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SPIN LEONARD PAIRS AND THE ZERO DIAGONAL SPACE*

KAZUMASA NOMURA† AND PAUL TERWILLIGER‡

Abstract. Let $\mathbb{F}$ denote a field, and let V denote a vector space over $\mathbb{F}$ with finite positive dimension. A Leonard pair on V is an ordered pair of diagonalizable linear maps $A : V \to V$ and $A^* : V \to V$ that each act on an eigenbasis of the other one in an irreducible tridiagonal fashion. A Leonard pair A, A^* on V is said to have spin whenever there exist invertible linear maps $W : V \to V$ and $W^* : V \to V$ such that $WA = AW$ and $W^*A^* = A^*W^*$ and $WA^*W^{-1} = (W^*)^{-1}AW^*$. Let $\{\theta^*_i\}_{i=0}^d$ denote a standard ordering of the eigenvalues of A^*. There is a related sequence of scalars $\{a_i\}_{i=0}^d$ called intersection numbers. The Leonard pair A, A^* is called self-dual whenever $\{\theta^*_i\}_{i=0}^d$ is a standard ordering of the eigenvalues of A. We obtain the following results under the assumption that $\mathbb{F}$ is algebraically closed and $d \geq 3$. We show that a Leonard pair A, A^* on V has spin if and only if both (i) A, A^* is self-dual; (ii) there exist scalars f_0, f_1, f_2, f_3 (not all zero) such that $f_0 + f_1\theta^*_i + f_2 a_i + f_3 a_i \theta^*_i = 0$ for $0 \leq i \leq d$. We also classify the Leonard pairs A, A^* on V that satisfy (ii) without assuming (i). To do this, we bring in the following maps. For $0 \leq i \leq d$, let $E^*_i : V \to V$ denote the projection onto the θ^*_i-eigenspace of A^*. Let $\mathcal{Z}(A, A^*)$ denote the set of elements X in $\mathrm{Span}\{I, A^*, A, AA^*\}$ such that $E^*_i X E^*_i = 0$ for $0 \leq i \leq d$. We call $\mathcal{Z}(A, A^*)$ the zero diagonal space of A, A^*. As we will see, $\mathcal{Z}(A, A^*) \neq 0$ if and only if the above condition (ii) holds. As we investigate the case $\mathcal{Z}(A, A^*) \neq 0$ in detail, we break the problem into 13 cases called types; these are the q-Racah type and its relatives. For each type, we give a necessary and sufficient condition for $\mathcal{Z}(A, A^*) \neq 0$. For each type, we give an explicit basis for $\mathcal{Z}(A, A^*)$.

Key words. Leonard pair, Leonard system, Spin Leonard pair.

AMS subject classifications. 05E30, 15B10.

1. Introduction.

The notion of a Leonard pair was introduced by the second author in [10]. A Leonard pair is described as follows. Let $\mathbb{F}$ denote a field, and let V denote a vector space over $\mathbb{F}$ with finite positive dimension. A Leonard pair on V is an ordered pair of diagonalizable linear maps $A : V \to V$ and $A^* : V \to V$ that each act on an eigenbasis of the other one in an irreducible tridiagonal fashion [10, Definition 1.1]. The Leonard pairs are classified in [10, Theorem 1.9], and the solutions correspond to the orthogonal polynomials in the terminating branch of the Askey scheme [12, Section 5], [13].

In the present paper, we consider a type of Leonard pair that is said to have spin. The spin condition is described as follows. According to [2, Definition 1.2], a Leonard pair A, A^* on V has spin whenever there exist invertible linear maps $W : V \to V$ and $W^* : V \to V$ such that $WA = AW$ and $W^*A^* = A^*W^*$ and $WA^*W^{-1} = (W^*)^{-1}AW^*$. The spin Leonard pairs are classified in [2, Theorem 1.13]. For more information about spin Leonard pairs, see [1–3, 7, 9].

As we describe the features of a spin Leonard pair, we will use the following parameters. Let A, A^* denote a Leonard pair on V. Let $\{\theta^*_i\}_{i=0}^d$ denote a standard ordering of the eigenvalues of A^* (see the paragraph above Definition 4). Associated with $\{\theta^*_i\}_{i=0}^d$ is a sequence of scalars $\{a_i\}_{i=0}^d$ called intersection numbers (see Definition 15). The Leonard pair A, A^* is said to be self-dual whenever $\{\theta^*_i\}_{i=0}^d$ is a standard ordering of the eigenvalues of A. According to [2, Theorem 1.9] and [3, Lemma 9.2], if A, A^* has spin, then

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†Institute of Science Tokyo, Kohnodai Ichikawa, 272-0827 Japan (knomura@pop11.odn.ne.jp).

‡Department of Mathematics, University of Wisconsin, 480 Lincoln Drive, Madison, Wisconsin, 53706 USA (terwilli@math.wisc.edu).

(i) A, A^* is self-dual.

By [2, Lemmas 1.7–1.11, Theorem 1.13], if A, A^* has spin then

(ii) there exist scalars f_0, f_1, f_2, f_3 (not all zero) such that

$$f_0 + f_1\theta^*_i + f_2 a_i + f_3 a_i \theta^*_i = 0 \qquad (0 \leq i \leq d).$$

In the present paper, we obtain the following results under the assumption that $\mathbb{F}$ is algebraically closed and $d \geq 3$. We show that a Leonard pair A, A^* on V has spin if and only if both (i), (ii) hold. Also, we classify the Leonard pairs A, A^* on V that satisfy (ii), without assuming (i). To this end, it is convenient to bring in the following maps. For $0 \leq i \leq d$ let $E^*_i : V \to V$ denote the projection onto the θ^*_i-eigenspace of A^*. An element $X \in \mathrm{Span}\{I, A, A^*, AA^*, A^*A\}$ is said to have zero diagonal whenever $E^*_i X E^*_i = 0$ for $0 \leq i \leq d$. As we will see in Lemma 26, the element $AA^* - A^*A$ has zero diagonal. Let $\mathcal{Z}(A, A^*)$ denote the set of elements in $\mathrm{Span}\{I, A^*, A, AA^*\}$ that have zero diagonal. Note that $\mathcal{Z}(A, A^*)$ is a subspace of $\mathrm{Span}\{I, A^*, A, AA^*\}$. We call $\mathcal{Z}(A, A^*)$ the zero diagonal space of A, A^*. We show that the dimension of $\mathcal{Z}(A, A^*)$ is at most 2. We show that $\mathcal{Z}(A, A^*) \neq 0$ if and only if condition (ii) holds above if and only if

$$(a_i - a_0)(\theta^*_i - \theta^*_d)(a_j - a_d)(\theta^*_j - \theta^*_0) = (a_i - a_d)(\theta^*_i - \theta^*_0)(a_j - a_0)(\theta^*_j - \theta^*_d),$$

for $0 \leq i, j \leq d$.

As we describe $\mathcal{Z}(A, A^*)$ in depth, we break the problem into 13 cases called types: q-Racah, q-Hahn, dual q-Hahn, quantum q-Krawtchouk, q-Krawtchouk, affine q-Krawtchouk, dual q-Krawtchouk, Racah, Hahn, dual Hahn, Krawtchouk, Bannai/Ito, and Orphan. For each type, A, A^* is described by some parameters such as s, s^*, r_1, r_2. For each type, we give a necessary and sufficient condition on these parameters for $\mathcal{Z}(A, A^*) \neq 0$. For each type, we give an explicit basis for $\mathcal{Z}(A, A^*)$. The main results of this paper are Theorems 66–68 and Theorem 81.

