ g \end{bmatrix}, JW_{-k} \right\rangle}{\langle W_k, JW_{-k}\rangle}, \quad C_{-k} = \frac{\left\langle \begin{bmatrix} f \\ g \end{bmatrix}, JW_{k} \right\rangle}{\langle W_{-k}, JW_{k}\rangle}, \quad k = 1, 2, \ldots,$$

similar to Case I.

If $\sqrt{\mu_k} \in i\mathbb{R} \setminus \{0\}$, we have

$$\begin{aligned}\langle W_k, JW_k\rangle &= \left\langle \begin{bmatrix} \widehat{f}_k \\ i\sqrt{-\mu_k}B^{-1}\widehat{f}_k - B^{-1}A\widehat{f}_k \end{bmatrix}, \begin{bmatrix} i\sqrt{-\mu_k}B^{-1}\widehat{f}_k - B^{-1}A\widehat{f}_k \\ -\widehat{f}_k \end{bmatrix} \right\rangle \\ &= \langle \widehat{f}_k, i\sqrt{-\mu_k}B^{-1}\widehat{f}_k - B^{-1}A\widehat{f}_k\rangle + \langle i\sqrt{-\mu_k}B^{-1}\widehat{f}_k - B^{-1}A\widehat{f}_k, -\widehat{f}_k\rangle \\ &= -2i\sqrt{-\mu_k}\langle B^{-1}\widehat{f}_k, \widehat{f}_k\rangle \neq 0,\end{aligned}$$

and

$$\begin{aligned}\langle W_k, JW_{-k}\rangle &= \left\langle \begin{bmatrix} -\widehat{f}_k \\ i\sqrt{-\mu_k}B^{-1}\widehat{f}_k + B^{-1}A\widehat{f}_k \end{bmatrix}, \begin{bmatrix} i\sqrt{-\mu_k}B^{-1}\widehat{f}_k + B^{-1}A\widehat{f}_k \\ \widehat{f}_k \end{bmatrix} \right\rangle \\ &= \langle -\widehat{f}_k, i\sqrt{-\mu_k}B^{-1}\widehat{f}_k + B^{-1}A\widehat{f}_k\rangle + \langle i\sqrt{-\mu_k}B^{-1}\widehat{f}_k + B^{-1}A\widehat{f}_k, \widehat{f}_k\rangle \\ &= 2i\sqrt{-\mu_k}\langle B^{-1}\widehat{f}_k, \widehat{f}_k\rangle \neq 0,\end{aligned}$$

because $\sqrt{\mu}_k \in i\mathbb{R} \setminus \{0\}$, B is invertible and definite, and $B^{-1}A$ is symmetric. Thus,

$$\begin{aligned}&\sum_{k=1}^{\infty}(C_kW_k + C_{-k}W_{-k}) \\ &= \sum_{k=1}^{\infty} \frac{\left\langle \begin{bmatrix} f \\ g \end{bmatrix}, \begin{bmatrix} i\sqrt{-\mu_k}B^{-1}\widehat{f}_k - B^{-1}A\widehat{f}_k \\ -\widehat{f}_k \end{bmatrix} \right\rangle}{-2i\sqrt{-\mu_k}\langle B^{-1}\widehat{f}_k, \widehat{f}_k\rangle} \begin{bmatrix} \widehat{f}_k \\ i\sqrt{-\mu_k}B^{-1}\widehat{f}_k - B^{-1}A\widehat{f}_k \end{bmatrix} \\ &\quad + \frac{\left\langle \begin{bmatrix} f \\ g \end{bmatrix}, \begin{bmatrix} i\sqrt{-\mu_k}B^{-1}\widehat{f}_k + B^{-1}A\widehat{f}_k \\ \widehat{f}_k \end{bmatrix} \right\rangle}{2\sqrt{\mu_k}\langle B^{-1}\widehat{f}_k, \widehat{f}_k\rangle} \begin{bmatrix} -\widehat{f}_k \\ i\sqrt{-\mu_k}B^{-1}\widehat{f}_k + B^{-1}A\widehat{f}_k \end{bmatrix} \\ &= \sum_{k=1}^{\infty} \frac{-\langle f, i\sqrt{-\mu_k}B^{-1}\widehat{f}_k + B^{-1}A\widehat{f}_k\rangle + \langle g, \widehat{f}_k\rangle}{2i\sqrt{-\mu_k}\langle B^{-1}\widehat{f}_k, \widehat{f}_k\rangle} \begin{bmatrix} \widehat{f}_k \\ i\sqrt{-\mu_k}B^{-1}\widehat{f}_k - B^{-1}A\widehat{f}_k \end{bmatrix} \\ &\quad + \sum_{k=1}^{\infty} \frac{\langle f, \sqrt{-\mu_k}B^{-1}\widehat{f}_k + B^{-1}A\widehat{f}_k\rangle + \langle g, \widehat{f}_k\rangle}{2i\sqrt{-\mu_k}\langle B^{-1}\widehat{f}_k, \widehat{f}_k\rangle} \begin{bmatrix} -\widehat{f}_k \\ \sqrt{-\mu_k}B^{-1}\widehat{f}_k + B^{-1}A\widehat{f}_k \end{bmatrix}\end{aligned}$$

$$= \sum_{k=1}^{\infty} \begin{bmatrix} \dfrac{\langle f, B^{-1}\widehat{f}_k\rangle}{\langle B^{-1}\widehat{f}_k, \widehat{f}_k\rangle}\widehat{f}_k \\ \dfrac{\langle f, B^{-1}A\widehat{f}_k\rangle + \langle g, \widehat{f}_k\rangle}{\langle B^{-1}\widehat{f}_k, \widehat{f}_k\rangle}B^{-1}\widehat{f}_k - \dfrac{\langle f, B^{-1}\widehat{f}_k\rangle}{\langle B^{-1}\widehat{f}_k, \widehat{f}_k\rangle}B^{-1}A\widehat{f}_k \end{bmatrix}, \tag{3.29}$$

which is identical to (3.24). Calculation similar to Case I leads to

$$\sum_{k=1}^{\infty}\langle C_kW_k + C_{-k}W_{-k}\rangle = \begin{bmatrix} f \\ g \end{bmatrix}.$$

The proof is now complete. □

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Similarly, we can obtain the following theorem.

