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FILTERS AND STRUCTURE PRESERVING TRANSFORMATIONS FOR QUADRATIC SYSTEMS*

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Abstract. The spectral equivalence of quadratic systems can be characterized using matrix polynomials called spectral filters. These filters were characterized in [8] for quadratic systems with nonsingular leading coefficient in terms of structure preserving transformations (SPTs) of three types of linearizations. This paper extends the results of [8] to quadratic systems with singular leading coefficients, using a new type of linearization called the *Rblock-symmetric linearization*. It is then shown that, for any two spectrally equivalent quadratic systems and any of their linearizations, there is a homeomorphism between their linear spectral filters and the set of SPTs of the chosen linearization. In addition, spectral filters for spectrally equivalent quadratic systems must satisfy a Bezout identity. With a view to the decoupling problem of quadratic systems using filters, we identify matrix polynomials of small degree that realize the Bezout identity for the given spectrally equivalent quadratic systems and their filters.

Key words. Quadratic systems, Matrix polynomials, Filters, Structure preserving transformations, Bezout identity.

AMS subject classifications. 15A22, 15A54, 93B17, 93B18.

1. Introduction.

This is the first part of an extensive investigation into the implementation of the *phase synchronization* technique for decoupling *diagonalizable* quadratic systems using *filters*. A quadratic system

$$M\ddot{x}(t) + D\dot{x}(t) + Kx(t) = f(t), \quad M, D, K \in \mathbb{R}^{n\times n}, \tag{1.1}$$

is said to be *decoupled* if M, D, and K are diagonal matrices; otherwise, it is said to be *coupled*. Equivalently, (1.1) is decoupled if the associated quadratic matrix polynomial $Q(s) = Ms^2 + Ds + K$ is diagonal. In this paper, quadratic matrix polynomials will also be referred to as *quadratic systems*. A quadratic system $Q(s)$ is diagonalizable if it is *unimodularly equivalent* to a diagonal quadratic matrix polynomial. That is to say (see Section 2), if there are unimodular matrices $U(s)$ and $V(s)$ (polynomial matrices whose determinants are nonzero constant polynomials) such that $U(s)Q(s)V(s)$ is a diagonal quadratic matrix polynomial.

Phase synchronization is a technique that has been developed in [14, 15, 17, 18, 11, 12] (see also [21]) for decoupling arbitrary quadratic systems (although M is assumed to be invertible except in [12]). It is shown in [17, 12] that, when the system is diagonalizable, then phase synchronization can be implemented using *structure preserving transformations* (SPTs). SPTs require the quadratic system to be in state-space form. In other words, it requires to choose an appropriate linearization of the quadratic system—that, in general, will have a precise structure (see Section 3)—and by similarity or strict equivalence obtain a linearization of a diagonal quadratic system with the same structure. The nonsingular matrices transforming the linearization of the original coupled system into the linearization of the decoupled system is an SPT. For example, if the quadratic system of (1.1) has $2n$ distinct eigenvalues, $\lambda_1, \ldots, \lambda_{2n}$ say, and $\Lambda_1 = \mathrm{Diag}(\lambda_1, \ldots, \lambda_n)$ and

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$\Lambda_2 = \mathrm{Diag}(\lambda_{n+1}, \ldots, \lambda_{2n})$, then $Q(s) = Ms^2+Ds+K$ and $Q_d(s) = I_n s^2-(\Lambda_1+\Lambda_2)s+\Lambda_1\Lambda_2$ are unimodularly equivalent and $Q_d(s)$ is diagonal. Then, (see Section 3),

$$\begin{bmatrix} M & 0 \\ 0 & I_n \end{bmatrix} s + \begin{bmatrix} D & K \\ -I_n & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} I_n & 0 \\ 0 & I_n \end{bmatrix} s + \begin{bmatrix} -(\Lambda_1+\Lambda_2) & \Lambda_1\Lambda_2 \\ -I_n & 0 \end{bmatrix},$$

are the first block-companion linearizations of $Q(s)$ and $Q_d(s)$, respectively. Since these matrices have the same eigenvalues and are all distinct, there are invertible matrices $S, R \in \mathbb{R}^{2n\times 2n}$ such that

$$S\left(\begin{bmatrix} M & 0 \\ 0 & I_n \end{bmatrix} s + \begin{bmatrix} D & K \\ -I_n & 0 \end{bmatrix}\right) = \left(\begin{bmatrix} I_n & 0 \\ 0 & I_n \end{bmatrix} s + \begin{bmatrix} -(\Lambda_1+\Lambda_2) & \Lambda_1\Lambda_2 \\ -I_n & 0 \end{bmatrix}\right) R.$$

Then, (S, R) is an SPT for the first block-companion linearizations of $Q(s)$ and $Q_d(s)$ (see Section 3 for the precise definition).

As mentioned in [8], there may be some drawbacks to working with quadratic systems in state-space form. For example, while a quadratic system remains quadratic if its matrix coefficients are subjected to perturbations, the structure of its linearizations may be lost if it is subjected to unstructured perturbations, in which case the resulting matrix (or pencil) may no longer represent a quadratic system. The alternative proposed in [8] is to use *coprime filters* to decouple diagonalizable quadratic systems. They were introduced in [8] for quadratic systems with invertible matrix M and extended to arbitrary matrix polynomials of any degree in [3], where they were called *spectral filters*. In this paper, we show that spectral filters and SPTs are closely related, but the former have some advantages. In particular, they act directly on the involved quadratic systems, and they do not depend on their linearizations. Specifically, while the SPTs strongly depend on the chosen linearizations, the set of spectral filters only depend on the given quadratic systems in the sense that it remains unchanged irrespective of their linearizations (see Remarks 4.5, 5.7 and 5.13).

The goal of this paper is to clarify the relationship between spectral filters and SPTs for nonsingular quadratic systems. This was done in [8] for the case $\det M \neq 0$, and we will extend this analysis to general nonsingular quadratic systems, including the case $\det M = 0$ (but, of course, $M \neq 0$). Additionally, we will prove some results that will be relevant to the implementing the phase synchronization method using spectral filters that may be of interest in their own right.

The rest of the paper is organized as follows. Section 2 is dedicated to introducing the notation and the necessary previous results used in the manuscript. These include the notions of coprime and spectral filters. While coprime filters preserve the finite structure of polynomial matrices, the more general concept of a spectral filter is needed to preserve the structure at infinity as well. Section 3 is dedicated to recalling the notions of linearization and SPT. In particular, the most commonly used linearizations are defined, and a new one is introduced that will be important in our developments: the *Rblock-symmetric linearization*. This preserves the finite and infinite structures of $Q(s) = Ms^2 + Ds + K$ provided that K is invertible. We aim to show that there is a bijection between the set of spectral filters of spectrally equivalent quadratic matrix systems $Q_1(s)$ and $Q_2(s)$ and the set of SPTs of their Rblock-symmetric linearizations. This was done in [8] when $\det M \neq 0$ and is revisited in Section 4. Coprime filters for unimodularly equivalent quadratic systems satisfy a Bezout identity. Matrices that realize such a Bezout identity are found in Section 4. These matrices are important to implement the method of phase synchronization. The results of Section 4 are generalized in Section 5 to the case of quadratic systems with $\det M = 0$ and $\det K \neq 0$. The case when both determinants are zero is covered in Section 5.1. Examples are provided to illustrate the main results of the paper.

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2. Preliminaries. Although the results of this manuscript are intended for application to real quadratic systems, they are purely algebraic and valid for arbitrary fields. Let $\mathbb{F}$ be a field and let $\overline{\mathbb{F}}$ denote its algebraic closure. The ring of polynomials in the indeterminate s with coefficients in $\mathbb{F}$ is denoted by $\mathbb{F}[s]$, and $\mathbb{F}(s)$ is the field of fractions of $\mathbb{F}[s]$, i.e., the field of rational functions over $\mathbb{F}$. A polynomial in $\mathbb{F}[s]$ is *monic* if its leading coefficient is 1. Given two polynomials $\alpha(s)$ and $\beta(s)$, by $\alpha(s) \mid \beta(s)$, we mean that $\alpha(s)$ is a divisor of $\beta(s)$.

As usual, $\mathbb{F}^{m\times n}$ and $\mathbb{F}[s]^{m\times n}$ denote the vector spaces over $\mathbb{F}$ of $m \times n$ matrices with elements in $\mathbb{F}$ and $\mathbb{F}[s]$, respectively. The elements of $\mathbb{F}[s]^{m\times n}$ are, strictly speaking, *polynomial matrices*; however, we will identify $\mathbb{F}[s]^{m\times n}$ with $\mathbb{F}^{m\times n}[s]$ and call the matrices of $\mathbb{F}[s]^{m\times n}$ *matrix polynomials*. The *rank* of a matrix polynomial over the field $\mathbb{F}(s)$, denoted $\operatorname{rank}(\cdot)$, is equal to the largest order of its non identically zero minors. It is also known in the literature as *normal rank*. A matrix polynomial is *regular* or *nonsingular* if it is square and its determinant is not identically zero; otherwise, it is said to be *singular*. The symbol $\deg(\cdot)$ will be used to denote the *degree* of a matrix polynomial, that is, the highest degree of its entries. The *leading coefficient* of a matrix polynomial is the matrix of the coefficients of the polynomial entries whose degrees are the degree of the matrix polynomial, that is, the coefficient of the term with the highest degree as matrix polynomial. The leading coefficient of a quadratic matrix polynomial is often referred to as the *mass matrix*, particularly if the quadratic matrix represents a vibrating system. Recall that a matrix $U(s) \in \mathbb{F}[s]^{n\times n}$ is said to be *unimodular* if it has inverse in $\mathbb{F}[s]^{n\times n}$, i.e., if it is polynomial and its determinat is a nonzero constant.

Two matrix polynomials $P_1(s), P_2(s) \in \mathbb{F}[s]^{m\times n}$ are *unimodularly equivalent* if there exist unimodular matrices $U_1(s) \in \mathbb{F}[s]^{m\times m}$ and $U_2(s) \in \mathbb{F}[s]^{n\times n}$ such that $P_2(s) = U_1(s)P_1(s)U_2(s)$. Let $P(s) \in \mathbb{F}[s]^{m\times n}$ of rank r. A canonical form for the unimodular equivalence is the *Smith normal form* (see, e.g., [13, Section 7.5]): If $P(s) \in \mathbb{F}[s]^{m\times n}$, then it is unimodularly equivalent to a diagonal matrix

$$\begin{bmatrix} \operatorname{Diag}\left(\alpha_1(s), \ldots, \alpha_r(s)\right) & 0 \\ 0 & 0 \end{bmatrix},$$

where $\alpha_1(s) \mid \cdots \mid \alpha_r(s)$ are monic polynomials. They are called the *invariant polynomials* of $P(s)$. The roots in $\overline{\mathbb{F}}$ of the invariant polynomials of $P(s)$ are the *finite eigenvalues* of $P(s)$.

For any $s_0 \in \overline{\mathbb{F}}$, the invariant polynomials can be uniquely factorized as $\alpha_i(s) = (s - s_0)^{d_i}\beta_i(s)$ with $d_i \geq 0$, $\beta_i(s_0) \neq 0$. The sequence $(d_1, \ldots, d_r)$, with $0 \leq d_1 \leq \cdots \leq d_r$, is called the *partial multiplicity sequence of $P(s)$ at s_0*.

The unimodular equivalence can be characterized using coprime filters. In order to introduce this notion, we recall when two matrix polynomials are said to be coprime. Two matrix polynomials $P(s) \in \mathbb{F}[s]^{m\times n}$ and $N(s) \in \mathbb{F}[s]^{p\times n}$ are *right coprime* if their common right divisors are unimodular matrices; that is, if there exist $\widetilde{P}(s) \in \mathbb{F}[s]^{m\times n}$, $\widetilde{N}(s) \in \mathbb{F}[s]^{p\times n}$, and $X(s) \in \mathbb{F}[s]^{n\times n}$ such that $P(s) = \widetilde{P}(s)X(s)$ and $N(s) = \widetilde{N}(s)X(s)$, then $X(s)$ is unimodular. Moreover, $P(s) \in \mathbb{F}[s]^{n\times m}$ and $N(s) \in \mathbb{F}[s]^{n\times p}$ are *left coprime* if their transposes $P(s)^T$ and $N(s)^T$ are right coprime. The following theorem gives several characterizations of this coprimeness notion.

THEOREM 2.1. *[20, Chapter 2, Theorem 6.1] The matrix polynomials $P(s) \in \mathbb{F}[s]^{n\times n}$ and $N(s) \in \mathbb{F}[s]^{n\times m}$ are left coprime if and only in one of the following equivalent conditions is satisfied.*

(i) $\operatorname{rank}\begin{bmatrix} P(s_0) & N(s_0) \end{bmatrix} = n$ *for all* $s_0 \in \overline{\mathbb{F}}$.
(ii) The Smith normal form of $\begin{bmatrix} P(s) & N(s) \end{bmatrix}$ *is* $\begin{bmatrix} I_n & 0 \end{bmatrix}$.

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(iii) There exist matrix polynomials $V(s) \in \mathbb{F}[s]^{m\times n}$, $W(s) \in \mathbb{F}[s]^{m\times m}$ *such that the Smith normal form of the matrix*

$$\begin{bmatrix} P(s) & N(s) \\ -V(s) & W(s), \end{bmatrix}$$

is I_{n+m}.

(iv) There exist right coprime matrix polynomials $X(s) \in \mathbb{F}[s]^{n\times n}$, $Y(s) \in \mathbb{F}[s]^{m\times n}$ *such that*

$$P(s)X(s) + N(s)Y(s) = I_n.$$

Coprime filters were introduced in [8] for quadratic matrix polynomials with nonsingular leading coefficients and extended in [3] for more general matrix polynomials.

DEFINITION 2.2. Let $P_1(s), P_2(s) \in \mathbb{F}[s]^{m\times n}$. Two matrices $F_1(s) \in \mathbb{F}[s]^{n\times n}, F_2(s) \in \mathbb{F}[s]^{m\times m}$ are *coprime filters from* $P_1(s)$ *to* $P_2(s)$ if

(i) $P_1(s)$ and $F_1(s)$ are right coprime,
(ii) $P_2(s)$ and $F_2(s)$ are left coprime,
(iii) $F_2(s)P_1(s) = P_2(s)F_1(s)$.

REMARK 2.3. *It follows from item (i) in Theorem 2.1 that when* $P_1(s)$ *and* $P_2(s)$ *are* $n \times n$ *nonsingular matrix polynomials, then* $P_1(s)$ *and* $F_1(s)$ *are right coprime if and only if* $\operatorname{rank}\begin{bmatrix} P_1(s_0) \\ F_1(s_0) \end{bmatrix} = n$ *for all* s_0 *eigenvalue of* $P_1(s)$. *Similarly,* $P_2(s)$ *and* $F_2(s)$ *are left coprime if and only if* $\operatorname{rank}\begin{bmatrix} P_2(s_0) & F_2(s_0) \end{bmatrix} = n$ *for all* s_0 *eigenvalue of* $P_2(s)$.

It is a key consequence of [19, Theorems 2 and 3] that two matrix polynomials are unimodularly equivalent if and only if there exist coprime filters connecting them.