This paper is organized as follows. Section 2 contains some preliminaries. In Section 3, we recall some basic results about a Leonard pair A, A^*. In Sections 4 and 5, we introduce the zero diagonal space $\mathcal{Z}(A, A^*)$ and discuss its basic properties. In Section 6, we describe $\mathcal{Z}(A, A^*)$ using the scalars $a^-_i = (a_i - a_0)(\theta^*_i - \theta^*_d)$ and $a^+_i = (a_i - a_d)(\theta^*_i - \theta^*_0)$ for $0 \leq i \leq d$. In Section 7, we describe the 13 types. In Sections 8 and 9, for each type we obtain a necessary and sufficient condition for $\mathcal{Z}(A, A^*) \neq 0$. In Section 10, we describe the Leonard pairs A, A^* such that $\mathcal{Z}(A, A^*)$ has dimension 2. In Sections 11 and 12, we describe the Leonard pairs A, A^* such that $\mathcal{Z}(A, A^*)$ has dimension 1. In Section 13, we describe the self-dual condition. In Section 14, we show that a Leonard pair A, A^* has spin if and only if A, A^* is self-dual and $\mathcal{Z}(A, A^*) \neq 0$.

2. Preliminaries.

Throughout the paper, the following assumptions and notational conventions are in force. Let $\mathbb{F}$ denote an algebraically closed field. By a *scalar*, we mean an element of $\mathbb{F}$. Let V denote a vector space over $\mathbb{F}$ with finite positive dimension. Let $\mathrm{End}(V)$ denote the $\mathbb{F}$-algebra consisting of the $\mathbb{F}$-linear maps $V \to V$. Let $I \in \mathrm{End}(V)$ denote the identity map. An element $A \in \mathrm{End}(V)$ is said to be *diagonalizable* whenever V is spanned by the eigenspaces of A. An element $A \in \mathrm{End}(V)$ is said to be *multiplicity-free* whenever A is diagonalizable and each eigenspace of A has dimension one. Assume that A is multiplicity-free, and let $\{V_i\}_{i=0}^d$ denote an ordering of the eigenspaces of A. The sum $V = \sum_{i=0}^d V_i$ is direct. For $0 \leq i \leq d$, let θ_i denote the eigenvalue of A for V_i. The scalars $\{\theta_i\}_{i=0}^d$ are mutually distinct. For $0 \leq i \leq d$, define $E_i \in \mathrm{End}(V)$ such that $(E_i - I)V_i = 0$ and $E_i V_j = 0$ if $j \neq i$ $(0 \leq j \leq d)$. Then, (i) $AE_i = E_i A = \theta_i E_i$ $(0 \leq i \leq d)$; (ii) $E_i E_j = \delta_{i,j} E_i$ $(0 \leq i, j \leq d)$; (iii) $\sum_{i=0}^d E_i = I$; (iv) $A = \sum_{i=0}^d \theta_i E_i$. We call E_i the *primitive idempotent of* A *associated with* θ_i $(0 \le i \le d)$. By linear algebra,

$$E_i = \prod_{\substack{0 \le j \le d \\ j \ne i}} \frac{A - \theta_j I}{\theta_i - \theta_j} \qquad (0 \le i \le d). \tag{2.1}$$

Let $\mathbb{F}^{d+1}$ denote the vector space over $\mathbb{F}$ consisting of the row vectors with $d+1$ coordinates and all entries in $\mathbb{F}$.

3. Leonard pairs and Leonard systems. In this section, we recall the definition and basic properties of a Leonard pair and a Leonard system. For more information see [10], [13], [14].

Definition 1. (See [10, Definition 1.1].) *A* Leonard pair on V *is an ordered pair* A, A^* *of elements in* $\mathrm{End}(V)$ *that satisfy the following* (i), (ii)*:*

(i) *there exists a basis for* V *with respect to which the matrix representing* A *is irreducible tridiagonal and the matrix representing* A^* *is diagonal;*
(ii) *there exists a basis for* V *with respect to which the matrix representing* A^* *is irreducible tridiagonal and the matrix representing* A *is diagonal.*

Lemma 2. (See [10, Lemma 1.3].) *Let* A, A^* *denote a Leonard pair on* V*. Then, each of* A, A^* *is multiplicity-free.*

Lemma 3. (See [13, Corollary 5.5].) *Let* A, A^* *denote a Leonard pair on* V*. Then, the elements* A, A^* *together generate* $\mathrm{End}(V)$*.*

We recall the notion of isomorphism for Leonard pairs. Let A, A^* denote a Leonard pair on V. Let V' denote a vector space over $\mathbb{F}$ with finite positive dimension, and let $A', A^{*\prime}$ denote a Leonard pair on V'. By an *isomorphism of Leonard pairs from* A, A^* *to* $A', A^{*\prime}$ we mean an $\mathbb{F}$-algebra isomorphism $\sigma : \mathrm{End}(V) \to \mathrm{End}(V')$ such that $A^\sigma = A'$ and $(A^*)^\sigma = A^{*\prime}$. The Leonard pairs A, A^* and $A', A^{*\prime}$ are said to be *isomorphic* whenever there exists an isomorphism of Leonard pairs from A, A^* to $A', A^{*\prime}$.

Next, we recall the notion of a Leonard system. We will use the following notation. Let A, A^* denote a Leonard pair on V. Let $\{\theta_i\}_{i=0}^d$ denote an ordering of the eigenvalues of A. For $0 \le i \le d$, let v_i denote an eigenvector for A corresponding to θ_i. We say that the ordering $\{\theta_i\}_{i=0}^d$ is *standard* whenever $\{v_i\}_{i=0}^d$ satisfies Definition 1(ii). For $0 \le i \le d$, let E_i denote the primitive idempotent of A associated with θ_i. The ordering $\{E_i\}_{i=0}^d$ is said to be *standard* whenever the ordering $\{\theta_i\}_{i=0}^d$ is standard. If an ordering $\{E_i\}_{i=0}^d$ is standard, then the ordering $\{E_{d-i}\}_{i=0}^d$ is standard and no further ordering is standard. A standard ordering of the primitive idempotents of A^* is similarly defined.

Definition 4. (See [6, Definition 3.1].) *By a* Leonard system on V*, we mean a sequence*

$$\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d),$$

that satisfies the following (i)–(iii)*:*

(i) A, A^* *is a Leonard pair on* V*;*
(ii) $\{E_i\}_{i=0}^d$ *is a standard ordering of the primitive idempotents of* A*;*
(iii) $\{E_i^*\}_{i=0}^d$ *is a standard ordering of the primitive idempotents of* A^**.*

We say that the Leonard pair A, A^* *and the Leonard system* Φ *are* associated*.*

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For the rest of this section, let

$$\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d),$$

denote a Leonard system on V.

Lemma 5. (See [15, Lemma 3.3].) *We have*

$$E_i A^* E_j = \begin{cases} 0 & \text{if } |i-j| > 1, \\ \neq 0 & \text{if } |i-j| = 1 \end{cases} \qquad (0 \leq i,j \leq d);$$

$$E_i^* A E_j^* = \begin{cases} 0 & \text{if } |i-j| > 1, \\ \neq 0 & \text{if } |i-j| = 1 \end{cases} \qquad (0 \leq i,j \leq d).$$

As we will see, our main results are meaningful only for $d \geq 3$. For the rest of this paper, we assume that $d \geq 3$.

Next we recall the notion of isomorphism for Leonard systems. Let V' denote a vector space over $\mathbb{F}$ with dimension $d+1$. For an algebra isomorphism $\sigma : \mathrm{End}(V) \to \mathrm{End}(V')$, define

$$\Phi^\sigma = (A^\sigma; \{E_i^\sigma\}_{i=0}^d; A^{*\sigma}; \{E_i^{*\sigma}\}_{i=0}^d).$$

Then, Φ^σ is a Leonard system on V'. For a Leonard system Φ' on V', by an *isomorphism of Leonard systems from* Φ *to* Φ', we mean an algebra isomorphism $\sigma : \mathrm{End}(V) \to \mathrm{End}(V')$ such that $\Phi^\sigma = \Phi'$. The Leonard systems Φ and Φ' are said to be *isomorphic* whenever there exists an isomorphism of Leonard systems from Φ to Φ'.