THEOREM 3.6. Let X be a separable space, and let $H = \begin{bmatrix} A & B \\ C & -A^* \end{bmatrix} : \mathcal{D}(H) \subset X \times X \to X \times X$ be an IDHO. Assume that C is invertible and definite, $\mathcal{D}(C) \subset \mathcal{D}(A)$, $C^{-1}A^*$ is symmetric, $B + AC^{-1}A^*$ is definite, and $C(B + AC^{-1}A^*)$ is an injective, simple operator. Then,

(i)

$$\sigma_p(H) = \{\lambda \in \mathbb{R} \cup i\mathbb{R} : \lambda \neq 0,\ \lambda^2 \in \sigma_p(C(B + AC^{-1}A^*))\}. \tag{3.30}$$

Thus, both H and $C(B + AC^{-1}A^*)$ have countably infinitely many eigenvalues.

(ii) v is an eigenvector of $C(B + AC^{-1}A^*)$ corresponding to λ^2 if and only if

$$\begin{bmatrix} \lambda C^{-1}v + C^{-1}A^*v \\ v \end{bmatrix}, \quad \begin{bmatrix} \lambda C^{-1}v - C^{-1}A^*v \\ -v \end{bmatrix}, \tag{3.31}$$

are eigenvectors of H corresponding to λ and $-\lambda$, respectively. Thus, H is a simple operator.

(iii) The eigenvectors of H are CCPV in $X \times X$ if and only if the eigenvectors of $C(B + AC^{-1}A^*)$ are complete in X.

We now give an example to illustrate the criterion.

EXAMPLE 3.7. Consider the partial differential equation with boundary conditions:

$$\begin{cases} s(x)\dfrac{\partial^2 u}{\partial t^2} + \dfrac{\partial}{\partial x}\left(p(x)\dfrac{\partial u}{\partial x}\right) = 0 \\ u(t,0) = u(t,1) = 0 \\ u(0,x) = \varphi(x), u_t(0,x) = \psi(x), \end{cases}$$

where $s(x) \in C[0,1]$, $p(x) \in C^1[0,1]$ and $s(x) > 0, p(x) > 0$ for all $x \in [0,1]$. The corresponding Hamiltonian system is

$$\frac{\partial}{\partial y}\begin{bmatrix} u \\ \sigma \end{bmatrix} = \begin{bmatrix} 0 & I \\ -s^{-1}(x)\dfrac{\partial}{\partial x}\left(p(x)\dfrac{\partial}{\partial x}\right) & 0 \end{bmatrix}\begin{bmatrix} u \\ \sigma \end{bmatrix}.$$

Let $X = L^2_s(0,1)$ be the L^2-space with weighted $s(x)$. The corresponding IDHO is

$$H = \begin{bmatrix} A & B \\ C & -A^* \end{bmatrix} = \begin{bmatrix} 0 & I \\ -s^{-1}(x)\dfrac{d}{dx}\left(p(x)\dfrac{d}{dx}\right) & 0 \end{bmatrix},$$

where

$$\mathcal{D}(H) = \left\{ \begin{bmatrix} u(x) \\ v(x) \end{bmatrix} \in X \times X : u' \text{ absolutely continuous}, u', u'' \in X, u(0) = u(1) = 0 \right\},$$

which is dense in $X \times X$. It is easy to check that

$$B(C + A^*B^{-1}A) = C = -s^{-1}(x)\frac{d}{dx}\left(p(x)\frac{d}{dx}\right),$$

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the Sturm–Liouville operator C is invertible and positive and is a simple operator, and C^{-1} is a compact self-adjoint operator. Therefore, the eigenvectors of C is complete in X. By Theorem 3.6, the eigensystem of H is CCPV in $X \times X$.

On the other hand, let $\{\lambda_k\}_{k=1}^{\infty}$ be the eigenvalues of C, and $\{u_k(x)\}_{k=1}^{\infty}$ the corresponding eigenvectors. Then, $\lambda_k > 0$, $k = 1, 2, \dots$, and the eigenvectors of H are given by

$$U_{\pm k}(x) = \begin{bmatrix} \pm u_k(x) \\ \sqrt{\lambda_k} u_k(x) \end{bmatrix}, \quad k = 1, 2, \dots,$$

and

$$\langle U_k(x), JU_{-k}(x)\rangle = \left\langle \begin{bmatrix} u_k(x) \\ \sqrt{\lambda_k} u_k(x) \end{bmatrix}, \begin{bmatrix} \sqrt{\lambda_k} u_k(x) \\ u_k(x) \end{bmatrix} \right\rangle = 2\sqrt{\lambda_k}\langle u_k(x), u_k(x)\rangle, \quad k = 1, 2, \dots,$$

For any $[f(x)\ g(x)]^\top \in X \times X$, let

$$C_k = \frac{\left\langle \begin{bmatrix} f(x) \\ g(x) \end{bmatrix}, JU_{-k}(x) \right\rangle}{2\sqrt{\lambda_k}\langle u_k(x), u_k(x)\rangle} = \frac{\langle f(x), \sqrt{\lambda_k} u_k(x)\rangle + \langle g(x), u_k(x)\rangle}{2\sqrt{\lambda_k}\langle u_k(x), u_k(x)\rangle}, \quad k = 1, 2, \dots,$$

and

$$C_{-k} = -\frac{\left\langle \begin{bmatrix} f(x) \\ g(x) \end{bmatrix}, JU_{k}(x) \right\rangle}{2\sqrt{\lambda_k}\langle u_k(x), u_k(x)\rangle} = \frac{-\langle f(x), \sqrt{\lambda_k} u_k(x)\rangle + \langle g(x), u_k(x)\rangle}{2\sqrt{\lambda_k}\langle u_k(x), u_k(x)\rangle}, \quad k = 1, 2, \dots.$$