Let $P(s) = P_d s^d + \cdots + P_1 s + P_0 \in \mathbb{F}[s]^{m\times n}$ with $P_d \neq 0$. The *reversal of* $P(s)$ is the matrix polynomial

$$\operatorname{rev} P(s) := s^d P(1/s) = P_0 s^d + P_1 s^{d-1} + \cdots + P_d.$$

If the leading coefficient of $P(s)$, P_d, is singular (as in [6] for example), then $P(s)$ has infinity as an eigenvalue (in fact, this means that $\operatorname{rev} P(s)$ has 0 as eigenvalue). The *partial multiplicity sequence of* $P(s)$ *at* $s_0 = \infty$ is the partial multiplicity sequence of $\operatorname{rev} P(s)$ at $s_0 = 0$. The set of eigenvalues of $P(s)$ (finite and infinite) is denoted $\Lambda(P)$. It follows from the definition that $\operatorname{rank} P(a) = \operatorname{rank} P(s)$ for $a \in \overline{\mathbb{F}}$ if and only if a is not an eigenvalue of $P(s)$.

We introduce now the spectral filters for matrix polynomials.

DEFINITION 2.4. Let $P_1(s), P_2(s) \in \mathbb{F}[s]^{m\times n}$ be matrix polynomials of degree d_i and leading coefficient P_{i,d_i}, $i = 1, 2$, respectively. Let $a \in \mathbb{F}$ such that $\operatorname{rank} P_i(a) = \operatorname{rank} P_i(s)$, $i = 1, 2$. Then, $P_1(s)$ and $P_2(s)$ are *spectrally equivalent* if there are matrix polynomials $F_1(s) \in \mathbb{F}[s]^{n\times n}$ and $F_2(s) \in \mathbb{F}[s]^{m\times m}$ satisfying:

(i) $\operatorname{rank}\begin{bmatrix} P_1(s_0) \\ F_1(s_0) \end{bmatrix} = n$ for all $s_0 \in \overline{\mathbb{F}} \setminus \{a\}$, and $\operatorname{rank}\begin{bmatrix} P_{1,d_1} \\ F_{1,n_1} \end{bmatrix} = n$,
(ii) $\operatorname{rank}\begin{bmatrix} F_2(s_0) & P_2(s_0) \end{bmatrix} = m$ for all $s_0 \in \overline{\mathbb{F}} \setminus \{a\}$, and $\operatorname{rank}\begin{bmatrix} F_{2,n_2} & P_{2,d_2} \end{bmatrix} = m$,
(iii) $F_2(s)P_1(s) = P_2(s)F_1(s)$,
(iv) $n_2 + d_1 = d_2 + n_1$,

where $n_i = \deg(F_i(s))$ and F_{i,n_i} is the leading coefficient of $F_i(s)$, $i = 1, 2$.

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DEFINITION 2.5. $F_1(s)$ and $F_2(s)$ in the previous definition are said to be *spectral filters from* $P_1(s)$ *to* $P_2(s)$.

It follows from [4, Theorem 5.3]) that $P_1(s)$ and $P_2(s)$ are spectrally equivalent if and only if they have the same finite and infinite eigenvalues with the same partial multiplicity sequences.

REMARK 2.6. *(i) As square matrix polynomials with invertible leading coefficients have no eigenvalues at infinity, unimodularly equivalent matrix polynomials with invertible leading coefficients are spectrally equivalent, and their coprime filters are spectral filters (see [4, Section 5.1]) .*

(ii) As in Remark 2.3, when $P_1(s)$ *and* $P_2(s)$ *are* $n \times n$ *nonsingular, items (i) and (ii) of Definition 2.4 can be replaced, respectively, by*

(i') $\operatorname{rank}\begin{bmatrix} P_1(s_0) \\ F_1(s_0) \end{bmatrix} = n$ *for all* $s_0 \in \Lambda(P_1(s))$, *and* $\operatorname{rank}\begin{bmatrix} P_{1,d_1} \\ F_{1,n_1} \end{bmatrix} = n$,

(ii') $\operatorname{rank}\begin{bmatrix} F_2(s_0) & P_2(s_0) \end{bmatrix} = n$ *for all* $s_0 \in \Lambda(P_2(s))$, *and* $\operatorname{rank}\begin{bmatrix} F_{2,n_2} & P_{2,d_2} \end{bmatrix} = n$.

Two matrix polynomials of degree at most one (sometimes called *pencils*), $A_1 s + B_1 \in \mathbb{F}[s]^{m\times n}$ and $A_2 s + B_2 \in \mathbb{F}[s]^{m\times n}$, are *strictly equivalent* if there exist invertible matrices $V \in \mathbb{F}^{m\times m}$ and $U \in \mathbb{F}^{n\times n}$ such that $(A_2 s + B_2) = V(A_1 s + B_1)U$. The canonical form under strict equivalence is the *Kronecker canonical form*. In particular, if $A_1 s + B_1 \in \mathbb{F}[s]^{n\times n}$ and $A_2 s + B_2 \in \mathbb{F}[s]^{n\times n}$ are nonsingular linear polynomials, then $A_1 s + B_1$ and $A_2 s + B_2$ are strictly equivalent if and only if they have the same finite and infinite eigenvalues with the same partial multiplicity sequences (see [7, Chapter XII, Theorem 2]).

3. Linearizations and SPTs. From now on, we consider nonsingular quadratic matrix polynomials of the form $Q(s) = Ms^2 + Ds + K$, with $M, D, K \in \mathbb{F}^{n\times n}$. Recall that a *linearization* of $Q(s)$ is defined to be any linear polynomial $As + B$, $A, B \in \mathbb{F}^{2n\times 2n}$, with the same invariant polynomials as those of $Q(s)$ apart from invariant polynomials equal to 1. In other words, $As + B$ is a linearization of $Q(s)$ if there are unimodular matrix polynomials $U_1(s), U_2(s)$ such that

$$U_1(s)(As + B)U_2(s) = \begin{bmatrix} Q(s) & 0 \\ 0 & I_n \end{bmatrix}.$$

All linearizations have the same finite eigenvalues as $Q(s)$ with the same partial multiplicity sequences. However, if M is singular, the more general notion of *strong linearization* is required: A linearization $As + B$ of $Q(s)$ is said to be strong if $\operatorname{rev}(As + B) = Bs + A$ is also a linearization of $\operatorname{rev} Q(s) = Ks^2 + Ds + M$. A consequence of this definition is that any strong linearization of $Q(s)$ have the same finite and infinite eigenvalues of $Q(s)$ with the same sequences of partial multiplicities. Therefore, if $Q_1(s)$ and $Q_2(s)$ are nonsingular spectrally equivalent and $A_1 s + B_1$ and $A_2 s + B_2$ are strong linearizations of $Q_1(s)$ and $Q_2(s)$, respectively, then $A_1 s + B_1$ and $A_2 s + B_2$ are strictly equivalent. And conversely, if $A_1 s + B_1$ and $A_2 s + B_2$ are strong linearizations of $Q_1(s)$ and $Q_2(s)$, respectively, and they are strictly equivalent then $Q_1(s)$ and $Q_2(s)$ are spectrally equivalent.

Distinguished strong linearizations of $Q(s) = Ms^2 + Ds + K$ (see, for instance, [5, 9, 22, 23]) are the *first block-companion linearization*

$$\begin{bmatrix} M & 0 \\ 0 & I_n \end{bmatrix} s + \begin{bmatrix} D & K \\ -I_n & 0 \end{bmatrix},$$

and the *second block-companion linearization*

$$\begin{bmatrix} M & 0 \\ 0 & I_n \end{bmatrix} s + \begin{bmatrix} D & -I_n \\ K & 0 \end{bmatrix}.$$

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When $\det M \neq 0$, the following matrices are also strong linearizations:

$$\begin{bmatrix} I_n & 0 \\ 0 & I_n \end{bmatrix} s + \begin{bmatrix} 0 & -I_n \\ M^{-1}K & M^{-1}D \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} I_n & 0 \\ 0 & I_n \end{bmatrix} s + \begin{bmatrix} 0 & KM^{-1} \\ -I_n & DM^{-1} \end{bmatrix},$$

are called in [8] *right block-companion linearization* and *left block-companion linearization*, respectively.

A large family of strong linearizations was obtained in [16, p. 989]: for any $a_1, a_2 \in \mathbb{F}$, not both zero and such that $-\frac{a_2}{a_1} \notin \Lambda(Q)$, every pencil of the form

$$\widetilde{L}(s) = \begin{bmatrix} a_1 M & a_2 M \\ a_2 M & a_2 D - a_1 K \end{bmatrix} s + \begin{bmatrix} a_1 D - a_2 M & a_1 K \\ a_1 K & a_2 K \end{bmatrix}, \tag{3.2}$$

is a strong linearization of $Q(s)$. Since

$$L(s) = \begin{bmatrix} 0 & I_n \\ I_n & 0 \end{bmatrix} \widetilde{L}(s) \begin{bmatrix} 0 & I_n \\ I_n & 0 \end{bmatrix} = \begin{bmatrix} a_2 D - a_1 K & a_2 M \\ a_2 M & a_1 M \end{bmatrix} s + \begin{bmatrix} a_2 K & a_1 K \\ a_1 K & a_1 D - a_2 M \end{bmatrix}, \tag{3.3}$$

$\widetilde{L}(s)$ and $L(s)$ are strictly equivalent and so the matrices $L(s)$ of (3.3) also form a family of strong linearizations of $Q(s)$ provided that $-\frac{a_2}{a_1} \notin \Lambda(Q)$ for any $a_1, a_2 \in \mathbb{F}$, not both zero.

Note that the first, second, right, and left block-companion linearizations are not included in these families of linear pencils. Note also that when $\det M \neq 0$, then $\infty \notin \Lambda(Q)$ and the matrix of (3.3) with $a_1 = 0$ and $a_2 = 1$ is a linearization of $Q(s)$. In this case, such a matrix is

$$L(s) = \begin{bmatrix} D & M \\ M & 0 \end{bmatrix} s + \begin{bmatrix} K & 0 \\ 0 & -M \end{bmatrix}. \tag{3.4}$$

This matrix was called the *block-symmetric linearization* of $Q(s)$ in [8].

On the other hand, taking $a_1 = 1$ and $a_2 = 0$ in (3.2), we get

$$L(s) = \begin{bmatrix} M & 0 \\ 0 & -K \end{bmatrix} s + \begin{bmatrix} D & K \\ K & 0 \end{bmatrix}, \tag{3.5}$$

whose reversal is

$$\operatorname{rev} L(s) = \begin{bmatrix} D & K \\ K & 0 \end{bmatrix} s + \begin{bmatrix} M & 0 \\ 0 & -K \end{bmatrix}. \tag{3.6}$$

Therefore, if $0 \notin \Lambda(Q)$, $L(s)$ of (3.5) is a strong linearization of $Q(s)$ whose reversal is the block-symmetric linearization of $\operatorname{rev} Q(s) = Ks^2 + Ds + M$. We will say that $L(s)$ of (3.5) is the *Rblock-symmetric linearization* of $Q(s)$ provided that $\det K \neq 0$.

Definition 3.1. Let $Q_1(s) = M_1 s^2 + D_1 s + K_1$ and $Q_2(s) = M_2 s^2 + D_2 s + K_2$ be two spectrally equivalent nonsingular quadratic matrix polynomials. Let $L_1(s)$ and $L_2(s)$ be the first block-companion (second block-companion, block-symmetric, Rblock-symmetric, right block-companion, left block-companion) linearizations of $Q_1(s)$ and $Q_2(s)$, respectively.

(i) A pair of nonsingular matrices (U, V), $U, V \in \mathbb{F}^{2n \times 2n}$, is said to be a *first block-companion (second block-companion, block-symmetric, Rblock-symmetric) SPT* from $Q_1(s)$ to $Q_2(s)$ if $UL_1(s) = L_2(s)V$.

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(ii) A nonsingular matrix $V \in \mathbb{F}^{2n\times 2n}$and their filters is said to be a *right block-companion (left block-companion) SPT* from $Q_1(s)$ to $Q_2(s)$ if $VL_1(s) = L_2(s)V$.

We are to show that spectral filters and the SPTs are very closely related. Actually, for unimodularly equivalent quadratic matrix polynomials with invertible mass matrices, it was shown in [8, Theorem 5.1] how to obtain coprime filters from right block-companion SPTs and conversely. We aim to generalize the result in [8] to spectrally equivalent nonsingular quadratic matrix polynomials with, possibly, singular mass matrix. Taking into account that the reversal of the Rblock-symmetric linearization of $Q(s)$ is the block-symmetric linearization of $\operatorname{rev} Q(s)$, it will be more convenient for our developments to consider the Rblock-symmetric linearizations of (3.5) when $\det K \neq 0$. As mentioned above, these are strong linearizations of $Q(s)$ if $\det K \neq 0$, but the only restriction on M is that it is not the zero matrix. Later on, we will remove the condition $\det K \neq 0$.

First of all, we need to adapt the results in [8] to block-symmetric linearizations because they will be used for the general case of quadratic matrix polynomials with, possibly, singular mass matrix.

4. Coprime filters and block-symmetric SPTs. In this section, we study the relationship between coprime filters and block-symmetric SPTs of unimodularly equivalent quadratic matrix polynomials with nonsingular leading coefficients.

First, we show that if (U, V) is a block-symmetric SPT connecting two unimodularly equivalent quadratic matrix polynomials with nonsingular leading coefficients $Q_1(s)$ and $Q_2(s)$, then $Q_1(s)$, $Q_2(s)$ and the n first rows of V determine completely V, U and linear coprime filters connecting $Q_1(s)$ and $Q_2(s)$.

PROPOSITION 4.1. *Let $Q_i(s) = M_i s^2 + D_i s + K_i \in \mathbb{F}[s]^{n\times n}$ with $\det M_i \neq 0$, $i = 1, 2$, and assume that $Q_1(s)$ and $Q_2(s)$ are unimodularly equivalent. Let*

$$A_i = \begin{bmatrix} D_i & M_i \\ M_i & 0 \end{bmatrix}, \ i = 1, 2. \tag{4.7}$$

Let (U, V) be a block-symmetric SPT from $Q_1(s)$ to $Q_2(s)$. Then,

(i)

$$V = \begin{bmatrix} V_1 & V_2 \\ -V_2M_1^{-1}K_1 & V_1 - V_2M_1^{-1}D_1 \end{bmatrix},$$

with $V_1, V_2 \in \mathbb{F}^{n\times n}$, and $U = A_2VA_1^{-1}$.

(ii) Write

$$U = \begin{bmatrix} U_1 & U_2 \\ U_3 & U_4 \end{bmatrix}, \ V = \begin{bmatrix} V_1 & V_2 \\ V_3 & V_4 \end{bmatrix}, \tag{4.8}$$

with $U_i, V_i \in \mathbb{F}^{n\times n}$, $i = 1, \ldots, 4$. Then, $F_1(s) = V_2s + V_1$ and $F_2(s) = U_3s + U_1$ are coprime filters from $Q_1(s)$ to $Q_2(s)$.

Proof. By item (iv) of Proposition 2.2 of [8], if (U, V) is a block-symmetric SPT from $Q_1(s)$ to $Q_2(s)$, then V is a right block-companion SPT from $Q_1(s)$ to $Q_2(s)$ and $U = A_2VA_1^{-1}$. Thus,

$$V \begin{bmatrix} 0 & I_n \\ -M_1^{-1}K_1 & -M_1^{-1}D_1 \end{bmatrix} = \begin{bmatrix} 0 & I_n \\ -M_2^{-1}K_2 & -M_2^{-1}D_2 \end{bmatrix} V.$$

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Let $V = \left[\begin{smallmatrix} V_1 & V_2 \\ V_3 & V_4 \end{smallmatrix}\right]$ with $V_i \in \mathbb{F}^{n\times n}$, $i = 1, \ldots, 4$. Then,

$$\begin{bmatrix} V_1 & V_2 \end{bmatrix} \begin{bmatrix} 0 & I_n \\ -M_1^{-1}K_1 & -M_1^{-1}D_1 \end{bmatrix} = \begin{bmatrix} V_3 & V_4 \end{bmatrix},$$

and so $V_3 = -V_2M_1^{-1}K_1$ and $V_4 = V_1 - V_2M_1^{-1}D_1$.