For scalars $\xi, \zeta, \xi^*, \zeta^* \in \mathbb{F}$ with $\xi\xi^* \neq 0$, the sequence

$$(\xi A + \zeta I; \{E_i\}_{i=0}^d; \xi^* A^* + \zeta^* I; \{E_i^*\}_{i=0}^d), \tag{3.2}$$

is a Leonard system on V, called an *affine transformation of* Φ. Also,

$$\Phi^* = (A^*; \{E_i^*\}_{i=0}^d; A; \{E_i\}_{i=0}^d),$$
$$\Phi^\downarrow = (A; \{E_i\}_{i=0}^d; A^*; \{E_{d-i}^*\}_{i=0}^d),$$
$$\Phi^\Downarrow = (A; \{E_{d-i}\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d),$$

are Leonard systems on V. By construction, the Leonard systems associated with A, A^* are Φ, $\Phi^\downarrow$, $\Phi^\Downarrow$, $\Phi^{\downarrow\Downarrow}$.

Next, we recall the parameter array of a Leonard system.

Definition 6. (See [10, Definition 1.8].) *For $0 \leq i \leq d$, let θ_i (resp. θ_i^*) denote the eigenvalue of A (resp. A^*) associated with E_i (resp. E_i^*). We call $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta_i^*\}_{i=0}^d$) the* eigenvalue sequence *(resp.* dual eigenvalue sequence*) of Φ.*

We have some comments about the eigenvalues and dual eigenvalues. We have

$$AE_i = \theta_i E_i = E_i A, \qquad A^* E_i^* = \theta_i^* E_i^* = E_i^* A^* \qquad (0 \leq i \leq d).$$

Let x denote an indeterminate. For $0 \leq i \leq d$, define a polynomial

$$\tau_i(x) = (x - \theta_0)(x - \theta_1) \cdots (x - \theta_{i-1}).$$

Note that $\tau_i(x)$ is monic with degree i.

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Definition 7. (See [13, Section 21].) *Pick $0 \neq u \in E_0^* V$. For $0 \leq i \leq d$, define the vector*

$$u_i = \tau_i(A)u.$$

The vectors $\{u_i\}_{i=0}^d$ form a basis for V. This basis is said to be Φ-split.

Lemma 8. (See [13, Section 21].) *With respect to a Φ-split basis of V, the matrices representing A and A^* have the form*

$$A: \begin{pmatrix} \theta_0 & & & & & \mathbf{0} \\ 1 & \theta_1 & & & & \\ & 1 & \theta_2 & & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ \mathbf{0} & & & & 1 & \theta_d \end{pmatrix}, \qquad A^*: \begin{pmatrix} \theta_0^* & \varphi_1 & & & & \mathbf{0} \\ & \theta_1^* & \varphi_2 & & & \\ & & \theta_2^* & \cdot & & \\ & & & \cdot & \cdot & \\ & & & & \cdot & \varphi_d \\ \mathbf{0} & & & & & \theta_d^* \end{pmatrix},$$

where $0 \neq \varphi_i \in \mathbb{F}$ for $1 \leq i \leq d$.

Definition 9. (See [11, Section 8].) *Referring to Lemma 8, the sequence $\{\varphi_i\}_{i=1}^d$ is called the* first split sequence of Φ. *By the* second split sequence of Φ, *we mean the first split sequence of $\Phi^{\Downarrow}$.*

Definition 10. (See [11, Definition 11.1].) *By the* parameter array of Φ, *we mean the sequence*

$$(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\varphi_i\}_{i=1}^d; \{\phi_i\}_{i=1}^d),$$

where $\{\varphi_i\}_{i=1}^d$ (resp. $\{\phi_i\}_{i=1}^d$) is the first split sequence (resp. second split sequence) of Φ.

Lemma 11. (See [13, Theorem 22.2].) *The Leonard system Φ is uniquely determined up to isomorphism by its parameter array.*

Note 12. *For a detailed description of the parameter arrays, see [14, Theorem 10.1] and [14, Appendix].*

Lemma 13. (See [4, Lemma 5.1].) *Consider the affine transformation of Φ shown in (3.2). This affine transformation has parameter array*

$$(\{\xi\theta_i + \zeta\}_{i=0}^d; \{\xi^*\theta_i^* + \zeta^*\}_{i=0}^d; \{\xi\xi^*\varphi_i\}_{i=1}^d; \{\xi\xi^*\phi_i\}_{i=1}^d).$$

Lemma 14. (See [10, Theorem 1.11].) *The following hold:*

(i) *the parameter array of $\Phi^{\downarrow}$ is*

$$(\{\theta_i\}_{i=0}^d; \{\theta_{d-i}^*\}_{i=0}^d; \{\phi_{d-i+1}\}_{i=1}^d; \{\varphi_{d-i+1}\}_{i=1}^d);$$

(ii) *the parameter array of $\Phi^{\Downarrow}$ is*

$$(\{\theta_{d-i}\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\phi_i\}_{i=1}^d; \{\varphi_i\}_{i=1}^d);$$

(iii) *the parameter array of Φ^* is*

$$(\{\theta_i^*\}_{i=0}^d; \{\theta_i\}_{i=0}^d; \{\varphi_i\}_{i=1}^d; \{\phi_{d-i+1}\}_{i=1}^d).$$

Definition 15. (See [13, Definition 7.1].) *Define*

$$a_i = tr(E_i^* A) \qquad (0 \leq i \leq d),$$

where tr means trace.

Lemma 16. (See [13, Lemma 7.5].) *We have*

$$E^*_i A E^*_i = a_i E^*_i \qquad (0 \le i \le d).$$

Lemma 17. (See [13, Lemma 23.6].) *We have*

$$\begin{aligned} a_0 &= \theta_0 + \frac{\varphi_1}{\theta^*_0 - \theta^*_1}, \\ a_i &= \theta_i + \frac{\varphi_i}{\theta^*_i - \theta^*_{i-1}} + \frac{\varphi_{i+1}}{\theta^*_i - \theta^*_{i+1}} \qquad (1 \le i \le d-1), \\ a_d &= \theta_d + \frac{\varphi_d}{\theta^*_d - \theta^*_{d-1}}. \end{aligned}$$

Definition 18. (See [13, Definition 10.3].) *Pick $0 \neq u \in E_0 V$. For $0 \le i \le d$, define the vector*

$$v^*_i = E^*_i u.$$

The vectors $\{v^*_i\}_{i=0}^d$ form a basis for V. This basis is said to be Φ-standard.

Lemma 19. (See [13, Definition 11.1].) *With respect to a Φ-standard basis for V, the matrices representing A and A^* have the form*

$$A: \begin{pmatrix} a_0 & b_0 & & & & \mathbf{0} \\ c_1 & a_1 & b_1 & & & \\ & c_2 & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & b_{d-1} \\ \mathbf{0} & & & & c_d & a_d \end{pmatrix}, \qquad A^*: \begin{pmatrix} \theta^*_0 & & & & & \mathbf{0} \\ & \theta^*_1 & & & & \\ & & \theta^*_2 & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ \mathbf{0} & & & & & \theta^*_d \end{pmatrix},$$

where $\{a_i\}_{i=0}^d$ are from Definition 15 and the scalars $\{c_i\}_{i=1}^d$, $\{b_i\}_{i=0}^{d-1}$ are nonzero.

The scalars $\{c_i\}_{i=1}^d$, $\{a_i\}_{i=0}^d$, $\{b_i\}_{i=0}^{d-1}$ in Lemma 19 are called the *intersection numbers of* Φ. See [10], [13], [14] for basic facts about the intersection numbers.

4. The zero diagonal space for a Leonard pair.

In this section, to each Leonard pair A, A^* on V, we associate a subspace of End(V) called the zero diagonal space for A, A^*.