Then,

$$\begin{aligned}
\sum_{k=1}^{\infty}\langle C_k U_k + C_{-k}U_{-k}\rangle &= \sum_{k=1}^{\infty} \begin{bmatrix} \dfrac{\langle f(x), u_k(x)\rangle}{\langle u_k(x), u_k(x)\rangle} u_k(x) \\ \\ \dfrac{\langle g(x), u_k(x)\rangle}{\langle u_k(x), u_k(x)\rangle} u_k(x) \end{bmatrix} \\
&= \sum_{k=1}^{\infty} \frac{\langle f(x), \sqrt{\lambda_k} u_k(x)\rangle + \langle g(x), u_k(x)\rangle}{2\sqrt{\lambda_k}\langle u_k(x), u_k(x)\rangle} \begin{bmatrix} u_k(x) \\ \sqrt{\lambda_k} u_k(x) \end{bmatrix} \\
&\quad + \frac{-\langle f(x), \sqrt{\lambda_k} u_k(x)\rangle + \langle g(x), u_k(x)\rangle}{2\sqrt{\lambda_k}\langle u_k(x), u_k(x)\rangle} \begin{bmatrix} -u_k(x) \\ \sqrt{\lambda_k} u_k(x) \end{bmatrix} \\
&= \begin{bmatrix} f(x) \\ g(x) \end{bmatrix},
\end{aligned}$$

i.e., the eigensystem of H is CCPV in $X \times X$.

4. Completeness of eigensystem of a 4×4 IDHO. When generalized eigenvectors exist and do not coincide with the eigensystem, it becomes essential to study the generalized eigensystem of IDHOs. We introduce several definitions and explore the conditions under which generalized eigensystems are complete.

Definition 4.1. Let X be a separable Hilbert space and let $T \in \mathcal{C}(X)$. For any $\lambda \in \sigma_p(T)$, suppose the order of generalized eigenvectors corresponding to λ is 1, meaning that $\dim N(T-\lambda I) = 1$ and $\dim L_\lambda(T) = 2$. Let $\sigma_p(T) = \{\lambda_{\pm k}\}_{k=1}^{\infty}$ with $\lambda_{\pm k} \neq 0$, $U_{\pm k} \in N(T - \lambda_{\pm k} I)$, and $\{V_{\pm k}\}_{k=1}^{\infty}$ be the generalized eigenvectors of order 1.

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For any $x \in X$, if there exist series $\{C_k\}_{k=1}^{\infty}, \{C_{-k}\}_{k=1}^{\infty}$, $\{D_k\}_{k=1}^{\infty}$, $\{D_{-k}\}_{k=1}^{\infty} \subset \mathbb{C}$ such that

$$x = \sum_{k=1}^{\infty}(C_kU_k + C_{-k}U_{-k} + D_kV_k + D_{-k}V_{-k}), \tag{4.32}$$

then the generalized eigensystem of T is said to be CCPV in X.

When $\sigma_p(T) = \{\lambda_{\pm k}\}_{k=1}^{\infty} \cup \{0\}$. Let U_0 and V_0 be the corresponding eigenvector and generalized eigenvector of 0, respectively. The generalized eigensystem of T is said to be CCPV in X, if for any $x \in X$, there exists series $\{C_k\}_{k=0}^{\infty}, \{C_{-k}\}_{k=1}^{\infty}$, $\{D_k\}_{k=0}^{\infty}, \{D_{-k}\}_{k=1}^{\infty}$ such that

$$x = C_0U_0 + D_0U_0 + \sum_{k=1}^{\infty}(C_kU_k + C_{-k}U_{-k} + D_kV_k + D_{-k}V_{-k}), \tag{4.33}$$

holds.

In this section, we study a special class of 4×4 IDHOs:

$$H = \begin{bmatrix} 0 & A_1 & B_1 & 0 \\ A_2 & 0 & 0 & B_2 \\ C_1 & 0 & 0 & -A_2^* \\ 0 & C_2 & -A_1^* & 0 \end{bmatrix},$$

which often appears in elasticity, where $C_i, B_i,\ i = 1, 2$ are self-adjoint. For instance, consider the plate bending equation:

$$\frac{\partial^2 M_x}{\partial x^2} - 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} = 0, \tag{4.34}$$

where M_x, M_y, M_{xy} denote bending moments, The corresponding IDHO [4, 25] is

$$H = \begin{bmatrix} 0 & v\frac{d}{dy} & D(1-v^2) & 0 \\ -\frac{d}{dy} & 0 & 0 & 2D(1-v) \\ 0 & 0 & 0 & -\frac{d}{dy} \\ 0 & -\frac{d^2}{Ddy^2} & v\frac{d}{dy} & 0 \end{bmatrix},$$

where D and v represent the flexural rigidity of the plate and Poisson's ratio ($0 \le v \le 0.5$), respectively. This system provides an illustration of how IDHOs appear in the context of plate bending and elasticity.

Lemma 4.2. Let $H = \begin{bmatrix} 0 & A_1 & B_1 & 0 \\ A_2 & 0 & 0 & B_2 \\ C_1 & 0 & 0 & -A_2^* \\ 0 & C_2 & -A_1^* & 0 \end{bmatrix}$ be an IDHO. Then, H is similar to $-H$ and $\widetilde{H}$, where

$$\widetilde{H} = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix},\ B = \begin{bmatrix} -C_1 & A_2^* \\ A_2 & B_2 \end{bmatrix},\ C = \begin{bmatrix} -B_1 & A_1 \\ A_1^* & C_2 \end{bmatrix}.$$

Thus, the eigenvalues of H is symmetric with respect to the origin.