To prove (ii), from $U = A_2VA_1^{-1}$, we have

$$\begin{aligned} \begin{bmatrix} U_1 \\ U_3 \end{bmatrix} &= \begin{bmatrix} (D_2V_2 + M_2V_4)M_1^{-1} \\ M_2V_2M_1^{-1} \end{bmatrix} = \begin{bmatrix} (D_2V_2 + M_2V_1 - M_2V_2M_1^{-1}D_1)M_1^{-1} \\ M_2V_2M_1^{-1} \end{bmatrix} \\ &= \begin{bmatrix} M_2(M_2^{-1}D_2V_2 - V_2M_1^{-1}D_1 + V_1)M_1^{-1} \\ M_2V_2M_1^{-1} \end{bmatrix}. \end{aligned}$$

By Theorem 5.1 of [8], $F_1(s) = V_2s + V_1$ and $F_2(s) = U_3s + U_1$ are coprime filters from $Q_1(s)$ to $Q_2(s)$. ∎

and their filters

The following proposition allows us to obtain block-symmetric SPTs from linear coprime filters of unimodularly equivalent quadratic matrix polynomials with nonsingular leading coefficients.

Proposition 4.2. *Let $Q_i(s) = M_is^2 + D_is + K_i \in \mathbb{F}[s]^{n\times n}$ with $\det M_i \neq 0$, $i = 1, 2$ and assume that $Q_1(s)$ and $Q_2(s)$ are unimodularly equivalent. Let $F_{11}s + F_{10} \in \mathbb{F}[s]^{n\times n}$ and $F_{21}s + F_{20} \in \mathbb{F}[s]^{n\times n}$ be coprime filters from $Q_1(s)$ to $Q_2(s)$. Let*

$$V = \begin{bmatrix} F_{10} & F_{11} \\ -F_{11}M_1^{-1}K_1 & F_{10} - F_{11}M_1^{-1}D_1 \end{bmatrix}, \tag{4.9}$$

and $U = A_2VA_1^{-1}$ where A_1, A_2 are the matrices of (4.7). Then, (U, V) is a block-symmetric SPT from $Q_1(s)$ to $Q_2(s)$. Moreover, considering U partitioned as in (4.8), $F_{21} = U_3$ and $F_{20} = U_1$.

Proof. Let

$$C_{R_i} = \begin{bmatrix} 0 & I_n \\ -M_i^{-1}K_i & -M_i^{-1}D_i \end{bmatrix}, \quad i = 1, 2.$$

The matrix $sI_{2n} - C_{R_i}$ is the right block-companion linearization of $Q_i(s)$, $i = 1, 2$. It is shown in Theorem 5.1 of [8] that (4.9) is an invertible matrix such that $VC_{R_1} = C_{R_2}V$. It follows from item (iv) of Proposition 2.2 of [8] that (U, V) is a block-symmetric SPT from $Q_1(s)$ to $Q_2(s)$. Now, from Proposition 4.1 *(ii)*, $F_{21} = U_3$ and $F_{20} = U_1$. ∎

and their filters

The previous propositions lead to a bijection between the set of linear coprime filters and the set of block-symmetric SPTs of unimodularly equivalent quadratic matrix polynomials with nonsingular leading coefficients. Let

$$\mathcal{F}_{deg.cop}(Q_1, Q_2) = \{(F_{11}, F_{10}, F_{21}, F_{20}) : F_{11}s + F_{10}, F_{21}s + F_{20} \text{ are coprime filters from } Q_1(s) \text{ to } Q_2(s)\}$$

and

$$\mathcal{S}(Q_1, Q_2) = \{(U, V) \text{ block-symmetric SPT from } Q_1(s) \text{ to } Q_2(s)\}.$$

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COROLLARY 4.3. *Let* $Q_i(s) = M_i s^2 + D_i s + K_i \in \mathbb{F}[s]^{n\times n}$ *with* $\det M_i \neq 0$, $i = 1, 2$ *and assume that* $Q_1(s)$ *and* $Q_2(s)$ *are unimodularly equivalent. Then,*

$$\alpha : \mathcal{F}_{deg.cop}(Q_1, Q_2) \longrightarrow \mathcal{S}(Q_1, Q_2) \tag{4.10}$$

defined by $\alpha(F_{11}, F_{10}, F_{21}, F_{20}) = (U, V)$ *with* V *as in* (4.9) *and* $U = A_2 V A_1^{-1}$, *where* A_1, A_2 *are as in* (4.7), *is a bijection.*

Proof. It follows from Proposition 4.2 that α is well defined and injective. In fact, if $\alpha(F_{11}, F_{10}, F_{21}, F_{20}) = (U, V)$, $\alpha(F'_{11}, F'_{10}, F'_{21}, F'_{20}) = (U', V')$ and $\alpha(F_{11}, F_{10}, F_{21}, F_{20}) = \alpha(F'_{11}, F'_{10}, F'_{21}, F'_{20})$ then $U = U'$, $V = V'$ and, therefore, $(F_{11}, F_{10}, F_{21}, F_{20}) = (F'_{11}, F'_{10}, F'_{21}, F'_{20})$. To see that α is surjective, we consider any block-symmetric SPT from $Q_1(s)$ to $Q_2(s)$, say (U, V). By Proposition 4.1,

$$V = \begin{bmatrix} V_1 & V_2 \\ -V_2 M_1^{-1} K_1 & V_1 - V_2 M_1^{-1} D_1 \end{bmatrix},$$

$U = A_2 V A_1^{-1}$, $F_1(s) = V_2 s + V_1$ and $F_2(s) = U_3 s + U_1$ are coprime filters from $Q_1(s)$ to $Q_2(s)$, and $\alpha(V_2, V_1, U_3, U_1) = (U, V)$. ∎

As mentioned in Remark 2.6, since matrix polynomials with invertible leading coefficients have no eigenvalues at infinity, their coprime filters are spectral filters. Therefore, the filters obtained in Proposition 4.1 from block-symmetric SPTs are spectral filters. Let

$$\begin{aligned}\mathcal{F}_{deg.spec}(Q_1, Q_2) = \{(F_{11}, F_{10}, F_{21}, F_{20}) : F_{11}s + F_{10},\ F_{21}s + F_{20} \\ \text{are spectral filters from } Q_1(s) \text{ to } Q_2(s)\}.\end{aligned} \tag{4.11}$$

Then, $\mathcal{F}_{deg.spec}(Q_1, Q_2) = \mathcal{F}_{deg.cop}(Q_1, Q_2)$ and the map in Corollary 4.3 becomes a bijection between $\mathcal{F}_{deg.spec}(Q_1, Q_2)$ and $\mathcal{S}(Q_1, Q_2)$.

REMARK 4.4. *Let* $\mathbb{F} = \mathbb{R}$ *or* $\mathbb{C}$ *be the field of real or complex numbers. Note that* $\mathcal{F}(Q_1, Q_2) = \{(F_{11}, F_{10}, F_{21}, F_{20}) : (F_{21}s + F_{20})Q_1(s) = Q_2(s)(F_{11}s + F_{10})\}$ *is a linear subspace of* $\mathbb{F}^{4n^2}$. *Note also that it follows from Remark 2.3 that the set of coprime filters from* $Q_1(s)$ *to* $Q_2(s)$ *is the intersection of a finite number of open sets. So,* $\mathcal{F}_{deg.cop}(Q_1, Q_2)$ *is an open set in* $\mathcal{F}(Q_1, Q_2)$. *Now, let* $L_1(s)$ *and* $L_2(s)$ *be block-symmetric linearizations of* $Q_1(s)$ *and* $Q_2(s)$, *respectively. The same properties hold for the sets* $\{(U, V) : UL_1(s) = L_2(s)V\}$ *and* $\mathcal{S}(Q_1, Q_2)$. *Now, by Propositions 4.2, the matrices* V *of* (4.9) *and* $U = A_2 V^{-1} A_1^{-1}$, *where* A_1, A_2 *are the matrices of* (4.7), *depend continuously on* $(F_{11}, F_{10}, F_{21}, F_{20}) \in \mathcal{F}_{deg.cop}(Q_1, Q_2)$. *And conversely, by Proposition 4.1,* $(F_{11}, F_{10}, F_{21}, F_{20}) \in \mathcal{F}_{deg.cop}(Q_1, Q_2)$ *depends continuously on* $(U, V) \in \mathcal{S}(Q_1, Q_2)$. *Therefore,* α *in* (4.10) *is a homeomorphism.*

REMARK 4.5. *Although the SPTs depend on the given linearizations of* $Q_1(s)$ *and* $Q_2(s)$, *the coprime filters do not depend on them. In fact, assuming that* $Q_1(s)$ *and* $Q_2(s)$ *are unimodularly equivalent, if, for* $i = 1, 2$, $N_i(s)$ *is an arbitrary linearization of* $Q_i(s)$ *and* $L_i(s)$ *is the block-symmetric linearization of* (3.4), *then all linearizations* $N_1(s)$, $N_2(s)$, $L_1(s)$, *and* $L_2(s)$ *are strictly equivalent. Thus, there are invertible matrices* $R, S, U, V, X_1, X_2, Y_1, Y_2 \in \mathbb{F}^{2n\times 2n}$ *such that*

$$UL_1(s) = L_2(s)V, \quad RN_1(s) = N_2(s)S, \quad \text{and} \quad X_i L_i(s) = N_i(s)Y_i, \quad i = 1, 2.$$

Let $\mathcal{S}_N(Q_1, Q_2) = \{(R, S) : RN_1(s) = N_2(s)S\}$ *be the set of SPTs from* $Q_1(s)$ *to* $Q_2(s)$ *for the given linearizations* $N_1(s)$ *and* $N_2(s)$. *Fix* X_i, Y_i *such that* $X_i L_i(s) = N_i(s)Y_i$, $i = 1, 2$. *If* $\sigma_N : \mathcal{S}(Q_1, Q_2) \longrightarrow \mathcal{S}_N(Q_1, Q_2)$ *is defined by* $\sigma_N(U, V) = (X_2 U X_1^{-1}, Y_2 V Y_1^{-1})$ *then* $\alpha_N = \sigma_N \circ \alpha : \mathcal{F}_{deg.cop}(Q_1, Q_2) \longrightarrow \mathcal{S}_N(Q_1, Q_2)$ *is a homeomorphism. In other words, the set of coprime filters remains unchanged irrespective of the chosen linearizations.*

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A property of coprime filters that is important in the decoupling problem of quadratic systems is the Bezout identity associated to the coprimeness condition of coprime filters (see item (iv) of Theorem 2.1): If $Q_1(s)$ and $Q_2(s)$ are unimodularly equivalent quadratic systems and $F_1(s)$ and $F_2(s)$ are coprime filters from $Q_1(s)$ to $Q_2(s)$, then there are matrices $X_1(s)$, $Y_1(s)$, $X_2(s)$, $Y_2(s)$ such that $X_1(s)Q_1(s) + Y_1(s)F_1(s) = I_n$ and $Q_2(s)X_2(s) + F_2(s)Y_2(s) = I_n$. To implement the phase synchronization technique for decoupling diagonalizable quadratic systems, matrices $X_i(s)$ and $Y_i(s)$, $i = 1, 2$, are required to have degrees not bigger than 0 and 1, respectively. This property is not considered in [8] but can be proved using a similar technique to the proof of Theorem 5.1 in that paper. It is based on the following basic unimodular transformation where $Q(s) = Ms^2 + Ds + K$ with $M \in \mathbb{F}^{n\times n}$ invertible and $B_1, B_2 \in \mathbb{F}^{n\times n}$ are arbitrary matrices:

$$\begin{bmatrix} I_n & sI_n \\ 0 & I_n \end{bmatrix} \begin{bmatrix} Ds+K & Ms & B_1 \\ Ms & -M & B_2 \end{bmatrix} = \begin{bmatrix} Q(s) & 0 & B_2 s + B_1 \\ 0 & I_n & 0 \end{bmatrix} \begin{bmatrix} I_n & 0 & 0 \\ Ms & -M & B_2 \\ 0 & 0 & I_n \end{bmatrix}. \tag{4.12}$$

THEOREM 4.6. *Let $Q_i(s) = M_i s^2 + D_i s + K_i \in \mathbb{F}[s]^{n\times n}$ with $\det M_i \neq 0$, $i = 1, 2$, and assume that $Q_1(s)$ and $Q_2(s)$ are unimodularly equivalent. Let $F_1(s) = F_{11}s + F_{10}$ and $F_2(s) = F_{21}s + F_{20}$ be coprime filters from $Q_1(s)$ to $Q_2(s)$. Let $(U, V) = \alpha(F_{11}, F_{10}, F_{21}, F_{20})$ where α is the mapping of (4.10), i.e., (U, V) is the block-symmetric SPT from $Q_1(s)$ to $Q_2(s)$ of Proposition 4.2. Assume that U and V are partitioned as in (4.8). Let $\widehat{U} = U^{-1}$ and $\widehat{V} = V^{-1}$ and write*

$$\widehat{U} = \begin{bmatrix} \widehat{U}_1 & \widehat{U}_2 \\ \widehat{U}_3 & \widehat{U}_4 \end{bmatrix}, \widehat{V} = \begin{bmatrix} \widehat{V}_1 & \widehat{V}_2 \\ \widehat{V}_3 & \widehat{V}_4 \end{bmatrix}, \quad \widehat{U}_i, \widehat{V}_i \in \mathbb{F}^{n\times n}, i = 1, \ldots, 4. \tag{4.13}$$

Then,

$$X_1(s) = -\widehat{V}_2 M_2^{-1} U_3,\ Y_1(s) = \widehat{V}_2 s + \widehat{V}_1,\ X_2(s) = -V_2 M_1^{-1} \widehat{U}_3,\ Y_2(s) = \widehat{U}_3 s + \widehat{U}_1 \tag{4.14}$$

are matrices satisfying $X_1(s)Q_1(s) + Y_1(s)F_1(s) = I_n$ and $Q_2(s)X_2(s) + F_2(s)Y_2(s) = I_n$.