Definition 20. *For a Leonard system $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E^*_i\}_{i=0}^d)$ on V, let $\mathcal{X}(\Phi)$ denote the subspace of End(V) consisting of the $X \in$ End(V) such that*

$$E_i X E_j = 0, \qquad E^*_i X E^*_j = 0 \qquad \text{if } |i-j| > 1 \qquad (0 \le i, j \le d).$$

Lemma 21. *Let $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E^*_i\}_{i=0}^d)$ denote a Leonard system on V, and let Φ' denote an affine transformation of Φ from (3.2). Then, $\mathcal{X}(\Phi') = \mathcal{X}(\Phi)$.*

Proof. The Φ and Φ' have the same $\{E_i\}_{i=0}^d$ and $\{E^*_i\}_{i=0}^d$. □

Lemma 22. *Let $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E^*_i\}_{i=0}^d)$ denote a Leonard system on V. Then, $\mathcal{X}(\Phi) = \mathcal{X}(\Phi^{\downarrow}) = \mathcal{X}(\Phi^{\Downarrow})$.*

Proof. By Definition 20. □

In view of Lemma 22, we make a definition.

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Definition 23. *For a Leonard pair A, A^* on V we define $\mathcal{X}(A, A^*) = \mathcal{X}(\Phi)$, where Φ is a Leonard system associated with A, A^*.*

Lemma 24. (See [5, Theorem 3.2].) *Let A, A^* denote a Leonard pair on V. Then, the following elements form a basis for $\mathcal{X}(A, A^*)$:*

$$I,\ A^*,\ A,\ AA^*,\ A^*A.$$

Definition 25. *Let $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a Leonard system on V. An element $X \in \mathcal{X}(\Phi)$ is said to have* zero diagonal *whenever $E_i^* X E_i^* = 0$ for $0 \leq i \leq d$.*

Lemma 26. *Let $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a Leonard system on V. Then, the element $AA^* - A^*A$ has zero diagonal.*

Proof. Let $\{\theta_i^*\}_{i=0}^d$ denote the dual eigenvalue sequence of Φ. For $0 \leq i \leq d$, we have $E_i^* A^* = \theta_i^* E_i^*$ and $A^* E_i^* = \theta_i^* E_i^*$. Therefore,

$$E_i^*(AA^* - A^*A)E_i^* = E_i^* AA^* E_i^* - E_i^* A^* A E_i^* = \theta_i^* E_i^* A E_i^* - \theta_i^* E_i^* A E_i^* = 0. \qquad \square$$

Lemma 27. *Let A, A^* denote a Leonard pair on V. Then, the following elements form a basis for $\mathcal{X}(A, A^*)$.*

$$I,\ A^*,\ A,\ AA^*,\ AA^* - A^*A.$$

Proof. By Lemma 24. □

Definition 28. *Let $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a Leonard system on V. Let $\mathcal{Z}(\Phi)$ denote the set of elements in* Span$\{I, A^*, A, AA^*\}$ *that have zero diagonal. Note that $\mathcal{Z}(\Phi)$ is a subspace of* Span$\{I, A^*, A, AA^*\}$*. We call $\mathcal{Z}(\Phi)$ the* zero diagonal space *for Φ.*

Lemma 29. *Let $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a Leonard system on V. Then, for $X \in \mathcal{X}(\Phi)$, the following are equivalent:*

(i) *X has zero diagonal;*
(ii) *$X \in$ Span$\{AA^* - A^*A\} + \mathcal{Z}(\Phi)$.*

Proof. By Lemmas 26, 27 and Definition 28. □

Lemma 30. *Let $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a Leonard system on V. Let Φ' denote an affine transformation of Φ from (3.2). Then, $\mathcal{Z}(\Phi') = \mathcal{Z}(\Phi)$.*

Proof. The Φ and Φ' have the same $\{E_i^*\}_{i=0}^d$. □

Lemma 31. *Let $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a Leonard system on V. Then, $\mathcal{Z}(\Phi) = \mathcal{Z}(\Phi^{\downarrow}) = \mathcal{Z}(\Phi^{\Downarrow})$.*

Proof. By Definition 28. □

In view of Lemma 31, we make a definition.

Definition 32. *For a Leonard pair A, A^* on V, we define $\mathcal{Z}(A, A^*) = \mathcal{Z}(\Phi)$, where Φ is a Leonard system associated with A, A^*. We call $\mathcal{Z}(A, A^*)$ the* zero diagonal space for *A, A^*.*

We mention a result for later use.

Lemma 33. *Let $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a Leonard system on V. Then, the following elements are linearly independent:*

$$(A - a_0 I)(A^* - \theta_d^* I), \qquad (A - a_d I)(A^* - \theta_0^* I).$$

Proof. By Lemma 24, the elements I, A^*, A, AA^* are linearly independent. The result follows from this and $\theta^*_0 \neq \theta^*_d$. ☐

5. The matrix M. For the rest of this paper, we fix a Leonard system

$$\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E^*_i\}_{i=0}^d) \tag{5.3}$$

on V with parameter array

$$(\{\theta_i\}_{i=0}^d; \{\theta^*_i\}_{i=0}^d; \{\varphi_i\}_{i=1}^d; \{\phi_i\}_{i=1}^d).$$

Let the scalars $\{a_i\}_{i=0}^d$ be from Definition 15. In this section, we introduce a matrix M and explain how the rank of M is related to the dimension of $\mathcal{Z}(\Phi)$.

Lemma 34. *For scalars f_0, f_1, f_2, f_3, the following are equivalent:*

(i) $f_0 I + f_1 A^* + f_2 A + f_3 AA^* \in \mathcal{Z}(\Phi)$;
(ii) *for* $0 \leq i \leq d$,

$$f_0 + f_1\theta^*_i + f_2 a_i + f_3 a_i \theta^*_i = 0.$$

Proof. For $0 \leq i \leq d$, we have

$$E^*_i(f_0 I + f_1 A^* + f_2 A + f_3 AA^*)E^*_i = (f_0 + f_1\theta^*_i + f_2 a_i + f_3 a_i\theta^*_i)E^*_i.$$

The result follows. ☐

Lemma 35. *The following are equivalent:*

(i) $\mathcal{Z}(\Phi) \neq 0$;
(ii) *there exist scalars f_0, f_1, f_2, f_3 (not all zero) such that*

$$f_0 + f_1\theta^*_i + f_2 a_i + f_3 a_i\theta^*_i = 0 \qquad (0 \leq i \leq d). \tag{5.4}$$

Proof. By Lemma 34. ☐

Definition 36. *Define a matrix $M = M(\Phi)$ of size $4 \times (d+1)$ by*

$$M = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \theta^*_0 & \theta^*_1 & \cdots & \theta^*_d \\ a_0 & a_1 & \cdots & a_d \\ a_0\theta^*_0 & a_1\theta^*_1 & \cdots & a_d\theta^*_d \end{pmatrix}.$$

Lemma 37. *We have* $2 \leq \operatorname{rank}(M) \leq 4$.

Proof. Clearly $\operatorname{rank}(M) \leq 4$. The top two rows of M are linearly independent, since $\{\theta^*_i\}_{i=0}^d$ are mutually distinct. Therefore, $\operatorname{rank}(M) \geq 2$. ☐

Definition 38. *Define an $\mathbb{F}$-linear map*

$$\psi : \operatorname{Span}\{I, A^*, A, AA^*\} \to \mathbb{F}^{d+1},$$

that sends

$$f_0 I + f_1 A^* + f_2 A + f_3 AA^* \mapsto (f_0, f_1, f_2, f_3)M.$$

Lemma 39. *The following hold:*

(i) $\operatorname{Ker}(\psi) = \mathcal{Z}(\Phi)$;
(ii) $\dim \operatorname{Im}(\psi) = \operatorname{rank}(M)$;
(iii) $\dim \mathcal{Z}(\Phi) = 4 - \operatorname{rank}(M)$.

Proof. (i) By Lemma 34.

(ii), (iii) By linear algebra. □

Lemma 40. *We have*

$$0 \leq \dim \mathcal{Z}(\Phi) \leq 2.$$

Proof. By Lemmas 37 and 39(iii). □

Lemma 41. *The following are equivalent:*

(i) $\dim \mathcal{Z}(\Phi) = 2$;
(ii) $\operatorname{rank}(M) = 2$;
(iii) $a_0 = a_1 = \cdots = a_d$.