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Proof. Let

$$J_1 = \begin{bmatrix} -I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & -I \end{bmatrix}, \quad J_2 = \begin{bmatrix} 0 & 0 & -I & 0 \\ 0 & I & 0 & 0 \\ I & 0 & 0 & 0 \\ 0 & 0 & 0 & I \end{bmatrix}.$$

It is easy to see that $J_1 = J_1^{-1}$ and $H = J_1(-H)J_1^{-1}$, i.e., H is similar to $-H$, and $J_2 H J_2^{-1} = \widetilde{H}$, i.e., H is similar to $\widetilde{H}$. ∎

THEOREM 4.3. Let $H = \begin{bmatrix} 0 & A_1 & B_1 & 0 \\ A_2 & 0 & 0 & B_2 \\ C_1 & 0 & 0 & -A_2^* \\ 0 & C_2 & -A_1^* & 0 \end{bmatrix}$ be an IDHO, where $B_i \geq M$, $C_i \geq 0$, and $M > 0$ is a scalar. Suppose that we have $B_i^{-1} P_i \mathcal{D}(H) \subset \mathcal{D}(A_i^*)$ $i = 1, 2$, where the operators $P_i : X^4 \to X$, $i = 1, 2$, are projections:

$$P_i\ [x_1\ x_2\ x_3\ x_4]^\top = x_i, \quad i = 1, 2, \tag{4.35}$$

and for each of $\lambda \in \sigma_p(H)$, there exists $\alpha^2 > \frac{1}{4}$ such that

$$(A_2^* B_2^{-1} - B_1^{-1} A_1) P_2 N(H - \lambda I) = \frac{2}{2\alpha - 1} B_1^{-1} P_1 N(H - \lambda I), \tag{4.36}$$

$$(B_2^{-1} A_2 - A_1^* B_1^{-1}) P_1 N(H - \lambda I) = \frac{2}{2\alpha + 1} B_2^{-1} P_2 N(H - \lambda I). \tag{4.37}$$

Then the following results hold:

(i) The order of generalized eigenvectors of H corresponding to λ is equal to 1.
(ii) Suppose that H is a simple operator. If the sets $\{P_1 U(\lambda) : U(\lambda) \in N(H - \lambda I)\}_{\lambda \in \sigma_p(H)}$ and $\{P_2 U(\lambda) : U(\lambda) \in N(H - \lambda I)\}_{\lambda \in \sigma_p(H)}$ are orthogonal bases of X, then the generalized eigensystem of H is CCPV in X^4.

Proof. (i) Let $\lambda \in \sigma_p(H)$ and $U = [f_1\ f_2\ g_1\ g_2]^\top \in N(H - \lambda I)$. Then, we have

$$g_1 = \lambda B_1^{-1} f_1 - B_1^{-1} A_1 f_2 \text{ and } g_2 = \lambda B_2^{-1} f_2 - B_2^{-1} A_2 f_1,$$

which leads to the following equations

$$(C_1 - \lambda^2 B_1^{-1} + A_2^* B_2^{-1} A_2) f_1 = \lambda (A_2^* B_2^{-1} - B_1^{-1} A_1) f_2, \tag{4.38}$$

$$(C_2 - \lambda^2 B_2^{-1} + A_1^* B_1^{-1} A_1) f_2 = \lambda (A_1^* B_1^{-1} - B_2^{-1} A_2) f_1. \tag{4.39}$$

Furthermore, using (4.36) and (4.37), we obtain

$$(A_2^* B_2^{-1} - B_1^{-1} A_1) f_2 = \frac{2\lambda}{2\alpha - 1} B_1^{-1} f_1, \tag{4.40}$$

$$(B_2^{-1} A_2 - A_1^* B_1^{-1}) f_1 = \frac{2\lambda}{2\alpha + 1} B_2^{-1} f_2. \tag{4.41}$$

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When $\lambda = 0$, by (4.40) and (4.41), we obtain

$$A_2^* B_2^{-1} f_2 = B_1^{-1} A_1 f_2,$$

$$B_2^{-1} A_2 f_1 = A_1^* B_1^{-1} f_1.$$

Let $V = \begin{bmatrix} f_1 \\ f_2 \\ B_1^{-1} f_1 - B_1^{-1} A_1 f_2 \\ B_2^{-1} f_2 - B_2^{-1} A_2 f_1 \end{bmatrix}$. Then, $V \in \mathcal{D}(H)$ and $HV = U$. When $\lambda \neq 0$, set

$$V = \begin{bmatrix} \frac{\alpha}{\lambda} f_1 \\ -\frac{\alpha}{\lambda} f_2 \\ (\alpha+1) B_1^{-1} f_1 + \frac{\alpha}{\lambda} B_1^{-1} A_1 f_2 \\ (1-\alpha) B_2^{-1} f_2 - \frac{\alpha}{\lambda} B_2^{-1} A_2 f_1 \end{bmatrix}.$$

Then, $V \in \mathcal{D}(H)$ and by (4.38)–(4.41), we have $(H - \lambda)V = U$. It shows that the order of generalized eigenvectors of H corresponding to λ is at least 1.

On the other hand, by (4.40) and (4.41), we have

$$\langle f_1, (A_2^* B_2^{-1} - B_1^{-1} A_1) f_2 \rangle = \frac{2\lambda}{2\alpha - 1} \langle f_1, B_1^{-1} f_1 \rangle,$$

$$\langle (B_2^{-1} A_2 - A_1^* B_1^{-1}) f_1, f_2 \rangle = \frac{2\overline{\lambda}}{2\alpha + 1} \langle B_2^{-1} f_2, f_2 \rangle,$$

which imply

$$\frac{\overline{\lambda}}{\lambda} = \frac{2\alpha+1}{2\alpha-1} \cdot \frac{\langle B_1^{-1} f_1, f_1 \rangle}{\langle B_2^{-1} f_2, f_2 \rangle} > 0.$$

Hence, $\sigma_p(H) \subset \mathbb{R}$ and

$$\langle B_2^{-1} f_2, f_2 \rangle = \frac{2\alpha+1}{2\alpha-1} \langle B_1^{-1} f_1, f_1 \rangle. \tag{4.42}$$

Case 1. When $\lambda = 0$, let

$$\widetilde{U} = \begin{bmatrix} -g_1 \\ g_2 \\ -f_1 \\ f_2 \end{bmatrix} \text{ and } \widetilde{V} = \begin{bmatrix} -B_1^{-1} f_1 + B_1^{-1} A_1 f_2 \\ B_2^{-1} f_2 - B_2^{-1} A_2 f_1 \\ -f_1 \\ f_2 \end{bmatrix}.$$

Then ,$J\widetilde{U}, J\widetilde{V} \in \mathcal{D}(H)$ and by $JHJ \subset H^*$, we have

$$H^* \widetilde{U} = JHJ\widetilde{U} = 0, \quad H^* \widetilde{V} = JHJ\widetilde{V} = \widetilde{U},$$

which imply

$$(H^*)^2 \widetilde{V} = 0, \quad \langle U, \widetilde{V} \rangle = -\langle B_1^{-1} f_1, f_1 \rangle - \langle B_2^{-1} f_2, f_2 \rangle \neq 0,$$

where

$$J = \begin{bmatrix} 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ -I & 0 & 0 & 0 \\ 0 & -I & 0 & 0 \end{bmatrix}.$$

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By Lemma 2.1, the order of generalized eigenvectors of H corresponding to $\lambda = 0$ is less than 2.