Proof. First of all, since (U, V) is a block-symmetric SPT from $Q_1(s)$ to $Q_2(s)$,

$$U \begin{bmatrix} D_1 s + K_1 & M_1 s \\ M_1 s & -M_1 \end{bmatrix} = \begin{bmatrix} D_2 s + K_2 & M_2 s \\ M_2 s & -M_2 \end{bmatrix} V.$$

Equivalently,

$$\widehat{U} \begin{bmatrix} D_2 s + K_2 & M_2 s \\ M_2 s & -M_2 \end{bmatrix} = \begin{bmatrix} D_1 s + K_1 & M_1 s \\ M_1 s & -M_1 \end{bmatrix} \widehat{V}. \tag{4.15}$$

Now, it follows from (4.12) that

$$\begin{bmatrix} D_2 s + K_2 & M_2 s & U_1 \\ M_2 s & -M_2 & U_3 \end{bmatrix} = \begin{bmatrix} I_n & -sI_n \\ 0 & I_n \end{bmatrix} \begin{bmatrix} Q_2(s) & 0 & U_3 s + U_1 \\ 0 & I_n & 0 \end{bmatrix} \begin{bmatrix} I_n & 0 & 0 \\ M_2 s & -M_2 & U_3 \\ 0 & 0 & I_n \end{bmatrix},$$

and

$$\begin{bmatrix} D_1 s + K_1 & M_1 s & I_n \\ M_1 s & -M_1 & 0 \end{bmatrix} = \begin{bmatrix} I_n & -sI_n \\ 0 & I_n \end{bmatrix} \begin{bmatrix} Q_1(s) & 0 & I_n \\ 0 & I_n & 0 \end{bmatrix} \begin{bmatrix} I_n & 0 & 0 \\ M_1 s & -M_1 & 0 \\ 0 & 0 & I_n \end{bmatrix}.$$

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Substituting these values in (4.15)

$$\begin{aligned}&\widehat{U}\begin{bmatrix} I_n & -sI_n \\ 0 & I_n \end{bmatrix}\begin{bmatrix} Q_2(s) & 0 & U_3s+U_1 \\ 0 & I_n & 0 \end{bmatrix}\begin{bmatrix} I_n & 0 & 0 \\ M_2s & -M_2 & U_3 \\ 0 & 0 & I_n \end{bmatrix}\\ &= \begin{bmatrix} I_n & -sI_n \\ 0 & I_n \end{bmatrix}\begin{bmatrix} Q_1(s) & 0 & I_n \\ 0 & I_n & 0 \end{bmatrix}\begin{bmatrix} I_n & 0 & 0 \\ M_1s & -M_1 & 0 \\ 0 & 0 & I_n \end{bmatrix}\begin{bmatrix} \widehat{V} & 0 \\ 0 & I_n \end{bmatrix}.\end{aligned} \tag{4.16}$$

Now,

$$\begin{bmatrix} Q_1(s) & 0 & I_n \\ 0 & I_n & 0 \end{bmatrix} = \begin{bmatrix} I_n & 0 & 0 \\ 0 & I_n & 0 \end{bmatrix}\begin{bmatrix} Q_1(s) & 0 & I_n \\ 0 & I_n & 0 \\ I_n & 0 & 0 \end{bmatrix}.$$

Substituting this identity in (4.16) and after some elementary manipulations, we get

$$\begin{bmatrix} I_n & sI_n \\ 0 & I_n \end{bmatrix} U \begin{bmatrix} I_n & -sI_n \\ 0 & I_n \end{bmatrix}\begin{bmatrix} I_n & 0 & 0 \\ 0 & I_n & 0 \end{bmatrix} = \begin{bmatrix} Q_2(s) & 0 & U_3s+U_1 \\ 0 & I_n & 0 \end{bmatrix} W_1(s) \tag{4.17}$$

where

$$\begin{aligned} W_1(s) &= \begin{bmatrix} I_n & 0 & 0 \\ M_2s & -M_2 & U_3 \\ 0 & 0 & I_n \end{bmatrix}\begin{bmatrix} V & 0 \\ 0 & I_n \end{bmatrix}\begin{bmatrix} I_n & 0 & 0 \\ sI_n & -M_1^{-1} & 0 \\ 0 & 0 & I_n \end{bmatrix}\begin{bmatrix} 0 & 0 & I_n \\ 0 & I_n & 0 \\ I_n & 0 & -Q_1(s) \end{bmatrix} \\ &= \begin{bmatrix} 0 & -V_2M_1^{-1} & V_2s+V_1 \\ U_3 & -M_2(V_2s-V_4)M_1^{-1} & M_2(V_2s^2+(V_1-V_4)s-V_3)-U_3Q_1(s) \\ I_n & 0 & -Q_1(s) \end{bmatrix}. \end{aligned}$$

On the other hand, for any matrix polynomial $A(s) \in \mathbb{F}[s]^{2n\times 2n}$

$$A(s)\begin{bmatrix} I_n & 0 & 0 \\ 0 & I_n & 0 \end{bmatrix} = \begin{bmatrix} I_n & 0 & 0 \\ 0 & I_n & 0 \end{bmatrix}\begin{bmatrix} A(s) & 0 \\ 0 & I_n \end{bmatrix}. \tag{4.18}$$

Using this property in (4.17), we get

$$\begin{bmatrix} I_n & 0 & 0 \\ 0 & I_n & 0 \end{bmatrix} = \begin{bmatrix} Q_2(s) & 0 & U_3s+U_1 \\ 0 & I_n & 0 \end{bmatrix} W_1(s)W_2(s),$$

where

$$W_2(s) = \begin{bmatrix} I_n & sI_n & 0 \\ 0 & I_n & 0 \\ 0 & 0 & I_n \end{bmatrix}\begin{bmatrix} \widehat{U} & 0 \\ 0 & I_n \end{bmatrix}\begin{bmatrix} I_n & -sI_n & 0 \\ 0 & I_n & 0 \\ 0 & 0 & I_n \end{bmatrix} = \begin{bmatrix} \widehat{U}_3s+\widehat{U}_1 & -\widehat{U}_3s^2-(\widehat{U}_1-\widehat{U}_4)s+\widehat{U}_2 & 0 \\ \widehat{U}_3 & -\widehat{U}_3s+\widehat{U}_4 & 0 \\ 0 & 0 & I_n \end{bmatrix}.$$

We are interested in finding matrices $X_2(s)$ and $Y_2(s)$ such that $Q_2(s)X_2(s)+(F_{21}s+F_{20})Y_2(s) = I_n$. By Proposition 4.2, $F_2(s) = U_3s+U_1$ and so we aim fo find $X_2(s)$ and $Y_2(s)$ such that $Q_2(s)X_2(s)+(U_3s+U_1)Y_2(s) = I_n$. To this end, we only have to compute the entries $(1,1)$ and $(3,1)$ of $W(s) = W_1(s)W_2(s)$. A simple computation shows that these are

$$W_{11}(s) = -V_2M_1^{-1}\widehat{U}_3, \quad W_{31}(s) = \widehat{U}_3s+\widehat{U}_1.$$

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Henceforth, putting $X_2(s) = -V_2 M_1^{-1}\widehat{U}_3$ and $Y_2(s) = \widehat{U}_3 s + \widehat{U}_1$, we have $Q_2(s)X_2(s) + F_2(s)Y_2(s) = I_n$ as desired.

Now, finding matrices $X_1(s), Y_1(s) \in \mathbb{F}[s]^{n\times n}$ such that $X_1(s)Q_1(s)+Y_1(s)F_1(s) = I_n$ is easy. First note that $F_1(s) = V_2 s+V_1$ and $F_2(s) = U_3 s+U_1$ are coprime filters from $Q_1(s)$ to $Q_2(s)$ if and only if $F_2(s)^T$ and $F_1(s)^T$ are coprime filters from $Q_2(s)^T$ to $Q_1(s)^T$. In particular, $(V_2^T s + V_1^T)Q_2(s)^T = Q_1(s)^T(U_3^T s + U_1^T)$ and

$$\begin{bmatrix} D_1^T s + K_1^T & M_1^T s \\ M_1^T s & -M_1^T \end{bmatrix} U^T = V^T \begin{bmatrix} D_2^T s + K_2^T & M_2^T s \\ M_2^T s & -M_2^T \end{bmatrix}.$$

The latter identity shows that (V^T, U^T) is a block-symmetric SPT from $Q_2(s)^T$ to $Q_1(s)^T$. Now, we can exchange the roles of $Q_1(s)$, $Q_2(s)$, $F_1(s)$, $F_2(s)$, $U = \left[\begin{smallmatrix} U_1 & U_2 \\ U_3 & U_4 \end{smallmatrix}\right]$ and $V = \left[\begin{smallmatrix} V_1 & V_2 \\ V_3 & V_4 \end{smallmatrix}\right]$ in our previous developments by $Q_2(s)^T$, $Q_1(s)^T$, $F_2(s)^T$, $F_1(s)^T$, $V^T = \left[\begin{smallmatrix} V_1^T & V_3^T \\ V_2^T & V_4^T \end{smallmatrix}\right]$ and $U^T = \left[\begin{smallmatrix} U_1^T & U_3^T \\ U_2^T & U_4^T \end{smallmatrix}\right]$, respectively (note that $\widehat{V}^T = (V^{-1})^T = \left[\begin{smallmatrix} \widehat{V}_1^T & \widehat{V}_3^T \\ \widehat{V}_2^T & \widehat{V}_4^T \end{smallmatrix}\right]$ and the same applies to $\widehat{U}^T$). Using this technique and putting $\widehat{X}_1(s) = -U_3^T M_2^{-T}\widehat{V}_2^T$ and $\widehat{Y}_1(s) = \widehat{V}_2^T s + \widehat{V}_1^T$, we can conclude that $Q_1(s)^T\widehat{X}_1(s) + F_1(s)^T\widehat{Y}_1(s) = I_n$. Therefore, if we put

$$X_1(s) = -\widehat{V}_2 M_2^{-1} U_3, \quad Y_1(s) = \widehat{V}_2 s + \widehat{V}_1$$

then $X_1(s)Q_1(s) + Y_1(s)F_1(s) = I_n$ and the proof is complete. ∎

Remark 4.7. *A simple computation shows that*

$$\begin{bmatrix} Q_2(s) & F_2(s) \\ -Y_1(s) & X_1(s) \end{bmatrix} \begin{bmatrix} X_2(s) & -F_1(s) \\ Y_2(s) & Q_1(s) \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ Z(s) & I_n \end{bmatrix}.$$

Then,

$$\begin{bmatrix} Q_2(s) & F_2(s) \\ -Y_1(s) & X_1(s) \end{bmatrix} \begin{bmatrix} X_2(s) + F_1(s)Z(s) & -F_1(s) \\ Y_2(s) - Q_1(s)Z(s) & Q_1(s) \end{bmatrix} = I_{2n}.$$

Theorem 4.6 provides explicit matrices $X_1(s), X_2(s), Y_1(s), Y_2(s)$, which allow to construct unimodular matrices, one the inverse of the other, from the matrices $\left[\begin{smallmatrix} Q_2(s) & F_2(s) \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} -F_1(s) \\ Q_1(s) \end{smallmatrix}\right]$ (see item (iii) of Theorem 2.1 and [10, Lemma 6.3-9]).

There is also a close relationship between the linear coprime filters from $Q_1(s)$ to $Q_2(s)$ and the linear coprime filters from $Q_2(s)$ to $Q_1(s)$. In fact, (U, V) is a block-symmetric SPT from $Q_1(s)$ to $Q_2(s)$ if and only if (U^{-1}, V^{-1}) is a block-symmetric SPT from $Q_2(s)$ to $Q_1(s)$ (see (4.15)).

Define the mapping

$$\widehat{\alpha} : \mathcal{F}_{deg.cop}(Q_2, Q_1) \longrightarrow \mathcal{S}(Q_2, Q_1), \tag{4.19}$$

as $\widehat{\alpha}(G_{21}, G_{20}, G_{11}, G_{10}) = (\widehat{U}, \widehat{V})$ where $\widehat{V} = \left[\begin{smallmatrix} G_{20} & G_{21} \\ -G_{21}M_2^{-1}K_2 & G_{20} - G_{21}M_2^{-1}D_2 \end{smallmatrix}\right]$ and $\widehat{U} = A_1\widehat{V}A_2^{-1}$ with A_1, A_2 as in (4.7). By Corollary 4.3 (exchanging the roles of $Q_1(s)$ and $Q_2(s)$, $(F_{11}, F_{10}, F_{21}, F_{20})$ and $(G_{21}, G_{20}, G_{11}, G_{10})$, U and $\widehat{U}$, and V and $\widehat{V}$), this is a bijection. In fact, it is a homeomorphism.

As a consequence, if $F_1(s) = F_{11}s + F_{10}$ and $F_2(s) = F_{21}s + F_{20}$ are coprime filters from $Q_1(s)$ to $Q_2(s)$ and $(U, V) = \alpha(F_{11}, F_{10}, F_{21}, F_{20})$ (α defined in Corollary 4.3), then (U^{-1}, V^{-1}) is a block-symmetric SPT from $Q_2(s)$ to $Q_1(s)$ and $G_2(s) = G_{21}s+G_{20}$ and $G_1(s) = G_{11}s+G_{10}$ are coprime filters from $Q_2(s)$ to $Q_1(s)$ where $(G_{21}, G_{20}, G_{11}, G_{10}) = \widehat{\alpha}^{-1}(U^{-1}, V^{-1})$. In other words, if $F_1(s) = F_{11}s + F_{10}$ and $F_2(s) = F_{21}s + F_{20}$

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are coprime filters from $Q_1(s)$ to $Q_2(s)$, V is the matrix in (4.9), $U = A_2VA_1^{-1}$ where A_1, A_2 are as in (4.7), and $\widehat{U} = U^{-1}$ and $\widehat{V} = V^{-1}$ are partitioned as in (4.13), then $G_2(s) = \widehat{V}_2 s + \widehat{V}_1$ and $G_1(s) = \widehat{U}_3 s + \widehat{U}_1$ are coprime filters from $Q_2(s)$ to $Q_1(s)$. And conversely, if $G_2(s) = G_{21}s + G_{20}$ and $G_1(s) = G_{11}s + G_{10}$ are coprime filters from $Q_2(s)$ to $Q_1(s)$, $\widehat{U}$ and $\widehat{V}$ are as in the definition of $\widehat{\alpha}$, and $U = \widehat{U}^{-1}$ and $V = \widehat{V}^{-1}$ are partitioned as in (4.8), then $F_1(s) = V_2 s + V_1$ and $F_2(s) = U_3 s + U_1$ are coprime filters from $Q_1(s)$ to $Q_2(s)$. Thus, exchanging the roles of $Q_1(s)$ and $Q_2(s)$, $F_1(s)$ and $G_2(s)$, $F_2(s)$ and $G_1(s)$, U and U^{-1}, and V and V^{-1}, we get the following result.

COROLLARY 4.8. *Let $Q_i(s) = M_i s^2 + D_i s + K_i \in \mathbb{F}[s]^{n\times n}$ with $\det M_i \neq 0$, $i = 1, 2$ and assume that $Q_1(s)$ and $Q_2(s)$ are unimodularly equivalent. Let $G_2(s) = G_{21}s + G_{20}$ and $G_1(s) = G_{11}s + G_{10}$ be coprime filters from $Q_2(s)$ to $Q_1(s)$ and $(\widehat{U}, \widehat{V}) = \widehat{\alpha}(G_{21}, G_{20}, G_{11}, G_{10})$, where $\widehat{\alpha}$ is the mapping of (4.19). Let $U = \widehat{U}^{-1}$ and $V = \widehat{V}^{-1}$ and assume that U and V are partitioned as in (4.8) and $\widehat{U}$ and $\widehat{V}$ as in (4.13). Then,*

$$X_1(s) = -\widehat{V}_2 M_2^{-1} U_3,\ Y_1(s) = U_3 s + U_1,\ X_2(s) = -V_2 M_1^{-1}\widehat{U}_3,\ Y_2(s) = V_2 s + V_1, \tag{4.20}$$

are matrices satisfying $Q_1(s)X_1(s) + G_1(s)Y_1(s) = I_n$ and $X_2(s)Q_2(s) + Y_2(s)G_2(s) = I_n$.