Proof. (i) ⇔ (ii) By Lemma 39(iii).

(ii) ⇒ (iii) The top two rows of M are linearly independent. So the 3rd row and the 4th row of M are contained in the span of the top two rows. Thus, there exist scalars α, β, γ, δ such that

$$a_i = \alpha + \beta\theta^*_i \qquad (0 \leq i \leq d),$$
$$a_i\theta^*_i = \gamma + \delta\theta^*_i \qquad (0 \leq i \leq d).$$

By these equations, we get

$$0 = \beta\theta^{*2}_i + (\alpha - \delta)\theta^*_i - \gamma \qquad (0 \leq i \leq d).$$

We assume $d \geq 3$ and the scalars $\{\theta^*_i\}_{i=0}^d$ are mutually distinct, so $\beta = 0$, $\alpha = \delta$, $\gamma = 0$. It follows that $a_0 = a_1 = \cdots = a_d$.

(iii) ⇒ (ii) Clear. □

Lemma 42. *Assume that* $\dim \mathcal{Z}(\Phi) = 2$. *Then, the scalars* f_0, f_1, f_2, f_3 *satisfy* (5.4) *if and only if*

$$f_0 + f_2 a_0 = 0, \qquad f_1 + f_3 a_0 = 0.$$

Proof. By Lemmas 34 and 41. □

Lemma 43. *Assume that* $\dim \mathcal{Z}(\Phi) = 2$. *Then, the following elements form a basis for* $\mathcal{Z}(\Phi)$*:*

$$A - a_0 I, \qquad AA^* - a_0 A^*.$$

Proof. By Lemmas 34 and 42. □

6. The scalars a^-_i and a^+_i.

We continue to discuss the Leonard system Φ from (5.3). Recall the zero diagonal space $\mathcal{Z}(\Phi)$ from Definition 28. In this section, we obtain a necessary and sufficient condition for $\mathcal{Z}(\Phi) \neq 0$.

Definition 44. *For* $0 \leq i \leq d$ *define*

$$a^-_i = (a_i - a_0)(\theta^*_i - \theta^*_d), \qquad a^+_i = (a_i - a_d)(\theta^*_i - \theta^*_0).$$

LEMMA 45. *We have* $a^-_0 = a^-_d = a^+_0 = a^+_d = 0$.

Proof. By Definition 44. □

DEFINITION 46. *Define a matrix* $L = L(\Phi)$ *of size* $4 \times (d+1)$ *by*

$$L = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \theta^*_0 & \theta^*_1 & \cdots & \theta^*_d \\ a^-_0 & a^-_1 & \cdots & a^-_d \\ a^+_0 & a^+_1 & \cdots & a^+_d \end{pmatrix}.$$

DEFINITION 47. *Define a matrix* T *by*

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ a_0\theta^*_d & -a_0 & -\theta^*_d & 1 \\ a_d\theta^*_0 & -a_d & -\theta^*_0 & 1 \end{pmatrix}.$$

LEMMA 48. *We have*

$$L = TM.$$

Proof. Routine verification. □

LEMMA 49. *The matrix* T *is invertible.*

Proof. We have $\det T = \theta^*_0 - \theta^*_d \neq 0$. □

COROLLARY 50. *The matrices* M *and* L *have the same rank.*

Proof. By Lemma 48 and linear algebra. □

PROPOSITION 51. *The following are equivalent:*

(i) $\mathcal{Z}(\Phi) \neq 0$;
(ii) $\operatorname{rank}(L) \leq 3$;
(iii) *the vectors* $(a^-_0, a^-_1, \ldots, a^-_d)$ *and* $(a^+_0, a^+_1, \ldots, a^+_d)$ *are linearly dependent;*
(iv) $a^-_i a^+_j = a^+_i a^-_j$ *for* $0 \leq i, j \leq d$;
(v) *the vectors* $(a^-_1, a^-_2, \ldots, a^-_{d-1})$ *and* $(a^+_1, a^+_1, \ldots, a^+_{d-1})$ *are linearly dependent;*
(vi) $a^-_i a^+_j = a^+_i a^-_j$ *for* $1 \leq i, j \leq d-1$.

Proof. (i) ⇔ (ii) By Lemma 39(iii) and Corollary 50.

(ii) ⇒ (iii) The rows of L are linearly dependent. Therefore, there exist scalars f_0, f_1, f_2, f_3 (not all zero) such that

$$f_0 + f_1\theta^*_i + f_2 a^-_i + f_3 a^+_i = 0 \qquad (0 \leq i \leq d).$$

By this and since $a^-_0 = a^-_d = a^+_0 = a^+_d = 0$,

$$f_0 + f_1\theta^*_0 = 0, \qquad f_0 + f_1\theta^*_d = 0.$$

We have $\theta^*_0 \neq \theta^*_d$, so $f_0 = f_1 = 0$. Observe that $f_2 \neq 0$ or $f_3 \neq 0$. Moreover,

$$f_2 a^-_i + f_3 a^+_i = 0 \qquad (0 \leq i \leq d).$$

This gives the linear dependency

$$f_2(a^-_0, a^-_1, \ldots, a^-_d) + f_3(a^+_0, a^+_1, \ldots, a^+_d) = 0.$$

(iii) ⇒ (ii) By linear algebra.

(iii) ⇔ (iv) By linear algebra.

(iii) ⇔ (v) Since $a_0^- = a_d^- = a_0^+ = a_d^+ = 0$.

(iv) ⇔ (vi) Since $a_0^- = a_d^- = a_0^+ = a_d^+ = 0$. □

7. The type of a Leonard system.

We continue to discuss the Leonard system Φ from (5.3). By [12, Theorem 5.16], Φ has one of the following types:

q-Racah,	q-Hahn,	dual q-Hahn,	quantum q-Krawtchouk,
q-Krawtchouk,	affine q-Krawtchouk,	dual q-Krawtchouk,	
Racah,	Hahn,	dual Hahn,	Krawtchouk,
Bannai/Ito,	Orphan.		

In this section, we describe each type in detail.

Definition 52. (See [12, Example 5.3].) *The Leonard system Φ is said to have* q-Racah type *whenever there exist scalars $q, h, h^*, s, s^*, r_1, r_2$ such that*

(i) *each of $q, h, h^*, s, s^*, r_1, r_2$ is nonzero;*
(ii) $r_1 r_2 = s s^* q^{d+1}$;
(iii) *none of $q^i, r_1 q^i, r_2 q^i, s^* q^i / r_1, s^* q^i / r_2$ is equal to 1 for $1 \leq i \leq d$;*
(iv) $sq^i \neq 1$, $s^* q^i \neq 1$ *for $2 \leq i \leq 2d$;*
(v) *for $0 \leq i \leq d$,*

$$\theta_i = \theta_0 + h(1-q^i)(1-sq^{i+1})q^{-i},$$
$$\theta^*_i = \theta^*_0 + h^*(1-q^i)(1-s^*q^{i+1})q^{-i};$$

(vi) *for $1 \leq i \leq d$,*

$$\varphi_i = hh^* q^{1-2i}(1-q^i)(1-q^{i-d-1})(1-r_1 q^i)(1-r_2 q^i),$$
$$\phi_i = hh^* q^{1-2i}(1-q^i)(1-q^{i-d-1})(r_1 - s^* q^i)(r_2 - s^* q^i)/s^*.$$

Definition 53. (See [12, Example 5.4].) *The Leonard system Φ is said to have* q-Hahn type *whenever there exist scalars q, h, h^*, s^*, r such that*