Case 2. When $\lambda \neq 0$, let

$$\widetilde{U} = \begin{bmatrix} -g_1 \\ g_2 \\ -f_1 \\ f_2 \end{bmatrix} \text{ and } \widetilde{V} = \begin{bmatrix} -B_1^{-1}f_1 - \alpha B_1^{-1}f_1 - \frac{\alpha}{\lambda}B_1^{-1}A_1f_2 \\ B_2^{-1}f_2 - \alpha B_2^{-1}f_2 - \frac{\alpha}{\lambda}B_2^{-1}A_2f_1 \\ -\frac{\alpha}{\lambda}f_1 \\ -\frac{\alpha}{\lambda}f_2 \end{bmatrix}.$$

Then, $J\widetilde{U}, J\widetilde{V} \in \mathcal{D}(H)$, and we have

$$(H^* - \overline{\lambda}I)\widetilde{U} = J(H + \lambda I)J\widetilde{U} = 0,$$

$$(H^* - \overline{\lambda}I)\widetilde{V} = J(H + \lambda I)J\widetilde{V} = \widetilde{U}.$$

Consequently, $\widetilde{V} \in N((H^* - \overline{\lambda}I)^2)$. By (4.38)–(4.41), we have

$$\begin{aligned} \langle U, \widetilde{V} \rangle &= -(2\alpha + 1)\langle B_1^{-1}f_1, f_1\rangle + (1 - 2\alpha)\langle B_2^{-1}f_2, f_2\rangle \\ &= -2(2\alpha + 1)\langle B_1^{-1}f_1, f_1\rangle \neq 0. \end{aligned}$$

By Lemma 2.1, the order of generalized eigenvectors of H corresponding to λ is less than 2, proving (i).

(ii) By Lemma 4.2, $\sigma_p(H)$ is symmetric with respect to the origin. When $0 \notin \sigma_p(H)$, let $\{\lambda_{\pm k}\}_{k=1}^{\infty}$, where $\lambda_{-k} = -\lambda_k$, are the eigenvalues of H and

$$U_k = \begin{bmatrix} f_1^{(k)} \\ f_2^{(k)} \\ \lambda_k B_1^{-1}f_1^{(k)} - B_1^{-1}A_1f_2^{(k)} \\ \lambda_k B_2^{-1}f_2^{(k)} - B_2^{-1}A_2f_1^{(k)} \end{bmatrix}, \quad U_{-k} = \begin{bmatrix} f_1^{(k)} \\ -f_2^{(k)} \\ -\lambda_k B_1^{-1}f_1^{(k)} + B_1^{-1}A_1f_2^{(k)} \\ \lambda_k B_2^{-1}f_2^{(k)} - B_2^{-1}A_2f_1^{(k)} \end{bmatrix},$$

are the corresponding eigenvectors of λ_k and λ_{-k}, $k = 1, 2, \ldots$ Then, the corresponding generalized eigenvectors are

$$V_k = \begin{bmatrix} \frac{\alpha}{\lambda_k}f_1^{(k)} \\ -\frac{\alpha}{\lambda_k}f_2^{(k)} \\ (\alpha+1)B_1^{-1}f_1^{(k)} + \frac{\alpha}{\lambda_k}B_1^{-1}A_1f_2^{(k)} \\ (1-\alpha)B_2^{-1}f_2^{(k)} - \frac{\alpha}{\lambda_k}B_2^{-1}A_2f_1^{(k)} \end{bmatrix}, \quad V_{-k} = \begin{bmatrix} -\frac{\alpha}{\lambda_k}f_1^{(k)} \\ -\frac{\alpha}{\lambda_k}f_2^{(k)} \\ (\alpha+1)B_1^{-1}f_1^{(k)} + \frac{\alpha}{\lambda_k}B_1^{-1}A_1f_2^{(k)} \\ (\alpha-1)B_2^{-1}f_2^{(k)} + \frac{\alpha}{\lambda_k}B_2^{-1}A_2f_1^{(k)} \end{bmatrix}.$$

By (ii), $\{f_1^{(k)}\}_{k=1}^{\infty}$ and $\{f_2^{(k)}\}_{k=1}^{\infty}$ form an orthogonal basis of X. Thus,

$$\langle f_1^{(k)}, f_1^{(j)}\rangle = \langle f_2^{(k)}, f_2^{(j)}\rangle = 0, \quad k \neq j, k, j = 1, 2, \ldots.$$

For any $U = \begin{bmatrix} x_1 & x_2 & y_1 & y_2 \end{bmatrix}^\top \in \mathcal{D}(H) \subset X^4$, set

$$C_k = \frac{\langle U, JV_{-k}\rangle}{\langle U_k, JV_{-k}\rangle}, \quad C_{-k} = \frac{\langle U, JV_k\rangle}{\langle U_{-k}, JV_k\rangle}, \quad k = 1, 2, \ldots,$$

$$D_k = \frac{\langle U, JU_{-k}\rangle}{\langle V_k, JU_{-k}\rangle}, \quad D_{-k} = \frac{\langle U, JU_k\rangle}{\langle V_{-k}, JU_k\rangle}, \quad k = 1, 2, \ldots.$$