5. Spectral filters and Rblock-symmetric SPTs. We consider now the case when the mass matrices of the given nonsingular quadratic matrix polynomials can be singular but nonzero. Let $Q_i(s) = M_i s^2 + D_i s + K_i$ and $R_i(s) = \operatorname{rev} Q_i(s) = K_i s^2 + D_i s + M_i$, $i = 1, 2$ and assume that $Q_1(s)$ and $Q_2(s)$ are spectrally equivalent. Assume also that $0 \notin \Lambda(Q_1) = \Lambda(Q_2)$, that is, $\det K_i \neq 0$, $i = 1, 2$. It was already mentioned that this restriction will be removed later on (see Section 5.1). Consider the Rblock-symmetric linearizations of $Q_i(s)$ for $i = 1, 2$, and their reversals

$$L_{Q_i}(s) = \begin{bmatrix} M_i & 0 \\ 0 & -K_i \end{bmatrix} s + \begin{bmatrix} D_i & K_i \\ K_i & 0 \end{bmatrix}, \quad L_{R_i}(s) = \begin{bmatrix} D_i & K_i \\ K_i & 0 \end{bmatrix} s + \begin{bmatrix} M_i & 0 \\ 0 & -K_i \end{bmatrix}. \tag{5.21}$$

Note that $L_{R_i}(s) = \operatorname{rev} L_{Q_i}(s)$ is the block-symmetric linearization of $R_i(s) = \operatorname{rev} Q_i(s)$, $i = 1, 2$. Recall that $L_{Q_1}(s)$ and $L_{Q_2}(s)$ are strong linearizations of $Q_1(s)$ and $Q_2(s)$, respectively. Hence, $L_{Q_1}(s)$ and $L_{Q_2}(s)$ are strictly equivalent, that is to say, there are invertible matrices $U, V \in \mathbb{F}^{2n\times 2n}$ such that $UL_{Q_1}(s) = L_{Q_2}(s)V$. This means that (U, V) is an Rblock-symmetric SPT from $Q_1(s)$ to $Q_2(s)$ (cf. Definition 3.1).

In the following result, we show how linear spectral filters from $Q_1(s)$ to $Q_2(s)$ can be determined from Rblock-symmetric SPTs.

PROPOSITION 5.1. *Let $Q_i(s) = M_i s^2 + D_i s + K_i$ with $\det K_i \neq 0$, $i = 1, 2$, and assume that $Q_1(s)$ and $Q_2(s)$ are spectrally equivalent. Let (U, V) be an Rblock-symmetric SPT from $Q_1(s)$ to $Q_2(s)$ with U and V partitioned as in (4.8). Then, $F_1(s) = V_1 s + V_2$ and $F_2(s) = U_1 s + U_3$ are spectral filters from $Q_1(s)$ to $Q_2(s)$. Moreover, $U_1 \neq 0$ and $V_1 \neq 0$ if $\det M_i = 0$, $i = 1, 2$.*

Proof. Let $R_i(s) = \operatorname{rev} Q_i(s)$, $i = 1, 2$. Since (U, V) is an Rblock-symmetric SPT from $Q_1(s)$ to $Q_2(s)$,

$$U \begin{bmatrix} M_1 & 0 \\ 0 & -K_1 \end{bmatrix} = \begin{bmatrix} M_2 & 0 \\ 0 & -K_2 \end{bmatrix} V,\ U \begin{bmatrix} D_1 & K_1 \\ K_1 & 0 \end{bmatrix} = \begin{bmatrix} D_2 & K_2 \\ K_2 & 0 \end{bmatrix} V. \tag{5.22}$$

∎

It follows from this and (5.21) that $UL_{R_1}(s) = L_{R_2}(s)V$. Since $\det K_i \neq 0$, (U, V) is a block-symmetric SPT from $R_1(s)$ to $R_2(s)$. It follows from Proposition 4.1 that $\widehat{F}_1(s) = V_2 s + V_1$ and $\widehat{F}_2(s) = U_3 s + U_1$ are coprime filters from $R_1(s)$ to $R_2(s)$. By Remark 2.6, $\widehat{F}_1(s)$ and $\widehat{F}_2(s)$ are spectral filters from $R_1(s)$ to

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$R_2(s)$ and $\deg(\widehat{F}_1(s)) = \deg(\widehat{F}_2(s))$ (as $\widehat{F}_2(s)R_1(s) = R_2(s)\widehat{F}_1(s)$ and, therefore, $U_3K_1 = K_2V_2$). Now, by [3, Proposition 5.1], $\operatorname{rev}\widehat{F}_1(s)$ and $\operatorname{rev}\widehat{F}_2(s)$ are coprime filters from $Q_1(s)$ to $Q_2(s)$.

If $\det M_i \neq 0$, $i = 1, 2$, by Remark 2.6 again, $\operatorname{rev}\widehat{F}_1(s)$ and $\operatorname{rev}\widehat{F}_2(s)$ are also spectral filters from $Q_1(s)$ to $Q_2(s)$. If $V_2 \neq 0$ and $U_3 \neq 0$, then $\operatorname{rev}\widehat{F}_1(s) = V_1s + V_2$ and $\operatorname{rev}\widehat{F}_2(s) = U_1s + U_3$. If $V_2 = 0$ and $U_3 = 0$, then $\operatorname{rev}\widehat{F}_1(s) = V_1$ and $\operatorname{rev}\widehat{F}_2(s) = U_1$. In this case, we apply Lemma 5.8 of [4] so that V_1s and U_1s are spectral filters from $Q_1(s)$ to $Q_2(s)$ as well. In any case, $F_1(s) = V_1s + V_2$ and $F_2(s) = U_1s + U_3$ are spectral filters from $Q_1(s)$ to $Q_2(s)$.

Assume now that $\det M_i = 0$, $i = 1, 2$. Since $\widehat{F}_1(s) = V_2s + V_1$ and $\widehat{F}_2(s) = U_3s + U_1$ are coprime filters from $\operatorname{rev} Q_1(s)$ to $\operatorname{rev} Q_2(s)$, it follows from [4, Lemma 5.9 (b)] that $V_1 \neq 0$ and $U_1 \neq 0$. On the other hand we have seen above that $\deg(\widehat{F}_1(s)) = \deg(\widehat{F}_2(s))$. Again by [4, Lemma 5.9 (b)], $F_1(s) = V_1s + V_2$ and $F_2(s) = U_1s + U_3$ are spectral filters from $Q_1(s)$ to $Q_2(s)$.

With the help of Proposition 4.2, we can see how Rblock-symmetric SPTs can be obtained from linear spectral filters.

PROPOSITION 5.2. *Let $Q_i(s) = M_is^2 + D_is + K_i$ with $\det K_i \neq 0$, $i = 1, 2$, and assume that $Q_1(s)$ and $Q_2(s)$ are spectrally equivalent. Let $F_{11}s + F_{10} \in \mathbb{F}[s]^{n\times n}$ and $F_{21}s + F_{20} \in \mathbb{F}[s]^{n\times n}$, with $F_{11} \neq 0$ and $F_{21} \neq 0$ if $\det M_i = 0$, $i = 1, 2$, be spectral filters from $Q_1(s)$ to $Q_2(s)$. Let*

$$\widehat{A}_i = \begin{bmatrix} D_i & K_i \\ K_i & 0 \end{bmatrix}, \quad i = 1, 2, \tag{5.23}$$

and define

$$V = \begin{bmatrix} F_{11} & F_{10} \\ -F_{10}K_1^{-1}M_1 & F_{11} - F_{10}K_1^{-1}D_1 \end{bmatrix}, \tag{5.24}$$

and $U = \widehat{A}_2V\widehat{A}_1^{-1}$. Then, (U, V) is an Rblock-symmetric SPT from $Q_1(s)$ to $Q_2(s)$. Moreover, $F_{21} = U_1$ and $F_{20} = U_3$.

Proof. If $F_{11}s + F_{10}$ and $F_{21}s + F_{20}$ are spectral filters from $Q_1(s)$ to $Q_2(s)$, then, by Proposition 5.1 of [3], $\operatorname{rev}(F_{11}s + F_{10})$ and $\operatorname{rev}(F_{21}s + F_{20})$ are coprime filters from $R_1(s) = \operatorname{rev} Q_1(s)$ to $R_2(s) = \operatorname{rev} Q_2(s)$.

If $\det M_i \neq 0$, $i = 1, 2$, by [4, Lemma 5.6], we know that $\deg(F_{11}s + F_{10}) = \deg(F_{21}s + F_{20})$. If $F_{11} \neq 0$ and $F_{21} \neq 0$, $\operatorname{rev}(F_{11}s + F_{10}) = F_{10}s + F_{11}$ and $\operatorname{rev}(F_{21}s + F_{20}) = F_{20}s + F_{21}$. If $F_{11} = 0$ and $F_{21} = 0$, we have that F_{10} and F_{20} are coprime filters from $R_1(s)$ to $R_2(s)$. By Remark 2.6, they are spectral filters, and from [4, Lemma 5.8], $F_{10}s$ and $F_{20}s$ are spectral filters, so coprime filters, from $R_1(s)$ to $R_2(s)$.

If $\det M_i = 0$, $i = 1, 2$, as we are assuming that $F_{11} \neq 0$ and $F_{21} \neq 0$, $\operatorname{rev}(F_{11}s + F_{10}) = F_{10}s + F_{11}$ and $\operatorname{rev}(F_{21}s + F_{20}) = F_{20}s + F_{21}$.

In any case, $F_{10}s + F_{11}$ and $F_{20}s + F_{21}$ are coprime filters from $R_1(s)$ to $R_2(s)$. Hence, by Proposition 4.2, if V is the matrix of (5.24) and $U = \widehat{A}_2V\widehat{A}_1^{-1}$, then $F_{21} = U_1$, $F_{20} = U_3$ and (U, V) is a block-symmetric SPT from $R_1(s)$ to $R_2(s)$. This means that (U, V) satisfies (5.22) and so (U, V) is an Rblock-symmetric SPT from $Q_1(s)$ to $Q_2(s)$. ∎

In this example, we show that the condition of filters being of exactly degree 1 when $\det M_i = 0$, $i = 1, 2$, is necessary for Proposition 5.2 to hold.

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EXAMPLE 5.3. *Let* $Q_1(s) = M_1s^2 + D_1s + K_1$ *and* $Q_2(s) = M_2s^2 + D_2s + K_2$ *be spectrally equivalent quadratic matrix polynomials with* $\det K_i \neq 0$, $i = 1, 2$, *and assume that there are invertible matrices* F_1 *and* F_2 *such that* $F_2Q_1(s) = Q_2(s)F_1$ *(this is the case if, for example,* $Q_1(s) = Q_2(s)$ *and* $F_1 = F_2 = I_n$*). Then, the matrix* V *of* (5.24) *is*

$$V = \begin{bmatrix} 0 & F_1 \\ -F_1K_1^{-1}M_1 & -F_1K_1^{-1}D_1 \end{bmatrix}.$$

This matrix is invertible if and only if $\det M_1 \neq 0$. *Therefore, if* $\det M_1 = 0$ *(in which case,* $\det M_2 = 0$ *as well),* (U, V) *as in Proposition 5.2 cannot be an Rblock-symmetric SPT from* $Q_1(s)$ *to* $Q_2(s)$.

We show an example of obtaining Rblock-symmetric SPTs from linear spectral filters following the ideas in Proposition 5.2.

EXAMPLE 5.4. *Consider the quadratic matrix polynomials*

$$Q_1(s) = \begin{bmatrix} s & 1 \\ 1 & s^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} s^2 + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} s + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

$$Q_2(s) = \begin{bmatrix} s^2 + s + 1 & 0 \\ 0 & s - 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} s^2 + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} s + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

These matrix polynomials are spectrally equivalent with singular leading coefficients and $\det K_i \neq 0$, $i = 1, 2$. *All possible spectral filters of degree 1 from* $Q_1(s)$ *to* $Q_2(s)$ *were given in [4, Example 6.4]:*

$$F_{11}s + F_{10} = \frac{1}{3}\left(\begin{bmatrix} 0 & -z - \sqrt{3}w \\ -3x & -y \end{bmatrix} s + \begin{bmatrix} z - \sqrt{3}w & 2z \\ y & -3x - y \end{bmatrix}\right)$$

,

$$F_{21}s + F_{20} = \frac{1}{3}\left(\begin{bmatrix} z - \sqrt{3}w & -z - \sqrt{3}w \\ -3x & 0 \end{bmatrix} s + \begin{bmatrix} 2z & z - \sqrt{3}w \\ 3x + y & -y \end{bmatrix}\right),$$

where x, y, z, w *are real numbers with* $x \neq 0$, $y \neq 0$, *and* $z^2 + w^2 \neq 0$. *Now, we can compute* V *and* U *of Proposition 5.2:*

$$V = \frac{1}{3}\begin{bmatrix} 0 & -z - \sqrt{3}w & z - \sqrt{3}w & 2z \\ -3x & -y & y & -3x - y \\ 0 & -z + \sqrt{3}w & -2z & -z - \sqrt{3}w \\ 0 & -y & y & -y \end{bmatrix},$$

$$U = \frac{1}{3}\begin{bmatrix} z - \sqrt{3}w & -z - \sqrt{3}w & -2z & -z + \sqrt{3}w \\ -3x & 0 & 2y & -3x \\ 2z & z - \sqrt{3}w & -z - \sqrt{3}w & -2z \\ 3x + y & -y & -y & -y \end{bmatrix}.$$

Therefore, for any choice of parameters x, y, z, w *(*$x \neq 0$, $y \neq 0$, $z^2 + w^2 \neq 0$*)* (U, V) *is an Rblock-symmetric SPT from* $Q_1(s)$ *to* $Q_2(s)$. *Note that* $\det V = \frac{1}{3}xy(z^2 + w^2) \neq 0$ *and* $\det U = -\frac{1}{3}xy(z^2 + w^2) \neq 0$. *In fact,* $U_1 = F_{21}$ *and* $U_3 = F_{20}$.

In view of Propositions 5.1 and 5.2, if the trailing and leading coefficients of $Q_1(s)$ and $Q_2(s)$ are nonsingular, we can define a bijection between the set of linear spectral filters from $Q_1(s)$ to $Q_2(s)$, $\mathcal{F}_{deg.spec}(Q_1, Q_2)$ (cf. (4.11)), and the set of Rblock-symmetric SPTs from $Q_1(s)$ to $Q_2(s)$

$$\mathcal{R}(Q_1, Q_2) = \{(U, V) \text{ Rblock-symmetric SPT from } Q_1(s) \text{ to } Q_2(s)\}.$$

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Furthermore, if the leading coefficients of $Q_1(s)$ and $Q_2(s)$ are singular, any (U,V) Rblock-symmetric SPT from $Q_1(s)$ to $Q_2(s)$ has the property $U_1 \neq 0$ and $V_1 \neq 0$ when U and V are partitioned as in (4.8) (see Proposition 5.1). Therefore, we can define a bijection between the set of spectral filters of degree exactly 1, say

$$\widehat{\mathcal{F}}_{deg.spec}(Q_1,Q_2) = \{(F_{11},F_{10},F_{21},F_{20}) : F_{11}s+F_{10},\ F_{21}s+F_{20} \text{ are spectral filters from } Q_1(s) \text{ to } Q_2(s) \text{ and } F_{11}\neq 0, F_{21}\neq 0\},$$

and the following subset of the set of Rblock-symmetric SPTs

$$\widehat{\mathcal{R}}(Q_1,Q_2) = \{(U,V) \text{ Rblock-symmetric SPT from } Q_1(s) \text{ to } Q_2(s) \text{ such that } U_1\neq 0, V_1\neq 0 \text{ when } U \text{ and } V \text{ are partitioned as in (4.8)}\}.$$

COROLLARY 5.5. *Let $Q_i(s) = M_is^2+D_is+K_i$ with $\det K_i \neq 0$, $i=1,2$, and assume that $Q_1(s)$ and $Q_2(s)$ are spectrally equivalent. Then,*

$$\beta_1 : \mathcal{F}_{deg.spec}(Q_1,Q_2) \longrightarrow \mathcal{R}(Q_1,Q_2),$$

if $\det M_i \neq 0$, $i=1,2$, and

$$\beta_2 : \widehat{\mathcal{F}}_{deg.spec}(Q_1,Q_2) \longrightarrow \widehat{\mathcal{R}}(Q_1,Q_2),$$

if $\det M_i = 0$, $i=1,2$, defined by $\beta_j(F_{11},F_{10},F_{21},F_{20}) = (U,V)$ with V as in (5.24) and $U = \widehat{A}_2V\widehat{A}_1^{-1}$ where $\widehat{A}_1,\widehat{A}_2$ are as in (5.23), is a bijection, $j=1,2$.