(i) *each of q, h, h^*, s^*, r is nonzero;*
(ii) *none of $q^i, rq^i, s^* q^i / r$ is equal to 1 for $1 \leq i \leq d$;*
(iii) $s^* q^i \neq 1$ *for $2 \leq i \leq 2d$;*
(iv) *for $0 \leq i \leq d$,*

$$\theta_i = \theta_0 + h(1-q^i)q^{-i},$$
$$\theta^*_i = \theta^*_0 + h^*(1-q^i)(1-s^*q^{i+1})q^{-i};$$

(v) *for $1 \leq i \leq d$,*

$$\varphi_i = hh^* q^{1-2i}(1-q^i)(1-q^{i-d-1})(1-rq^i),$$
$$\phi_i = -hh^* q^{1-i}(1-q^i)(1-q^{i-d-1})(r - s^* q^i).$$

Definition 54. (See [12, Example 5.5].) *The Leonard system Φ is said to have* dual q-Hahn type *whenever there exist scalars q, h, h^*, s, r such that*

- (i) *each of q, h, h^*, s, r is nonzero;*
- (ii) *none of $q^i, rq^i, sq^i/r$ is equal to 1 for $1 \le i \le d$;*
- (iii) *$sq^i \ne 1$ for $2 \le i \le 2d$;*
- (iv) *for $0 \le i \le d$,*

$$\theta_i = \theta_0 + h(1-q^i)(1-sq^{i+1})q^{-i},$$
$$\theta^*_i = \theta^*_0 + h^*(1-q^i)q^{-i};$$

- (v) *for $1 \le i \le d$,*

$$\varphi_i = hh^*q^{1-2i}(1-q^i)(1-q^{i-d-1})(1-rq^i),$$
$$\phi_i = hh^*q^{d+2-2i}(1-q^i)(1-q^{i-d-1})(s-rq^{i-d-1}).$$

Definition 55. (See [12, Example 5.6].) *The Leonard system Φ is said to have* quantum q-Krawtchouk type *whenever there exist scalars q, h^*, s, r such that*

- (i) *each of q, h^*, s, r is nonzero;*
- (ii) *$q^i \ne 1$, $sq^i/r \ne 1$ for $1 \le i \le d$;*
- (iii) *for $0 \le i \le d$,*

$$\theta_i = \theta_0 - sq(1-q^i),$$
$$\theta^*_i = \theta^*_0 + h^*(1-q^i)q^{-i};$$

- (iv) *for $1 \le i \le d$,*

$$\varphi_i = -rh^*q^{1-i}(1-q^i)(1-q^{i-d-1}),$$
$$\phi_i = h^*q^{d+2-2i}(1-q^i)(1-q^{i-d-1})(s-rq^{i-d-1}).$$

Definition 56. (See [12, Example 5.7].) *The Leonard system Φ is said to have* q-Krawtchouk type *whenever there exist scalars q, h, h^*, s^* such that*

- (i) *each of q, h, h^*, s^* is nonzero;*
- (ii) *$q^i \ne 1$ for $1 \le i \le d$;*
- (iii) *$s^*q^i \ne 1$ for $2 \le i \le 2d$;*
- (iv) *for $0 \le i \le d$,*

$$\theta_i = \theta_0 + h(1-q^i)q^{-i},$$
$$\theta^*_i = \theta^*_0 + h^*(1-q^i)(1-s^*q^{i+1})q^{-i};$$

- (v) *for $1 \le i \le d$,*

$$\varphi_i = hh^*q^{1-2i}(1-q^i)(1-q^{i-d-1}),$$
$$\phi_i = hh^*s^*q(1-q^i)(1-q^{i-d-1}).$$

Definition 57. (See [12, Example 5.8].) *The Leonard system Φ is said to have* affine q-Krawtchouk type *whenever there exist scalars q, h, h^*, r such that*

- (i) *each of q, h, h^*, r is nonzero;*

(ii) *$q^i \neq 1$, $rq^i \neq 1$ for $1 \leq i \leq d$;*
(iii) *for $0 \leq i \leq d$,*

$$\theta_i = \theta_0 + h(1-q^i)q^{-i},$$
$$\theta^*_i = \theta^*_0 + h^*(1-q^i)q^{-i};$$

(iv) *for $1 \leq i \leq d$,*

$$\varphi_i = hh^* q^{1-2i}(1-q^i)(1-q^{i-d-1})(1-rq^i),$$
$$\phi_i = -hh^* r q^{1-i}(1-q^i)(1-q^{i-d-1}).$$

Definition 58. (See [12, Example 5.9].) *The Leonard system Φ is said to have* dual q-Krawtchouk type *whenever there exist scalars q, h, h^*, s such that*

(i) *each of q, h, h^*, s is nonzero;*
(ii) *$q^i \neq 1$ for $1 \leq i \leq d$;*
(iii) *$sq^i \neq 1$ for $2 \leq i \leq 2d$;*
(iv) *for $0 \leq i \leq d$,*

$$\theta_i = \theta_0 + h(1-q^i)(1-sq^{i+1})q^{-i},$$
$$\theta^*_i = \theta^*_0 + h^*(1-q^i)q^{-i};$$

(v) *for $1 \leq i \leq d$,*

$$\varphi_i = hh^* q^{1-2i}(1-q^i)(1-q^{i-d-1}),$$
$$\phi_i = hh^* s q^{d+2-2i}(1-q^i)(1-q^{i-d-1}).$$

Definition 59. (See [12, Example 5.10].) *The Leonard system Φ is said to have* Racah type *whenever there exist scalars h, h^*, s, s^*, r_1, r_2 such that*

(i) *each of h, h^* is nonzero;*
(ii) *$r_1 + r_2 = s + s^* + d + 1$;*
(iii) *the characteristic of $\mathbb{F}$ is 0 or a prime greater than d;*
(iv) *none of $r_1, r_2, s^* - r_1, s^* - r_2$ is equal to $-i$ for $1 \leq i \leq d$;*
(v) *$s \neq -i$, $s^* \neq -i$ for $2 \leq i \leq 2d$;*
(vi) *for $0 \leq i \leq d$,*

$$\theta_i = \theta_0 + hi(i+1+s),$$
$$\theta^*_i = \theta^*_0 + h^* i(i+1+s^*);$$

(vii) *for $1 \leq i \leq d$,*

$$\varphi_i = hh^* i(i-d-1)(i+r_1)(i+r_2),$$
$$\phi_i = hh^* i(i-d-1)(i+s^*-r_1)(i+s^*-r_2).$$

Definition 60. (See [12, Example 5.11].) *The Leonard system Φ is said to have* Hahn type *whenever there exist scalars h^*, s, s^*, r such that*

(i) *each of h^*, s is nonzero;*
(ii) *the characteristic of $\mathbb{F}$ is 0 or a prime greater than d;*

(iii) *neither of* $r, s^* - r$ *is equal to* $-i$ *for* $1 \leq i \leq d$*;*
(iv) $s^* \neq -i$ *for* $2 \leq i \leq 2d$*;*
(v) *for* $0 \leq i \leq d$*,*

$$\theta_i = \theta_0 + si,$$
$$\theta^*_i = \theta^*_0 + h^* i(i + 1 + s^*);$$

(vi) *for* $1 \leq i \leq d$*,*

$$\varphi_i = h^* si(i - d - 1)(i + r),$$
$$\phi_i = -h^* si(i - d - 1)(i + s^* - r).$$

Definition 61. (See [12, Example 5.12].) *The Leonard system* Φ *is said to have* dual Hahn type *whenever there exist scalars* h, s, s^*, r *such that*

(i) *each of* h, s^* *is nonzero;*
(ii) *the characteristic of* $\mathbb{F}$ *is* 0 *or a prime greater than* d*;*
(iii) *neither of* $r, s - r$ *is equal to* $-i$ *for* $1 \leq i \leq d$*;*
(iv) $s \neq -i$ *for* $2 \leq i \leq 2d$*;*
(v) *for* $0 \leq i \leq d$*,*

$$\theta_i = \theta_0 + hi(i + 1 + s),$$
$$\theta^*_i = \theta^*_0 + s^* i;$$

(vi) *for* $1 \leq i \leq d$*,*

$$\varphi_i = hs^* i(i - d - 1)(i + r),$$
$$\phi_i = hs^* i(i - d - 1)(i + r - s - d - 1).$$