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Then,

$$\sum_{k=1}^{\infty}(C_kU_k + C_{-k}U_{-k} + D_kV_k + D_{-k}V_{-k}) = \sum_{k=1}^{\infty}\begin{bmatrix} \dfrac{\langle x_1, B_1^{-1}f_1^{(k)}\rangle}{\langle f_1^{(k)}, B_1^{-1}f_1^{(k)}\rangle}f_1^{(k)} \\ \dfrac{\langle x_2, B_2^{-1}f_2^{(k)}\rangle}{\langle f_2^{(k)}, B_2^{-1}f_2^{(k)}\rangle}f_2^{(k)} \\ \dfrac{\langle y_1, f_1^{(k)}\rangle}{\langle f_1^{(k)}, B_1^{-1}f_1^{(k)}\rangle}B_1^{-1}f_1^{(k)} + \triangle_1 \\ \dfrac{\langle y_2, f_2^{(k)}\rangle}{\langle f_2^{(k)}, B_2^{-1}f_2^{(k)}\rangle}B_2^{-1}f_2^{(k)} + \triangle_2 \end{bmatrix},$$

where

$$\begin{aligned}\triangle_1 = \frac{\langle x_2, B_2^{-1}A_2f_1^{(k)}\rangle}{\langle f_1^{(k)}, B_1^{-1}f_1^{(k)}\rangle}B_1^{-1}f_1^{(k)} - \frac{2\lambda_k}{2\alpha+1}\cdot\frac{\langle x_2, B_2^{-1}f_2^{(k)}\rangle}{\langle f_1^{(k)}, B_1^{-1}f_1^{(k)}\rangle}B_1^{-1}f_1^{(k)} \\ -\frac{(2\alpha-1)}{2\alpha+1)}\cdot\frac{\langle x_2, B_2^{-1}f_2^{(k)}\rangle}{\langle f_1^{(k)}, B_1^{-1}f_1^{(k)}\rangle}B_1^{-1}A_1f_2^{(k)},\end{aligned}$$

and

$$\begin{aligned}\triangle_2 = \frac{\langle x_1, B_1^{-1}A_1f_2^{(k)}\rangle}{\langle f_2^{(k)}, B_2^{-1}f_2^{(k)}\rangle}B_2^{-1}f_2^{(k)} + \frac{2\lambda_k}{(2\alpha-1)}\cdot\frac{\langle x_1, B_1^{-1}f_1^{(k)}\rangle}{\langle f_2^{(k)}, B_2^{-1}f_2^{(k)}\rangle}B_2^{-1}f_2^{(k)} \\ -\frac{2\alpha+1}{2\alpha-1}\cdot\frac{\langle x_1, B_1^{-1}f_1^{(k)}\rangle}{(2\alpha-1)\langle f_2^{(k)}, B_2^{-1}f_2^{(k)}\rangle}B_2^{-1}A_2f_2^{(k)}.\end{aligned}$$

By Lemma 2.2, for $k \neq j$, where $k, j = 1, 2, \dots$, we have

$$\langle U_k, JU_j\rangle = (\lambda_k - \lambda_j)[\langle B_1^{-1}f_1^k, f_1^j\rangle + \langle B_2^{-1}f_2^k, f_2^j\rangle] = 0,$$

which implies

$$\langle B_1^{-1}f_1^{(k)}, f_1^{(j)}\rangle = \langle B_2^{-1}f_2^{(k)}, f_2^{(j)}\rangle = 0, \quad k \neq j, \quad k, j = 1, 2, \dots.$$

For any $k, j = 1, 2, \dots$, we have

$$\left\langle B_1^{-1}f_1^{(k)} - \frac{\langle B_1^{-1}f_1^{(k)}, f_1^{(k)}\rangle}{\langle f_1^{(k)}, f_1^{(k)}\rangle}f_1^{(k)}, f_1^{(j)}\right\rangle = \langle B_1^{-1}f_1^{(k)}, f_1^{(k)}\rangle - \langle B_1^{-1}f_1^{(k)}, f_1^{(k)}\rangle = 0.$$

Hence,

$$B_1^{-1}f_1^{(k)} = \frac{\langle B_1^{-1}f_1^{(k)}, f_1^{(k)}\rangle}{\langle f_1^{(k)}, f_1^{(k)}\rangle}f_1^{(k)}, \quad k = 1, 2, \dots.$$

Similarly, we also have

$$B_2^{-1}f_2^{(k)} = \frac{\langle B_2^{-1}f_2^{(k)}, f_2^{(k)}\rangle}{\langle f_2^{(k)}, f_2^{(k)}\rangle}f_2^{(k)}, \quad k = 1, 2, \dots.$$

As a result,

$$\sum_{k=1}^{\infty}\frac{\langle x_1, B_1^{-1}f_1^{(k)}\rangle}{\langle f_1^{(k)}, B_1^{-1}f_1^{(k)}\rangle}f_1^{(k)} = \sum_{k=1}^{\infty}\frac{\langle x_1, f_1^k\rangle}{\langle f_1^k, f_1^k\rangle}f_1^{(k)} = x_1,$$

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and

$$\sum_{k=1}^{\infty} \frac{\langle x_2, B_2^{-1} f_2^{(k)} \rangle}{\langle f_2^{(k)}, B_2^{-1} f_2^{(k)} \rangle} f_2^{(k)} = \sum_{k=1}^{\infty} \frac{\langle x_2, f_2^k \rangle}{\langle f_2^k, f_2^k \rangle} f_2^{(k)} = x_2.$$

Similarly,

$$\sum_{k=1}^{\infty} \frac{\langle y_1, f_1^{(k)} \rangle}{\langle f_1^{(k)}, B_1^{-1} f_1^{(k)} \rangle} B_1^{-1} f_1^{(k)} = y_1, \quad \sum_{k=1}^{\infty} \frac{\langle y_2, f_2^{(k)} \rangle}{\langle f_2^{(k)}, B_2^{-1} f_2^{(k)} \rangle} B_2^{-1} f_2^{(k)} = y_2.$$