Proof. It follows from Proposition 5.2 that β_1 and β_2 are well defined and injective. That they are surjective is derived from Proposition 5.1. ∎

REMARK 5.6. *Taking into account item (ii) of Remark 2.6, we can see as in Remark 4.4 that both β_1 and β_2 are homeomorphisms when $\mathbb{F}=\mathbb{R}$ or $\mathbb{C}$.*

REMARK 5.7. *We can see that the set of spectral filters remains unchanged irrespective of the chosen linearization as in Remark 4.5. Let $N_1(s)$ and $N_2(s)$ be arbitrary given strong linearizations of spectrally equivalently $Q_1(s) = M_1s^2+D_1s+K_1$ and $Q_2(s) = M_2s^2+D_2s+K_2$, respectively, with $\det K_1 \neq 0$ and $\det K_2 \neq 0$. Let $L_1(s)$ and $L_2(s)$ be Rblock-symmetric linearizations of $Q_1(s)$ and $Q_2(s)$, respectively. Fix $X_i, Y_i \in \mathbb{F}^{2n\times 2n}$ such that $X_iL_i(s) = N_i(s)Y_i$, $i=1,2$.*

If $\det M_i \neq 0$, $i=1,2$, let $\mathcal{R}_N(Q_1,Q_2) = \{(R,S) : RN_1(s) = N_2(s)S\}$ and

$$\delta_N : \mathcal{R}(Q_1,Q_2) \longrightarrow \mathcal{R}_N(Q_1,Q_2)$$

defined by $\delta_N(U,V) = (X_2UX_1^{-1}, Y_2VY_1^{-1})$. Then, $\beta_{1N} = \delta_N\circ\beta_1 : \mathcal{F}_{deg.spec}(Q_1,Q_2) \longrightarrow \mathcal{R}_N(Q_1,Q_2)$ is a homeomorphism.

If $\det M_i = 0$, $i=1,2$, let $\widehat{\mathcal{R}}_N(Q_1,Q_2) = \{(R,S) : RN_1(s) = N_2(s)S,\ (X_2^{-1}RX_1)_{11}\neq 0,\ (Y_2^{-1}SY_1)_{11}\neq 0\}$, where $(Z)_{11}$ stands for the $n\times n$ upper left block of matrix Z. Let also

$$\widehat{\delta}_N : \widehat{\mathcal{R}}(Q_1,Q_2) \longrightarrow \widehat{\mathcal{R}}_N(Q_1,Q_2)$$

be defined by $\widehat{\delta}_N(U,V) = (X_2UX_1^{-1}, Y_2VY_1^{-1})$. Then, $\beta_{2N} = \widehat{\delta}_N\circ\beta_2 : \widehat{\mathcal{F}}_{deg.spec}(Q_1,Q_2) \longrightarrow \widehat{\mathcal{R}}_N(Q_1,Q_2)$ is a homeomorphism.

EXAMPLE 5.8. *Let $Q_i(s) = M_is^2+D_is+K_i \in \mathbb{F}[s]^{n\times n}$ with $\det K_i \neq 0$, $i=1,2$, and assume that $Q_1(s)$ and $Q_2(s)$ are spectrally equivalent. Let $L_i(s)$ be the Rblock-symmetric linearization of $Q_i(s)$ and let $N_i(s) =$*

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$\left[\begin{smallmatrix} M_i & 0 \\ 0 & I_n \end{smallmatrix}\right]s + \left[\begin{smallmatrix} D_i & K_i \\ -I_n & 0 \end{smallmatrix}\right]$ *be the first block-companion linearization of* $Q_i(s)$, $i = 1, 2$. *Take* $X_i = \left[\begin{smallmatrix} I_n & 0 \\ 0 & -K_i^{-1} \end{smallmatrix}\right]$ *and* $Y_i = I_{2n}$ *so that* $X_i L_i(s) = N_i(s) Y_i, i = 1, 2$. *Let*

$$\widehat{\mathcal{R}}_N(Q_1, Q_2) = \{(R, S) \text{ first block-companion SPT from } Q_1(s) \text{ to } Q_2(s) \text{ such that } (R)_{11} \neq 0, (S)_{11} \neq 0\}.$$

Then, $\widehat{\delta}_N : \widehat{\mathcal{F}}_{deg.spec}(Q_1, Q_2) \longrightarrow \widehat{\mathcal{R}}_N(Q_1, Q_2)$ *defined by* $\widehat{\delta}_N(F_{11}, F_{10}, F_{21}, F_{20}) = (X_2 U X_1^{-1}, V)$ *with* V *as in* (5.24) *and* $U = \widehat{A}_2 V \widehat{A}_1^{-1}$, *where* $\widehat{A}_1, \widehat{A}_2$ *are the matrices of* (5.23), *is a homeomorphism.*

If $F_1(s)$ and $F_2(s)$ are spectral filters from $Q_1(s)$ to $Q_2(s)$, they are, in particular, coprime filters (see [4, Section 5]). Hence, (see item (iv) of Theorem 2.1), there exist matrix polynomials $X_1(s), X_2(s), Y_1(s), Y_2(s) \in \mathbb{F}[s]^{n\times n}$ such that $X_1(s)Q_1(s) + Y_1(s)F_1(s) = I_n$ and $Q_2(s)X_2(s) + F_2(s)Y_2(s) = I_n$. We proceed now, as in the previous subsection, to find a quadruple of such matrices satisfying the degree restrictions $\deg X_i(s) \leq 0$ and $\deg Y_i(s) \leq 1$, $i = 1, 2$. The technique is the same as in Theorem 4.6 but the starting basic unimodular transformations are different: Let $Q(s) = Ms^2 + Ds + K \in \mathbb{F}[s]^{n\times n}$ with $\det K \neq 0$ and let $B_1, B_2 \in \mathbb{F}^{n\times n}$ be arbitrary matrices. Then,

(5.25)
$$\begin{bmatrix} I_n & 0 \\ sI_n & I_n \end{bmatrix} \begin{bmatrix} Ms + D & K & B_1 \\ K & -Ks & B_2 \end{bmatrix} \begin{bmatrix} 0 & I_n & 0 \\ K^{-1} & -K^{-1}(Ms + D) & -K^{-1}B_1 \\ 0 & 0 & I_n \end{bmatrix} = \begin{bmatrix} I_n & 0 & 0 \\ 0 & Q(s) & B_1 s + B_2 \end{bmatrix};$$

and also,

$$\begin{bmatrix} I_n & -(Ms + D)K^{-1} \\ 0 & K^{-1} \end{bmatrix} \begin{bmatrix} Ms + D & K & I_n \\ K & -Ks & 0 \end{bmatrix} \begin{bmatrix} sI_n & I_n & 0 \\ I_n & 0 & 0 \\ 0 & 0 & I_n \end{bmatrix} = \begin{bmatrix} Q(s) & 0 & I_n \\ 0 & I_n & 0 \end{bmatrix}. \tag{5.26}$$

THEOREM 5.9. *Let* $Q_i(s) = M_i s^2 + D_i s + K_i \in \mathbb{F}[s]^{n\times n}$ *with* $\det K_i \neq 0$, $i = 1, 2$, *and assume that* $Q_1(s)$ *and* $Q_2(s)$ *are spectrally equivalent. Let* $F_1(s) = F_{11}s + F_{10}$ *and* $F_2(s) = F_{21}s + F_{20}$ *be spectral filters from* $Q_1(s)$ *to* $Q_2(s)$ *with* $F_{11} \neq 0$ *and* $F_{21} \neq 0$ *if* $\det M_i = 0$. *Let* $(U, V) = \beta_1(F_{11}, F_{10}, F_{21}, F_{20})$ *if* $\det M_i \neq 0$ *and* $(U, V) = \beta_2(F_{11}, F_{10}, F_{21}, F_{20})$ *if* $\det M_i = 0$, $i = 1, 2$, *where* β_j *is the mapping of Corollary 5.5,* $j = 1, 2$. *Assume that* U *and* V *are partitioned as in* (4.8) *and let* $\widehat{U} = U^{-1}$ *and* $\widehat{V} = V^{-1}$ *be partitioned as in* (4.13). *Then*

(5.27)
$$X_1(s) = \widehat{V}_4 K_2^{-1} U_1,\ Y_1(s) = \widehat{V}_3 - \widehat{V}_4 K_2^{-1}(M_2 s + D_2),\ X_2(s) = V_1 K_1^{-1} \widehat{U}_4,\ Y_2(s) = \widehat{U}_2 - (M_1 s + D_1)K_1^{-1}\widehat{U}_4$$

are matrices satisfying $X_1(s)Q_1(s) + Y_1(s)F_1(s) = I_n$ *and* $Q_2(s)X_2(s) + F_2(s)Y_2(s) = I_n$.

Proof. By Proposition 5.2, $F_{21} = U_1$ and $F_{20} = U_3$. Since (U, V) is an Rblock-symmetric SPT from $Q_1(s)$ to $Q_2(s)$, it follows from (5.22) that

$$\widehat{U} \begin{bmatrix} M_2 s + D_2 & K_2 \\ K_2 & -K_2 s \end{bmatrix} = \begin{bmatrix} M_1 s + D_1 & K_1 \\ K_1 & -K_1 s \end{bmatrix} \widehat{V}. \tag{5.28}$$

This means that $(\widehat{U}, \widehat{V})$ is an Rblock-symmetric SPT from $Q_2(s)$ to $Q_1(s)$. The proof now is the same as that of Theorem 4.6 but using the basic unimodular transformations of (5.25) and (5.26). We provide the details for readers' convenience. From (5.25) and (5.26)

$$\begin{bmatrix} M_2 s + D_2 & K_2 & U_1 \\ K_2 & -K_2 s & U_3 \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ -sI_n & I_n \end{bmatrix} \begin{bmatrix} I_n & 0 & 0 \\ 0 & Q_2(s) & U_1 s + U_3 \end{bmatrix} \begin{bmatrix} M_2 s + D_2 & K_2 & U_1 \\ I_n & 0 & 0 \\ 0 & 0 & I_n \end{bmatrix}, \tag{5.29}$$

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and

$$\begin{bmatrix} M_1 s + D_1 & K_1 & I_n \\ K_1 & -K_1 s & 0 \end{bmatrix} = \begin{bmatrix} I_n & M_1 s + D_1 \\ 0 & K_1 \end{bmatrix} \begin{bmatrix} Q_1(s) & 0 & I_n \\ 0 & I_n & 0 \end{bmatrix} \begin{bmatrix} 0 & I_n & 0 \\ I_n & -sI_n & 0 \\ 0 & 0 & I_n \end{bmatrix}. \tag{5.30}$$

Substituting these values in (5.28)

$$\begin{aligned} &\widehat{U} \begin{bmatrix} I_n & 0 \\ -sI_n & I_n \end{bmatrix} \begin{bmatrix} I_n & 0 & 0 \\ 0 & Q_2(s) & U_1 s + U_3 \end{bmatrix} \begin{bmatrix} M_2 s + D_2 & K_2 & U_1 \\ I_n & 0 & 0 \\ 0 & 0 & I_n \end{bmatrix} \\ &= \begin{bmatrix} I_n & M_1 s + D_1 \\ 0 & K_1 \end{bmatrix} \begin{bmatrix} Q_1(s) & 0 & I_n \\ 0 & I_n & 0 \end{bmatrix} \begin{bmatrix} 0 & I_n & 0 \\ I_n & -sI_n & 0 \\ 0 & 0 & I_n \end{bmatrix} \begin{bmatrix} \widehat{V} & 0 \\ 0 & I_n \end{bmatrix}. \end{aligned} \tag{5.31}$$

But

$$\begin{bmatrix} Q_1(s) & 0 & I_n \\ 0 & I_n & 0 \end{bmatrix} = \begin{bmatrix} I_n & 0 & 0 \\ 0 & I_n & 0 \end{bmatrix} \begin{bmatrix} Q_1(s) & 0 & I_n \\ 0 & I_n & 0 \\ I_n & 0 & 0 \end{bmatrix}.$$

Substituting this identity in (5.31) and after some elementary manipulations, we get

$$\begin{bmatrix} I_n & 0 \\ sI_n & I_n \end{bmatrix} U \begin{bmatrix} I_n & M_1 s + D_1 \\ 0 & K_1 \end{bmatrix} \begin{bmatrix} I_n & 0 & 0 \\ 0 & I_n & 0 \end{bmatrix} = \begin{bmatrix} I_n & 0 & 0 \\ 0 & Q_2(s) & U_1 s + U_3 \end{bmatrix} W_1(s), \tag{5.32}$$

where

$$\begin{aligned} W_1(s) &= \begin{bmatrix} M_2 s + D_2 & K_2 & U_1 \\ I_n & 0 & 0 \\ 0 & 0 & I_n \end{bmatrix} \begin{bmatrix} V & 0 \\ 0 & I_n \end{bmatrix} \begin{bmatrix} sI_n & I_n & 0 \\ I_n & 0 & 0 \\ 0 & 0 & I_n \end{bmatrix} \begin{bmatrix} 0 & 0 & I_n \\ 0 & I_n & 0 \\ I_n & 0 & -Q_1(s) \end{bmatrix} \\ &= \begin{bmatrix} U_1 & (M_2 s + D_2)V_1 + K_2 V_3 & (M_2 s + D_2)(V_1 s + V_2) + K_2(V_3 s + V_4) - U_1 Q_1(s) \\ 0 & V_1 & V_1 s + V_2 \\ I_n & 0 & -Q_1(s) \end{bmatrix}. \end{aligned}$$

We use now (4.18) in (5.32) to obtain:

$$\begin{bmatrix} I_n & 0 & 0 \\ 0 & I_n & 0 \end{bmatrix} = \begin{bmatrix} I_n & 0 & 0 \\ 0 & Q_2(s) & U_1 s + U_3 \end{bmatrix} W_1(s) W_2(s),$$

where

$$\begin{aligned} W_2(s) &= \begin{bmatrix} I_n & -(M_1 s + D_1)K_1^{-1} & 0 \\ 0 & K_1^{-1} & 0 \\ 0 & 0 & I_n \end{bmatrix} \begin{bmatrix} \widehat{U} & 0 \\ 0 & I_n \end{bmatrix} \begin{bmatrix} I_n & 0 & 0 \\ -sI_n & I_n & 0 \\ 0 & 0 & I_n \end{bmatrix} \\ &= \begin{bmatrix} W_{2_{11}}(s) & \widehat{U}_2 - (M_1 s + D_1)K_1^{-1}\widehat{U}_4 & 0 \\ K_1^{-1}(\widehat{U}_3 - \widehat{U}_4 s) & K_1^{-1}\widehat{U}_4 & 0 \\ 0 & 0 & I_n \end{bmatrix}, \end{aligned}$$

and $W_{2_{11}}(s) = \widehat{U}_1 - (M_1 s + D_1)K_1^{-1}\widehat{U}_3 - \widehat{U}_2 s + (M_1 s + D_1)K_1^{-1}\widehat{U}_4 s$.