Definition 62. (See [12, Example 5.13].) *The Leonard system* Φ *is said to have* Krawtchouk type *whenever there exist scalars* s, s^*, r *such that*

(i) *each of* s, s^*, r *is nonzero;*
(ii) *the characteristic of* $\mathbb{F}$ *is* 0 *or a prime greater than* d*;*
(iii) $r \neq ss^*$*;*
(iv) *for* $0 \leq i \leq d$*,*

$$\theta_i = \theta_0 + si,$$
$$\theta^*_i = \theta^*_0 + s^* i;$$

(v) *for* $1 \leq i \leq d$*,*

$$\varphi_i = ri(i - d - 1),$$
$$\phi_i = (r - ss^*)i(i - d - 1).$$

Definition 63. (See [12, Example 5.14].) *The Leonard system* Φ *is said to have* Bannai/Ito type *whenever there exist scalars* h, h^*, s, s^*, r_1, r_2 *such that*

(i) *each of* h, h^* *is nonzero;*
(ii) *the characteristic of* $\mathbb{F}$ *is* 0 *or an odd prime greater than* $d/2$*;*

(iii) $r_1 + r_2 = -s - s^* + d + 1$;
(iv) *neither of* r_1, $-s^* - r_1$ *is equal to* $-i$ *for* $1 \le i \le d$, $d - i$ *even;*
(v) *neither of* r_2, $-s^* - r_2$ *is equal to* $-i$ *for* $1 \le i \le d$, i *odd;*
(vi) *neither of* s, s^* *is equal to* $2i$ *for* $1 \le i \le d$;
(vii) *for* $0 \le i \le d$,

$$\theta_i = \theta_0 + h\big(s - 1 + (1 - s + 2i)(-1)^i\big),$$
$$\theta^*_i = \theta^*_0 + h^*\big(s^* - 1 + (1 - s^* + 2i)(-1)^i\big);$$

(viii) *for* $1 \le i \le d$,

$$\varphi_i = \begin{cases} -4hh^* i(i + r_1) & \text{if } i \text{ even, } d \text{ even,} \\ -4hh^*(i - d - 1)(i + r_2) & \text{if } i \text{ odd, } d \text{ even,} \\ -4hh^* i(i - d - 1) & \text{if } i \text{ even, } d \text{ odd,} \\ -4hh^*(i + r_1)(i + r_2) & \text{if } i \text{ odd, } d \text{ odd,} \end{cases}$$

$$\phi_i = \begin{cases} 4hh^* i(i - s^* - r_1) & \text{if } i \text{ even, } d \text{ even,} \\ 4hh^*(i - d - 1)(i - s^* - r_2) & \text{if } i \text{ odd, } d \text{ even,} \\ -4hh^* i(i - d - 1) & \text{if } i \text{ even, } d \text{ odd,} \\ -4hh^*(i - s^* - r_1)(i - s^* - r_2) & \text{if } i \text{ odd, } d \text{ odd.} \end{cases}$$

Definition 64. (See [12, Example 5.15].) *The Leonard system* Φ *is said to have* Orphan type *whenever* $d = 3$ *and the characteristic of* $\mathbb{F}$ *is* 2, *and there exist scalars* h, h^*, s, s^*, r *such that*

(i) *each of* h, h^*, s, s^*, r *is nonzero;*
(ii) $s \ne 1$, $s^* \ne 1$;
(iii) r *is equal to none of* $s + s^*, s(1 + s^*), s^*(1 + s)$;
(iv)

$$\theta_1 = \theta_0 + h(s + 1), \qquad \theta_2 = \theta_0 + h, \qquad \theta_3 = \theta_0 + hs,$$
$$\theta^*_1 = \theta^*_0 + h^*(s^* + 1), \qquad \theta^*_2 = \theta^*_0 + h^*, \qquad \theta^*_3 = \theta^*_0 + h^*s^*;$$

(v)

$$\varphi_1 = hh^*r, \qquad \varphi_2 = hh^*, \qquad \varphi_3 = hh^*(r + s + s^*),$$
$$\phi_1 = hh^*(r + s + ss^*), \qquad \phi_2 = hh^*, \qquad \phi_3 = hh^*(r + s^* + ss^*).$$

8. About the equation Proposition 51(vi). We continue to discuss the Leonard system Φ from (5.3). By Proposition 51, we have $\mathcal{Z}(\Phi) \ne 0$ if and only if the following equation holds for $1 \le i, j \le d - 1$:

$$(a_i - a_0)(\theta^*_i - \theta^*_d)(a_j - a_d)(\theta^*_j - \theta^*_0) = (a_i - a_d)(\theta^*_i - \theta^*_0)(a_j - a_0)(\theta^*_j - \theta^*_d). \tag{8.5}$$

In this section, we evaluate the left-hand side of (8.5) minus the right-hand side of (8.5). We use the following expression:

$$\frac{(\theta^*_0 - \theta^*_i)(\theta^*_0 - \theta^*_j)(\theta^*_0 - \theta^*_d)(\theta^*_i - \theta^*_j)(\theta^*_i - \theta^*_d)(\theta^*_j - \theta^*_d)}{(\theta^*_0 - \theta^*_1)(\theta^*_{i-1} - \theta^*_i)(\theta^*_i - \theta^*_{i+1})(\theta^*_{j-1} - \theta^*_j)(\theta^*_j - \theta^*_{j+1})(\theta^*_{d-1} - \theta^*_d)}. \tag{8.6}$$

Proposition 65. *For $1 \leq i, j \leq d-1$, the left-hand side of* (8.5) *minus the right-hand side of* (8.5) *is equal to* (8.6) *times the factor given in the table below:*

Type	*Factor*
q-Racah	$h^2 h^{*2} q^{-3-d}(q-1)^4(q^2-1)^2(s^* - r_1^2)(s^* - r_2^2)/s^*$
q-Hahn	$h^2 h^{*2} q^{-3-d}(q-1)^4(q^2-1)^2(s^* - r^2)$
dual q-Hahn	$-h^2 h^{*2} q^{-3-d}(q-1)^4(q^2-1)^2 r^2$
quantum q-Krawtchouk	$-h^{*2} q^{-3-d}(q-1)^4(q^2-1)^2 r^2$
q-Krawtchouk	$h^2 h^{*2} q^{-3-d}(q-1)^4(q^2-1)^2 s^*$
affine q-Krawtchouk	$-h^2 h^{*2} q^{-3-d}(q-1)^4(q^2-1)^2 r^2$
dual q-Krawtchouk	0
Racah	$4h^2 h^{*2}(s^* - 2r_1)(s^* - 2r_2)$
Hahn	0
dual Hahn	$-4h^2 s^{*2}$
Krawtchouk	0
Bannai/Ito	$64(-1)^{d+1} h^2 h^{*2}(s^* + 2r_1)(s^* + 2r_2)$
Orphan	$h^2 h^{*2}(s^{*2} + 1)$

Proof. Routine verification using the data in Section 7. □

9. A necessary and sufficient condition for $\mathcal{Z}(\Phi) \neq 0$. We continue to discuss the Leonard system Φ from (5.3). In this section, we find a necessary and sufficient condition for $\mathcal{Z}(\Phi) \neq 0$.

Theorem 66. *Assume that Φ has one of the following types:*

dual q-Krawtchouk, Hahn, Krawtchouk.

Then $\mathcal{Z}(\Phi) \neq 0$.

Proof. By Propositions 51 and 65. □

Theorem 67. *Assume that Φ has one of the following types:*

dual q-Hahn, quantum q-Krawtchouk, q-Krawtchouk,
affine q-Krawtchouk, dual Hahn, Orphan.

Then $\mathcal{Z}(\Phi) = 0$.