Next, we prove that $\sum_{k=1}^{\infty} \triangle_1 = 0$. By (4.42),

$$\begin{aligned}\sum_{k=1}^{\infty} \triangle_1 &= \sum_{k=1}^{\infty} \Bigg[\frac{\langle A_2^* B_2^{-1} x_2, f_1^{(k)} \rangle}{\langle f_1^{(k)}, B_1^{-1} f_1^{(k)} \rangle} B_1^{-1} f_1^{(k)} - \frac{\langle x_2, (B_2^{-1} A_2 - A_1^* B_1^{-1}) f_1^k \rangle}{\langle f_1^{(k)}, B_1^{-1} f_1^{(k)} \rangle} \\ &\qquad - \frac{\langle x_2, B_2^{-1} f_2^k \rangle}{\langle f_2^{(k)}, B_2^{-1} f_2^{(k)} \rangle} B_1^{-1} A_1 f_2^k \Bigg] \\ &= A_2^* B_2^{-1} x_2 - (A_2^* B_2^{-1} - B_1^{-1} A_1) x_2 - B_1^{-1} A_1 x_2 \\ &= 0.\end{aligned}$$

Similarly, it can be shown that $\sum_{k=1}^{\infty} \triangle_2 = 0$. Therefore,

$$\sum_{k=1}^{\infty} (C_k U_k + C_{-k} U_{-k} + D_k V_k + D_{-k} V_{-k}) = [x_1 \ x_2 \ y_1 \ y_2]^\top.$$

When $0 \in \sigma_p(H)$, let

$$U_0 = \begin{bmatrix} f_1^{(0)} \\ f_2^{(0)} \\ -B_1^{-1} A_1 f_2^{(0)} \\ -B_2^{-1} A_2 f_1^{(0)} \end{bmatrix}, \quad V_0 = \begin{bmatrix} f_1^{(0)} \\ f_2^{(0)} \\ B_1^{-1} f_1^{(0)} - B_1^{-1} A_1 f_2^{(0)} \\ B_2^{-1} f_2^{(0)} - B_2^{-1} A_2 f_1^{(0)} \end{bmatrix},$$

be the corresponding eigenvector and generalized eigenvector. Set

$$C_0 = \frac{\langle U, JV_0 \rangle}{\langle U_0, JV_0 \rangle}, \quad D_0 = \frac{\langle U, JU_0 \rangle}{\langle V_0, JU_0 \rangle},$$

$$C_k = \frac{\langle U, JV_{-k} \rangle}{\langle U_k, JV_{-k} \rangle}, \quad C_{-k} = \frac{\langle U, JV_k \rangle}{\langle U_{-k}, JV_k \rangle}, \quad k = 1, 2, \ldots,$$

$$D_k = \frac{\langle U, JU_{-k} \rangle}{\langle V_k, JU_{-k} \rangle}, \quad D_{-k} = \frac{\langle U, JU_k \rangle}{\langle V_{-k}, JU_k \rangle}, \quad k = 1, 2, \ldots.$$

It can then be similarly proved that

$$C_0 U_0 + D_0 V_0 + \sum_{k=1}^{\infty} (C_k U_k + C_{-k} U_{-k} + D_k V_k + D_{-k} V_{-k}) = [x_1 \ x_2 \ y_1 \ y_2]^\top$$

holds for any $[x_1 \ x_2 \ y_1 \ y_2]^\top \in \mathcal{D}(H)$. Since $\overline{\mathcal{D}(H)} = X^4$, the proof is complete. ∎

The following theorem shows that the condition (ii) in Theorem 4.3 can be expressed in an alternative form.

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THEOREM 4.4. Let $H = \begin{bmatrix} 0 & A_1 & B_1 & 0 \\ A_2 & 0 & 0 & B_2 \\ C_1 & 0 & 0 & -A_2^* \\ 0 & C_2 & -A_1^* & 0 \end{bmatrix}$ be an IDHO with $B_i \geq M, C_i \geq 0,\ i = 1, 2,$ where $M > 0$ is a scalar. If for each $\lambda \in \sigma_p(H)$, we have

$$B_i^{-1} P_i \mathcal{D}(H) \subset \mathcal{D}(A_i^*), \quad i = 1, 2,$$

and there exists $\alpha^2 > \frac{1}{4}$ such that the following conditions hold:

$$(A_2^* B_2^{-1} - B_1^{-1} A_1) P_2 N(H - \lambda I) = \frac{2\lambda}{2\alpha - 1} B_1^{-1} P_1 N(H - \lambda I), \tag{4.43}$$

$$(B_2^{-1} A_2 - A_1^* B_1^{-1}) P_1 N(H - \lambda I) = \frac{2\lambda}{2\alpha + 1} B_2^{-1} P_2 N(H - \lambda I), \tag{4.44}$$

where the operators $P_i : X^4 \to X,\ i = 1, 2$ are defined as in (4.35). In addition, let H be a simple operator, and assume that $\{\lambda^m P_1 U(\lambda) : U(\lambda) \in N(H - \lambda I)\}_{\lambda \in \sigma_p(H)}$ and $\{\lambda^m P_2 U(\lambda) : U(\lambda) \in N(H - \lambda I)\}_{\lambda \in \sigma_p(H)}$ are orthogonal bases of X, where m is an integer. Then the generalized eigensystem of H is CCPV in X^4.

Proof. If the sequences $\{f_1^{(k)}\}_{k=1}^{\infty}$ and $\{f_2^{(k)}\}_{k=1}^{\infty}$ of vectors in Theorem 4.3 are replaced by $\{\lambda_k^m f_1^{(k)}\}_{k=1}^{\infty}$ and $\{\lambda_k^m f_2^{(k)}\}_{k=1}^{\infty}$, respectively, then it can be shown analogously that the generalized eigensystem of H is CCPV in X^4. ∎

We now give an example to illustrate the criterion.