To find matrices $X_2(s)$ and $Y_2(s)$ such that $Q_2(s)X_2(s) + (U_1 s + U_3)Y_2(s) = I_n$, we only have to compute the entries $(2,2)$ and $(3,2)$ of $W(s) = W_1(s)W_2(s)$. A simple computation shows that these are

$$W_{22}(s) = V_1 K_1^{-1}\widehat{U}_4, \quad W_{32}(s) = \widehat{U}_2 - (M_1 s + D_1)K_1^{-1}\widehat{U}_4.$$

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Henceforth, putting $X_2(s) = V_1K_1^{-1}\widehat{U}_4$ and $Y_2(s) = \widehat{U}_2 - (M_1s + D_1)K_1^{-1}\widehat{U}_4$, we have $Q_2(s)X_2(s) + F_2(s)Y_2(s) = I_n$ as desired.

So we aim to find matrices $X_1(s), Y_1(s) \in \mathbb{F}[s]^{n\times n}$ such that $X_1(s)Q_1(s)+Y_1(s)F_1(s) = I_n$ by transposition as in the last part of the proof of Theorem 4.6. By using that technique and defining $\widehat{X}_1(s) = U_1^TK_2^{-T}\widehat{V}_4^T$ and $\widehat{Y}_1(s) = -M_2^TK_2^{-T}\widehat{V}_4^Ts + \widehat{V}_3^T - D_2^TK_2^{-T}\widehat{V}_4^T$, we obtain $Q_1(s)^T\widehat{X}_1(s) + F_1(s)^T\widehat{Y}_1(s) = I_n$. Therefore, if we put

$$X_1(s) = \widehat{V}_4K_2^{-1}U_1, \quad Y_1(s) = \widehat{V}_3 - \widehat{V}_4K_2^{-1}(M_2s + D_2),$$

then $X_1(s)Q_1(s) + Y_1(s)F_1(s) = I_n$ and the proof is complete. ∎

Now we present a similar result to Corollary 4.8 for spectral filters from $Q_2(s)$ to $Q_1(s)$. For that, we define

$$\begin{aligned} &\widehat{\beta}_1 : \mathcal{F}_{deg.spec}(Q_2, Q_1) \longrightarrow \mathcal{R}(Q_2, Q_1), \quad \text{if } \det M_i \neq 0, \\ &\widehat{\beta}_2 : \widehat{\mathcal{F}}_{deg.spec}(Q_2, Q_1) \longrightarrow \widehat{\mathcal{R}}(Q_2, Q_1), \quad \text{if} \det M_i = 0, \end{aligned} \tag{5.33}$$

$i = 1, 2$, as $\widehat{\beta}_j(G_{21}, G_{20}, G_{11}, G_{10}) = (\widehat{U}, \widehat{V})$ with $\widehat{V} = \begin{bmatrix} G_{21} & G_{20} \\ -G_{20}K_2^{-1}M_2 & G_{21}-G_{20}K_2^{-1}D_2 \end{bmatrix}$ and $\widehat{U} = \widehat{A}_1\widehat{V}\widehat{A}_2^{-1}$, where $\widehat{A}_1$,$\widehat{A}_2$ are as in (5.23), $j = 1, 2$. As in Corollary 5.5, $\widehat{\beta}_j$ is a bijection, $j = 1, 2$. In fact, it is a homeomorphism.

As for coprime filters, there is a close relationship between the spectral filters from $Q_1(s)$ to $Q_2(s)$ and the spectral filters from $Q_2(s)$ to $Q_1(s)$. It is a consequence of the fact that (U, V) is an Rblock-symmetric SPT from $Q_1(s)$ to $Q_2(s)$ if and only if (U^{-1}, V^{-1}) is an Rblock-symmetric SPT from $Q_2(s)$ to $Q_1(s)$. If $F_1(s) = F_{11}s+F_{10}$ and $F_2(s) = F_{21}s+F_{20}$ are spectral filters from $Q_1(s)$ to $Q_2(s)$ and $(U, V) = \beta_j(F_{11}, F_{10}, F_{21}, F_{20})$, then (U^{-1}, V^{-1}) is an Rblock-symmetric SPT from $Q_2(s)$ to $Q_1(s)$ and $G_2(s) = G_{21}s + G_{20}$ and $G_1(s) = G_{11}s+G_{10}$ are spectral filters from $Q_2(s)$ to $Q_1(s)$ where $(G_{21}, G_{20}, G_{11}, G_{10}) = \widehat{\beta}_j^{-1}(U^{-1}, V^{-1})$, $j = 1, 2$. In other words, if $F_1(s) = F_{11}s + F_{10}$ and $F_2(s) = F_{21}s + F_{20}$ are spectral filters from $Q_1(s)$ to $Q_2(s)$, V is the matrix in (5.24), $U = \widehat{A}_2V\widehat{A}_1^{-1}$ where $\widehat{A}_1, \widehat{A}_2$ are as in (5.23), and $\widehat{U} = U^{-1}$ and $\widehat{V} = V^{-1}$ are partitioned as in (4.13), then $G_2(s) = \widehat{V}_1s + \widehat{V}_2$ and $G_1(s) = \widehat{U}_1s + \widehat{U}_3$ are spectral filters from $Q_2(s)$ to $Q_1(s)$.

The following example shows how to obtain spectral filters from $Q_2(s)$ to $Q_1(s)$ from spectral filters from $Q_1(s)$ to $Q_2(s)$.

Example 5.10. *Consider Example 5.4 and choose $x = y = z = 1$ and $w = 0$. Spectral filters from $Q_1(s)$ to $Q_2(s)$ are*

$$F_1(s) = F_{11}s+F_{10} = \begin{bmatrix} 0 & -1/3 \\ -1 & -1/3 \end{bmatrix} s + \begin{bmatrix} 1/3 & 2/3 \\ 1/3 & -4/3 \end{bmatrix}, \quad F_2(s) = F_{21}s+F_{20} = \begin{bmatrix} 1/3 & -1/3 \\ -1 & 0 \end{bmatrix} s + \begin{bmatrix} 2/3 & 1/3 \\ 4/3 & -1/3 \end{bmatrix},$$

and the corresponding Rblock-symmetric SPT from $Q_1(s)$ to $Q_2(s)$ is

$$U = \begin{bmatrix} 1/3 & -1/3 & -2/3 & -1/3 \\ -1 & 0 & 2/3 & -1 \\ 2/3 & 1/3 & -1/3 & -2/3 \\ 4/3 & -1/3 & -1/3 & -1/3 \end{bmatrix}, \quad V = \begin{bmatrix} 0 & -1/3 & 1/3 & 2/3 \\ -1 & -1/3 & 1/3 & -4/3 \\ 0 & -1/3 & -2/3 & -1/3 \\ 0 & -1/3 & 1/3 & -1/3 \end{bmatrix}.$$

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Then, (U^{-1},V^{-1}) *is an Rblock-symmetric SPT from* $Q_2(s)$ *to* $Q_1(s)$*, with*

$$U^{-1}=\begin{bmatrix}-4/7 & -1/7 & 1/7 & 5/7\\ -5/7 & -3/7 & 10/7 & -6/7\\ -9/7 & 3/7 & -3/7 & 6/7\\ -2/7 & -4/7 & -3/7 & -1/7\end{bmatrix},\quad V^{-1}=\begin{bmatrix}-1 & -1 & 0 & 2\\ -1 & 0 & -1 & -1\\ 0 & 0 & -1 & 1\\ 1 & 0 & 0 & -1\end{bmatrix}.$$

Therefore, $\widehat{\beta}_2^{-1}(U^{-1},V^{-1})=(G_{21},G_{20},G_{11},G_{10})$*, and*

$$G_2(s)=G_{21}s+G_{20}=\begin{bmatrix}-1 & -1\\ -1 & 0\end{bmatrix}s+\begin{bmatrix}0 & 2\\ -1 & -1\end{bmatrix},\quad G_1(s)=G_{11}s+G_{10}=\begin{bmatrix}-4/7 & -1/7\\ -5/7 & -3/7\end{bmatrix}s+\begin{bmatrix}-9/7 & 3/7\\ -2/7 & -4/7\end{bmatrix},$$

are spectral filters from $Q_2(s)$ *to* $Q_1(s)$*.*

COROLLARY 5.11. *Let* $Q_i(s)=M_is^2+D_is+K_i\in\mathbb{F}[s]^{n\times n}$ *with* $\det K_i\neq 0$*,* $i=1,2$*, and assume that* $Q_1(s)$ *and* $Q_2(s)$ *are spectrally equivalent. Let* $G_2(s)=G_{21}s+G_{20}$ *and* $G_1(s)=G_{11}s+G_{10}$*, with* $G_{21}\neq 0$ *and* $G_{11}\neq 0$ *if* $\det M_i=0$*, be spectral filters from* $Q_2(s)$ *to* $Q_1(s)$*, and let* $(\widehat{U},\widehat{V})=\widehat{\beta}_1(G_{21},G_{20},G_{11},G_{10})$ *if* $\det M_i\neq 0$ *and* $(\widehat{U},\widehat{V})=\widehat{\beta}_2(G_{21},G_{20},G_{11},G_{10})$ *if* $\det M_i=0$*,* $i=1,2$*, where* $\widehat{\beta}_j$ *is the mapping of* (5.33)*,* $j=1,2$*. Let* $U=\widehat{U}^{-1}$ *and* $V=\widehat{V}^{-1}$ *and assume that* U *and* V *are partitioned as in* (4.8) *and* $\widehat{U}$ *and* $\widehat{V}$ *as in* (4.13)*. Then,*

(5.34)

$$X_1(s)=\widehat{V}_1K_2^{-1}U_4,\ Y_1(s)=U_2-(M_2s+D_2)K_2^{-1}U_4,\ X_2(s)=V_4K_1^{-1}\widehat{U}_1,\ Y_2(s)=V_3-V_4K_1^{-1}(M_1s+D_1)$$

are matrices satisfying $Q_1(s)X_1(s)+G_1(s)Y_1(s)=I_n$ *and* $X_2(s)Q_2(s)+Y_2(s)G_2(s)=I_n$*.*

5.1. Removing the restriction $\det K\neq 0$**.** Let $Q(s)=Ms^2+Ds+K\in\mathbb{F}[s]^{n\times n}$ be nonsingular and assume that there is an element $a\in\mathbb{F}$, such that $a\notin\Lambda(Q)$. Then, the pencil $L(s)$ of (3.2) with $a_1=1$ and $a_2=-a$ is

$$L_a(s)=\begin{bmatrix}M & -aM\\ -aM & -aD-K\end{bmatrix}s+\begin{bmatrix}D+aM & K\\ K & -aK\end{bmatrix}. \tag{5.35}$$

This pencil is a strong linearization of $Q(s)$ whether or not $\det M=0$ or $\det K=0$. It turns out that this pencil is strictly equivalent to the pencil of (3.5) when it is written in the binomial basis generated by $(s-a)$ for the nonsingular quadratic matrix polynomial $Q(s)$ written in the same binomial basis. In other words, write

$$Q(s)=\widetilde{M}(s-a)^2+\widetilde{D}(s-a)+\widetilde{K}$$

and

$$\widetilde{L}_a(s)=\begin{bmatrix}\widetilde{M} & 0\\ 0 & -\widetilde{K}\end{bmatrix}(s-a)+\begin{bmatrix}\widetilde{D} & \widetilde{K}\\ \widetilde{K} & 0\end{bmatrix}=\begin{bmatrix}\widetilde{M} & 0\\ 0 & -\widetilde{K}\end{bmatrix}s+\begin{bmatrix}\widetilde{D}-a\widetilde{M} & \widetilde{K}\\ \widetilde{K} & a\widetilde{K}\end{bmatrix}, \tag{5.36}$$

and note that $\widetilde{M}=M$, $\widetilde{D}=\dot{Q}(a)=D+2aM$, where $\dot{Q}(a)$ stands for the first derivative of $Q(s)$ evaluated at $s=a$, and $\widetilde{K}=Q(a)=a^2M+aD+K$. Put $P_1=\left[\begin{smallmatrix}I_n & 0\\ aI_n & I_n\end{smallmatrix}\right]$ and $P_2=\left[\begin{smallmatrix}I_n & aI_n\\ 0 & I_n\end{smallmatrix}\right]$. Then,

$$P_1L_a(s)P_2=\begin{bmatrix}M & 0\\ 0 & -(a^2M+aD+K)\end{bmatrix}s+\begin{bmatrix}D+aM & a^2M+aD+K\\ a^2M+aD+K & a(a^2M+aD+K)\end{bmatrix}=\widetilde{L}_a(s).$$

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It follows that $\widetilde{L}_a(s)$ and $L_a(s)$ are strictly equivalent and so $\widetilde{L}_a(s)$ is a strong linearization of $Q(s)$.

Now $\det \widetilde{K} \neq 0$ because $a \notin \Lambda(Q)$, and $\widetilde{L}_a(s)$ looks like an "Rblock-symmetric linearization" of $Q(s)$ in the binomial basis generated by $(s-a)$. We will call it *$a-$Rblock-symmetric linearization* of $Q(s)$ and we can apply to $\widetilde{L}_a(s)$ and $Q(s)$ the results of Section 5. Note that a is just a shift parameter that can be chosen conveniently as long as it satisfies $a \notin \Lambda(Q)$.

First, we introduce the notion of *$a-$Rblock-symmetric SPT*: Let $Q_i(s) = M_i s^2 + D_i s + K_i \in \mathbb{F}[s]^{n\times n}$, $i = 1, 2$, be nonsingular and assume that $Q_1(s)$ and $Q_2(s)$ are spectrally equivalent. Assume also that $a \notin \Lambda(Q_1) = \Lambda(Q_2)$. For $i = 1, 2$, write $Q_i(s)$ in the binomial basis generated by $(s-a)$:

$$Q_i(s) = \widetilde{M}_i(s-a)^2 + \widetilde{D}_i(s-a) + \widetilde{K}_i \tag{5.37}$$

and define

$$\widetilde{L}_{Q_i}(s) = \begin{bmatrix} \widetilde{M}_i & 0 \\ 0 & -\widetilde{K}_i \end{bmatrix}(s-a) + \begin{bmatrix} \widetilde{D}_i & \widetilde{K}_i \\ \widetilde{K}_i & 0 \end{bmatrix}. \tag{5.38}$$

We will say that (U, V), $U, V \in \mathbb{F}^{2n\times 2n}$ and invertible, is an $a-$Rblock-symmetric SPT from $Q_1(s)$ to $Q_2(s)$ if $U\widetilde{L}_{Q_1}(s) = \widetilde{L}_{Q_2}(s)V$, that is,

$$U \begin{bmatrix} \widetilde{M}_1 & 0 \\ 0 & -\widetilde{K}_1 \end{bmatrix} = \begin{bmatrix} \widetilde{M}_2 & 0 \\ 0 & -\widetilde{K}_2 \end{bmatrix} V \quad \text{and} \quad U \begin{bmatrix} \widetilde{D}_1 & \widetilde{K}_1 \\ \widetilde{K}_1 & 0 \end{bmatrix} = \begin{bmatrix} \widetilde{D}_2 & \widetilde{K}_2 \\ \widetilde{K}_2 & 0 \end{bmatrix} V. \tag{5.39}$$

Although the matrices $\widetilde{D}_i$, $\widetilde{K}_i$, $\widetilde{L}_{Q_i}(s)$, U, and V depend on a, we will not write this dependence explicitly for notational simplicity.