EXAMPLE 4.5. Consider the basic equation of plate strip [4]

$$\frac{\partial^2 M_x}{\partial x^2} - 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} = 0, \tag{4.45}$$

and the corresponding infinite dimensional Hamiltonian system $\dot{\Phi} = H\Phi$, with homogeneous boundary conditions:

$$\phi_x = 0, \text{ when } y = 0 \text{ or } y = b,$$

$$\frac{1}{D}\phi_y - v k_y = 0, \text{ when } y = 0 \text{ or } y = b,$$

where

$$\Phi = [\phi_x\ \phi_y\ k_y\ k_{xy}]^\top.$$

Let $X = L^2(0, b)$. Then,

$$H = \begin{bmatrix} 0 & v\frac{d}{dy} & D(1 - v^2) & 0 \\ -\frac{d}{dy} & 0 & 0 & 2D(1 - v) \\ 0 & 0 & 0 & -\frac{d}{dy} \\ 0 & -\frac{d^2}{Ddy^2} & v\frac{d}{dy} & 0 \end{bmatrix},$$

and

$$\mathcal{D}(H) = \left\{ \begin{bmatrix} \psi_1(y) \\ \psi_2(y) \\ \psi_3(y) \\ \psi_4(y) \end{bmatrix} \in X^4 : \begin{bmatrix} \psi_1(0) = \psi_1(b) = 0, \frac{d\psi_2(0)}{Ddx} - v\psi_3(0) = 0, \\ \frac{d\psi_2(b)}{Ddx} - v\psi_3(b) = 0 \\ \psi_i(y) \in AC(0, b), \psi_i'(x) \in L^2(0, b), i = 1, 3, 4 \\ \psi_2'(y) \in AC(0, b), \psi_2''(y) \in L^2(0, b) \end{bmatrix} \right\}.$$

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By direct calculation, the eigenvalues of H are

$$\lambda_k = \frac{k\pi}{b}, \quad k = 0, \pm 1, \pm 2, \ldots,$$

and the corresponding eigenvectors are

$$U_0 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad U_k = \begin{bmatrix} \frac{D(1-v)}{\lambda_k}\sin(\lambda_k y) \\ \frac{D(1-v)}{\lambda_k}\cos(\lambda_k y) \\ \sin(\lambda_k y) \\ \cos(\lambda_k y) \end{bmatrix}, \quad k = \pm 1, \pm 2, \ldots.$$

Let $\alpha = \frac{3+v}{2(v-1)}$. Since Poisson's ratio v satisfies $0 \le v \le 0.5$, we have $\alpha^2 > \frac{1}{4}$ and

$$(A_2^* B_2^{-1} - B_1^{-1} A_1) f_2^{(k)} = \frac{1-v}{2(1+v)} \sin(\lambda_k y) = \frac{2\lambda_k}{2\alpha - 1} B_1^{-1} f_1^{(k)},$$

$$(A_1^* B_1^{-1} - B_2^{-1} A_2) f_1^{(k)} = \frac{1-v}{2(v+1)} \cos(\lambda_k y) = -\frac{2\lambda_k}{2\alpha + 1} B_2^{-1} f_2^{(k)}.$$

In addition,

$$\overline{\mathrm{Span}\,\{\lambda P_1 N(H-\lambda I) \cup P_1 N(H) : \lambda \in \sigma_p(H)\}} = \overline{\mathrm{Span}\,\{D(1-v)\sin \lambda_k y\}_{k=1}^{\infty}} = X,$$

$$\overline{\mathrm{Span}\,\{\lambda P_2 N(H-\lambda I) \cup P_2 N(H) : \lambda \in \sigma_p(H)\}} = \overline{\mathrm{Span}\,\{D(1-v)\cos \lambda_k y\}_{k=0}^{\infty}} = X.$$

By Theorem 4.4, the order of generalized eigenvectors of H is equal to 1, and the generalized eigensystem of H is CCPV in X^4.

On the other hand, by [4, Theorem 1], we have

$$V_0 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \frac{1}{2D(1-v)} \end{bmatrix}, \quad V_k = \begin{bmatrix} -\frac{D(3+v)}{2\lambda_k^2}\sin(\lambda_k y) \\ \frac{D(3+v)}{2\lambda_k^2}\cos(\lambda_k y) \\ -\frac{1}{2\lambda_k}\sin(\lambda_k y) \\ \frac{1}{2\lambda_k}\cos(\lambda_k y) \end{bmatrix}, \quad k = \pm 1, \pm 2, \ldots,$$

For any $U = [\psi_1(y)\ \psi_2(y)\ \psi_3(y)\ \psi_4(y)]^\top \in \mathcal{D}(H) \subset X^4$, let

$$\begin{aligned} C_0 &= \frac{1}{b}\int_0^b \psi_2(y)dy - \frac{2D(1-v)}{b}\int_0^b \psi_4(y)\,dy, \\ D_0 &= \frac{2D(1-v)}{b}\int_0^b \psi_4(y)\,dy, \\ C_n &= \frac{1}{4b^3 D}\int_0^b \Big[b\sin\frac{n\pi y}{b}(n\pi\psi_1(y) + bD(3+v)\psi_3(y)) \\ &\quad + b\cos\frac{n\pi y}{b}(n\pi\psi_2(y) + bD(3+v)\psi_4(y)) \Big]\,dy, \\ C_{-n} &= \frac{1}{4b^3 D}\int_0^b \Big[b\sin\frac{n\pi y}{b}(n\pi\psi_1(y) - bD(3+v)\psi_3(y)) \\ &\quad - b\cos\frac{n\pi y}{b}(n\pi\psi_2(y) - bD(3+v)\psi_4(y)) \Big]\,dy, \end{aligned}$$

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$$D_n = \frac{-n\pi}{2b^3 D}\int_0^b \Big[b\sin\frac{n\pi y}{b}(n\pi\psi_1(y) + bD(1-v)\psi_3(y)) \\ - \cos\frac{n\pi y}{b}(n\pi\psi_2(y) + bD(1-v)\psi_4(y))\Big]\,dy,$$

$$D_{-n} = \frac{n\pi}{2b^3 D}\int_0^b \Big[b\sin\frac{n\pi y}{b}(n\pi\psi_1(y) - bD(1-v)\psi_3(y)) \\ + \cos\frac{n\pi y}{b}(n\pi\psi_2(y) - bD(1-v)\psi_4(y))\Big]\,dy.$$

Then, we have

$$U = C_0U_0 + D_0V_0 + \sum_{k=1}^{\infty}(C_kU_k + C_{-k}U_{-k} + D_kV_k + D_{-k}V_{-k}),$$

i.e., the generalized eigensystem of H is CCPV in X^4.

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