THEOREM 5.12. *Let $Q_i(s) = M_i s^2 + D_i s + K_i \in \mathbb{F}[s]^{n\times n}$ be nonsingular, $i = 1, 2$, and assume that $a \notin \Lambda(Q_1) = \Lambda(Q_2)$ and that $Q_1(s)$ and $Q_2(s)$ are spectrally equivalent. Let $\widetilde{K}_i = Q_i(a)$, $\widetilde{D}_i = \dot{Q}_i(a)$ and $\widetilde{M}_i = M_i$, $i = 1, 2$. Then,*

(i) Let (U, V) be an $a-$Rblock-symmetric SPT from $Q_1(s)$ to $Q_2(s)$ with U and V partitioned as in (4.8). Then $F_1(s) = V_1 s + (V_2 - aV_1)$ and $F_2(s) = U_1 s + (U_3 - aU_1)$ are spectral filters from $Q_1(s)$ to $Q_2(s)$. Moreover, $U_1 \neq 0$ and $V_1 \neq 0$ if $\det M_i = 0$, $i = 1, 2$.

(ii) Let $F_{11}s + F_{10} \in \mathbb{F}[s]^{n\times n}$ and $F_{21}s + F_{20} \in \mathbb{F}[s]^{n\times n}$, with $F_{11} \neq 0$ and $F_{21} \neq 0$ if $\det M_i = 0$, $i = 1, 2$, be spectral filters from $Q_1(s)$ to $Q_2(s)$ and

$$\widetilde{A}_i = \begin{bmatrix} \widetilde{D}_i & \widetilde{K}_i \\ \widetilde{K}_i & 0 \end{bmatrix}, \quad i = 1, 2. \tag{5.40}$$

If

$$V = \begin{bmatrix} F_{11} & F_{11}a + F_{10} \\ -(F_{11}a + F_{10})\widetilde{K}_1^{-1}\widetilde{M}_1 & F_{11} - (F_{11}a + F_{10})\widetilde{K}_1^{-1}\widetilde{D}_1 \end{bmatrix}, \tag{5.41}$$

and $U = \widetilde{A}_2 V \widetilde{A}_1^{-1}$ then (U, V) is an $a-$Rblock-symmetric SPT from $Q_1(s)$ to $Q_2(s)$. Moreover, $F_{21} = U_1$ and $F_{20} = U_3 - aU_1$.

(iii) Let

$$\mathcal{R}_a(Q_1, Q_2) = \{(U, V)\ a-\text{Rblock-symmetric SPT from } Q_1(s) \text{ to } Q_2(s)\},$$

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$$\widehat{\mathcal{R}}_a(Q_1,Q_2)=\{(U,V)\ a-Rblock\text{-}symmetric\ SPT\ from\ Q_1(s)\ to\ Q_2(s)\\ such\ that\ U_1\neq 0, V_1\neq 0\ when\ U\ and\ V\ are\ partitioned\ as\ in\ (4.8)\}.$$

Then,

$$\gamma_{a,1}:\mathcal{F}_{deg.spec}(Q_1,Q_2)\longrightarrow\mathcal{R}_a(Q_1,Q_2),$$

if $\det M_i\neq 0$, $i=1,2$, *and*

$$\gamma_{a,2}:\widehat{\mathcal{F}}_{deg.spec}(Q_1,Q_2)\longrightarrow\widehat{\mathcal{R}}_a(Q_1,Q_2),$$

if $\det M_i=0$, $i=1,2$, *defined by* $\gamma_{a,j}(F_{11},F_{10},F_{21},F_{20})=(U,V)$, *with* U *and* V *as in the previous item, is a bijection,* $j=1,2$.

(iv) Let $F_1(s)=F_{11}s+F_{10}\in\mathbb{F}[s]^{n\times n}$ *and* $F_2(s)=F_{21}s+F_{20}\in\mathbb{F}[s]^{n\times n}$, *with* $F_{11}\neq 0$ *and* $F_{21}\neq 0$ *if* $\det M_i=0$, *be spectral filters from* $Q_1(s)$ *to* $Q_2(s)$, *and let* $(U,V)=\gamma_{a,1}(F_{11},F_{10},F_{21},F_{20})$ *if* $\det M_i\neq 0$ *and* $(U,V)=\gamma_{a,2}(F_{11},F_{10},F_{21},F_{20})$ *if* $\det M_i=0$, $i=1,2$, *where* $\gamma_{a,j}$ *is the mapping of the previous item,* $j=1,2$. *Assume that* U, V *are partitioned as in* (4.8) *and* $\widehat{U}=U^{-1}$, $\widehat{V}=V^{-1}$ *as in* (4.13). *Then,*

$$\begin{array}{ll}X_1(s)=\widehat{V}_4\widetilde{K}_2^{-1}U_1, & Y_1(s)=\widehat{V}_3-\widehat{V}_4\widetilde{K}_2^{-1}(\widetilde{M}_2(s-a)+\widetilde{D}_2),\\ X_2(s)=V_1\widetilde{K}_1^{-1}\widehat{U}_4, & Y_2(s)=\widehat{U}_2-(\widetilde{M}_1(s-a)+\widetilde{D}_1)\widetilde{K}_1^{-1}\widehat{U}_4,\end{array}\tag{5.42}$$

are matrices satisfying $X_1(s)Q_1(s)+Y_1(s)F_1(s)=I_n$ *and* $Q_2(s)X_2(s)+F_2(s)Y_2(s)=I_n$.

(v) The maps

$$\begin{array}{l}\widehat{\gamma}_{a,1}:\mathcal{F}_{deg.spec}(Q_2,Q_1)\longrightarrow\mathcal{R}_a(Q_2,Q_1),\quad if\ \det M_i\neq 0,\\ \widehat{\gamma}_{a,2}:\widehat{\mathcal{F}}_{deg.spec}(Q_2,Q_1)\longrightarrow\widehat{\mathcal{R}}_a(Q_2,Q_1),\quad if\ \det M_i=0,\end{array}\tag{5.43}$$

$i=1,2$, *defined by*

$$\widehat{\gamma}_{a,j}(G_{21},G_{20},G_{11},G_{10})=(\widehat{U},\widehat{V})$$

with $\widehat{V}=\begin{bmatrix}G_{21} & G_{21}a+G_{20}\\ -(G_{21}a+G_{20})\widetilde{K}_2^{-1}\widetilde{M}_2 & G_{21}-(G_{21}a+G_{20})\widetilde{K}_2^{-1}\widetilde{D}_2\end{bmatrix}$ *and* $\widehat{U}=\widetilde{A}_2\widehat{V}\widetilde{A}_1^{-1}$, *where* $\widetilde{A}_1$, $\widetilde{A}_2$ *are as in (5.40), are bijections for* $j=1,2$. *Moreover, let* $G_{21}s+G_{20}\in\mathbb{F}[s]^{n\times n}$ *and* $G_{11}s+G_{10}\in\mathbb{F}[s]^{n\times n}$, *with* $G_{21}\neq 0$ *and* $G_{11}\neq 0$ *if* $\det M_i=0$, *be spectral filters from* $Q_2(s)$ *to* $Q_1(s)$. *Assume that* $\widehat{U}$, $\widehat{V}$ *are partitioned as in* (4.13) *and* $U=\widehat{U}^{-1}$, $V=\widehat{V}^{-1}$ *as in* (4.8). *Then,*

$$\begin{array}{ll}X_1(s)=\widehat{V}_1\widetilde{K}_2^{-1}U_4, & Y_1(s)=U_2-(\widetilde{M}_2(s-a)+\widetilde{D}_2)\widetilde{K}_2^{-1}U_4,\\ X_2(s)=V_4\widetilde{K}_1^{-1}\widehat{U}_1, & Y_2(s)=V_3-V_4\widetilde{K}_1^{-1}(\widetilde{M}_1(s-a)+\widetilde{D}_1)\end{array}\tag{5.44}$$

are matrices satisfying $Q_1(s)X_1(s)+G_1(s)Y_1(s)=I_n$ *and* $X_2(s)Q_2(s)+Y_2(s)G_2(s)=I_n$.

Proof. Note that if $\widetilde{Q}_i(s)=\widetilde{M}_is^2+\widetilde{D}_is+\widetilde{K}_i$, then $Q_i(s)=\widetilde{Q}_i(s-a)$, $i=1,2$. As $\widetilde{Q}_1(s-a)$ and $\widetilde{Q}_2(s-a)$ are spectrally equivalent, they have the same eigenvalues and their respective partial multiplicity sequences. By [1, Section 6.1], $\widetilde{Q}_1(s)$ and $\widetilde{Q}_2(s)$ have the same eigenvalues and their respective partial multiplicity sequences as well, and therefore, they are spectrally equivalent. Moreover, $\det\widetilde{K}_i\neq 0, i=1,2$, as $a\notin\Lambda(Q_1)=\Lambda(Q_2)$.

(i) (U,V) is an $a-$Rblock-symmetric SPT from $Q_1(s)$ to $Q_2(s)$ if and only if U and V satisfy (5.39), that is, if and only if (U,V) is an Rblock-symmetric SPT from $\widetilde{Q}_1(s)$ to $\widetilde{Q}_2(s)$. It follows from Proposition 5.1 that $H_1(s)=V_1s+V_2$ and $H_2(s)=U_1s+U_3$ are spectral filters from $\widetilde{Q}_1(s)$ to $\widetilde{Q}_2(s)$. Moreover, $U_1\neq 0$ and $V_1\neq 0$ if $\det M_i=0$, $i=1,2$. Since the coprimeness property keeps invariant under translations and dilations (see [2, Theorem 5.4]), $F_1(s)=H_1(s-a)$ and $F_2(s)=H_2(s-a)$ are spectral filters from $Q_1(s)=\widetilde{Q}_1(s-a)$ to $Q_2(s)=\widetilde{Q}_2(s-a)$.

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(ii) As in the previous item, if $F_{11}s+F_{10}$ and $F_{21}s+F_{20}$ are spectral filters from $Q_1(s)$ to $Q_2(s)$, then $F_{11}(s+a)+F_{10}$ and $F_{21}(s+a)+F_{20}$ are spectral filters from $\widetilde{Q}_1(s)=Q_1(s+a)$ to $\widetilde{Q}_2(s)=Q_2(s+a)$. By Proposition 5.2, if V is the matrix of (5.41) and $U=\widetilde{A}_2V\widetilde{A}_1^{-1}$, then (U,V) is an Rblock-symmetric SPT from $\widetilde{Q}_1(s)$ to $\widetilde{Q}_2(s)$ and $F_{21}=U_1$ and $F_{20}=U_3-aU_1$. Hence, (U,V) is an $a-$Rblock-symmetric SPT from $Q_1(s)$ to $Q_2(s)$.

(iii) It is a consequence of the two previous items.

(iv) Again, if $F_1(s)=F_{11}s+F_{10}$ and $F_2(s)=F_{21}s+F_{20}$ are spectral filters from $Q_1(s)$ to $Q_2(s)$, then $\widetilde{F}_1(s)=F_1(s+a)$ and $\widetilde{F}_2(s)=F_2(s+a)$ are spectral filters from $\widetilde{Q}_1(s)=Q_1(s+a)$ to $\widetilde{Q}_2(s)=Q_2(s+a)$. Also, if $(U,V)=\gamma_{a,1}(F_{11},F_{10},F_{21},F_{20})$ in the case that $\det M_i\neq 0$ or $(U,V)=\gamma_{a,2}(F_{11},F_{10},F_{21},F_{20})$ otherwise, $i=1,2$, by the beginning of the proof of item *(i)*, (U,V) is an Rblock-symmetric SPT from $\widetilde{Q}_1(s)$ to $\widetilde{Q}_2(s)$. By Theorem 5.9, if

$$\begin{aligned}\widetilde{X}_1(s)&=\widehat{V}_4\widetilde{K}_2^{-1}U_1, \quad \widetilde{Y}_1(s)=\widehat{V}_3-\widehat{V}_4\widetilde{K}_2^{-1}(\widetilde{M}_2s+\widetilde{D}_2)\\ \widetilde{X}_2(s)&=V_1\widetilde{K}_1^{-1}\widehat{U}_4, \quad \widetilde{Y}_2(s)=\widehat{U}_2-(\widetilde{M}_1s+\widetilde{D}_1)\widetilde{K}_1^{-1}\widehat{U}_4,\end{aligned}$$

then $\widetilde{X}_1(s)\widetilde{Q}_1(s)+\widetilde{Y}_1(s)\widetilde{F}_1(s)=I_n$ and $\widetilde{Q}_2(s)\widetilde{X}_2(s)+\widetilde{F}_2(s)\widetilde{Y}_2(s)=I_n$. Since $X_i(s)=\widetilde{X}_i(s-a)$ and $Y_i(s)=\widetilde{Y}_i(s-a)$, $i=1,2$, we get $X_1(s)Q_1(s)+Y_1(s)F_1(s)=\widetilde{X}_1(s-a)\widetilde{Q}_1(s-a)+\widetilde{Y}_1(s-a)\widetilde{F}_1(s-a)=I_n$ and $Q_2(s)X_2(s)+F_2(s)Y_2(s)=\widetilde{Q}_2(s-a)\widetilde{X}_2(s-a)+\widetilde{F}_2(s-a)\widetilde{Y}_2(s-a)=I_n$.

(v) The proof follows from *(iii)* and *(iv)*. ∎

REMARK 5.13. *The maps $\gamma_{a,i}$ and $\widehat{\gamma}_{a,i}$, $i=1,2$, are homeomorphisms when $\mathbb{F}=\mathbb{R}$ or $\mathbb{C}$. Moreover, when $a\notin\Lambda(Q_1)=\Lambda(Q_2)$, the set of spectral filters remains unchanged irrespective of the chosen linearization.*

6. Conclusions. Phase synchronization is a technique that enables the decoupling of quadratic systems of multivariable linear differential equations of the form (1.1). When the system is diagonalizable, then this technique can be implemented using SPTs. This implementation requires the quadratic system to be linearized, thereby increasing the size of the system by a factor of two when $\det M\neq 0$. In addition, linearizations are not unique and under unstructured perturbations they may not become the linearization of any quadratic system. These drawbacks can be avoided using spectral filters. To implement the phase synchronization technique by using filters, the relationship between filters and SPTs must be clarified. This paper analyzes this relationship, showing a bijection between spectral filters and Rblock-symmetric SPTs. While the spectral filters remain the same regardless of the chosen linearizations, the SPTs are closely related to the linearizations. It has been seen that for any pair of spectral filters, the matrices that realize the associated Bezout identity are blocks of the corresponding Rblock-symmetric SPT. It will be shown in an upcoming paper how to use this Bezout identity and the Rblock-symmetric SPT associated to the spectral filters of a given diagonalizable quadratic system to identify the initial conditions and the right-hand side function that allows the decoupling.

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