Proof. By Propositions 51 and 65. □

Theorem 68. *For the following types, $\mathcal{Z}(\Phi) \neq 0$ if and only if the given condition is satisfied.*

Type	Condition
q-Racah	$s^* = r_1^2$ *or* $s^* = r_2^2$
q-Hahn	$s^* = r^2$
Racah	$s^* = 2r_1$ *or* $s^* = 2r_2$
Bannai/Ito	$s^* = -2r_1$ *or* $s^* = -2r_2$

Proof. By Propositions 51 and 65. ∎

10. The Leonard systems with $\dim \mathcal{Z}(\Phi) = 2$. We continue to discuss the Leonard system Φ from (5.3). Recall by Lemma 41 that $\dim \mathcal{Z}(\Phi) = 2$ if and only if $a_0 = a_1 = \cdots = a_d$. In this section, we examine this condition for each of the types listed in Section 7.

PROPOSITION 69. (See [8, Section 17].) *If* $\dim \mathcal{Z}(\Phi) = 2$, *then* Φ *has one of the types:*

q-Racah, dual q-Krawtchouk, Hahn, Krawtchouk, Bannai/Ito with d even.

For each of the above types, $\dim \mathcal{Z}(\Phi) = 2$ *if and only if the following condition holds:*

Type	Condition
q-Racah	$s^* = r_1^2$ *and* $s = -q^{-d-1}$
dual q-Krawtchouk	$s = -q^{-d-1}$
Hahn	$s^* = 2r$
Krawtchouk	$ss^* = 2r$
Bannai/Ito, d even	$s^* = -2r_1$ *and* $s = d+1$

11. A relation between a_i^- and a_i^+. We continue to discuss the Leonard system Φ from (5.3). Recall the scalars $\{a_i^-\}_{i=0}^d$ and $\{a_i^+\}_{i=0}^d$ from Definition 44. In this section, we assume $\mathcal{Z}(\Phi) \neq 0$ and obtain a relation between a_i^- and a_i^+ for $0 \le i \le d$.

PROPOSITION 70. *Assume that* $\mathcal{Z}(\Phi) \neq 0$. *Then,* a_i^- *and* a_i^+ *are related as follows for* $0 \le i \le d$:

Case	*Relation between a_i^- and a_i^+*
q-Racah, $s^* = r_1^2$	$a_i^- q^d (r_1+1)(r_1 q+1) = a_i^+ (r_1 q^d + 1)(r_1 q^{d+1}+1)$
q-Racah, $s^* = r_2^2$	$a_i^- q^d (r_2+1)(r_2 q+1) = a_i^+ (r_2 q^d + 1)(r_2 q^{d+1}+1)$
q-Hahn, $s^* = r^2$	$a_i^- q^d (r+1)(rq+1) = a_i^+ (rq^d + 1)(rq^{d+1}+1)$
dual q-Krawtchouk	$a_i^- q^d = a_i^+$
Racah, $s^* = 2r_1$	$a_i^- = a_i^+$
Racah, $s^* = 2r_2$	$a_i^- = a_i^+$
Hahn	$a_i^- s^*(s^*+2) = a_i^+ (s^*+2d)(s^*+2d+2)$
Krawtchouk	$a_i^- = a_i^+$
Bannai/Ito, d even, $s^* = -2r_1$	$a_i^- (r_1+1) = a_i^+ (r_1+d+1)$
Bannai/Ito, d even, $s^* = -2r_2$	$a_i^- r_2 = a_i^+ (r_2+d)$
Bannai/Ito, d odd, $s^* = -2r_1$	$a_i^- r_1 = -a_i^+ (r_1+d+1)$
Bannai/Ito, d odd, $s^* = -2r_2$	$a_i^- r_2 = -a_i^+ (r_2+d+1)$

Proof. Routine verification using Lemma 17 and the data in Section 7. ∎

12. The Leonard systems Φ such that dim $\mathcal{Z}(\Phi) = 1$. We continue to discuss the Leonard system Φ from (5.3). In Lemma 43, we gave a basis for $\mathcal{Z}(\Phi)$ under the assumption that $\dim \mathcal{Z}(\Phi) = 2$. In this section, we give a basis for $\mathcal{Z}(\Phi)$ under the assumption that $\dim \mathcal{Z}(\Phi) = 1$.

Proposition 71. *Assume that $\mathcal{Z}(\Phi) \neq 0$. Then, the following element is nonzero and contained in $\mathcal{Z}(\Phi)$.*

Case	*A nonzero element in $\mathcal{Z}(\Phi)$*
q-Racah, $s^* = r_1^2$	$(A - a_0 I)(A^* - \theta^*_d I)q^d(r_1 + 1)(r_1 q + 1)$ $-(A - a_d I)(A^* - \theta^*_0 I)(r_1 q^d + 1)(r_1 q^{d+1} + 1)$
q-Racah, $s^* = r_2^2$	$(A - a_0 I)(A^* - \theta^*_d I)q^d(r_2 + 1)(r_2 q + 1)$ $-(A - a_d I)(A^* - \theta^*_0 I)(r_2 q^d + 1)(r_2 q^{d+1} + 1)$
q-Hahn, $s^* = r^2$	$(A - a_0 I)(A^* - \theta^*_d I)q^d(r + 1)(rq + 1)$ $-(A - a_d I)(A^* - \theta^*_0 I)(rq^d + 1)(rq^{d+1} + 1)$
dual q-Krawtchouk	$(A - a_0 I)(A^* - \theta^*_d I)q^d - (A - a_d I)(A^* - \theta^*_0 I)$
Racah, $s^* = 2r_1$	$(A - a_0 I)(A^* - \theta^*_d I) - (A - a_d I)(A^* - \theta^*_0 I)$
Racah, $s^* = 2r_2$	$(A - a_0 I)(A^* - \theta^*_d I) - (A - a_d I)(A^* - \theta^*_0 I)$
Hahn	$(A - a_0 I)(A^* - \theta^*_d I)s^*(s^* + 2)$ $-(A - a_d I)(A^* - \theta^*_0 I)(s^* + 2d)(s^* + 2d + 2)$
Krawtchouk	$(A - a_0 I)(A^* - \theta^*_d I) - (A - a_d I)(A^* - \theta^*_0 I)$
Bannai/Ito, d even, $s^* = -2r_1$	$(A - a_0 I)(A^* - \theta^*_d I)(r_1 + 1)$ $-(A - a_d I)(A^* - \theta^*_0 I)(r_1 + d + 1)$
Bannai/Ito, d even, $s^* = -2r_2$	$(A - a_0 I)(A^* - \theta^*_d I)r_2$ $-(A - a_d I)(A^* - \theta^*_0 I)(r_2 + d)$
Bannai/Ito, d odd, $s^* = -2r_1$	$(A - a_0 I)(A^* - \theta^*_d I)r_1$ $+(A - a_d I)(A^* - \theta^*_0 I)(r_1 + d + 1)$
Bannai/Ito, d odd, $s^* = -2r_2$	$(A - a_0 I)(A^* - \theta^*_d I)r_2$ $+(A - a_d I)(A^* - \theta^*_0 I)(r_2 + d + 1)$

Moreover, the above element is a basis of $\mathcal{Z}(\Phi)$, provided that $\dim \mathcal{Z}(\Phi) = 1$.

Proof. The given element is contained in $\mathcal{Z}(\Phi)$ by Lemma 34, Definition 44, and Proposition 70. The given element is nonzero, by Lemma 33 and the inequalities in Section 7. The result follows. ∎

13. Self-dual Leonard pairs and Leonard systems. We are done with our first topic. We now consider our second topic, concerning spin Leonard pairs. These Leonard pairs have a property called self-dual. In this section, we describe the self-dual property.

Definition 72. (See [3, Definition 7.1], [6, Definition 3.4].) *A Leonard pair A, A^* on V is said to be* self-dual *whenever A, A^* is isomorphic to the Leonard pair A^*, A.*

Definition 73. (See [6, Definition 3.4].) *A Leonard system Φ on V is said to be* self-dual *whenever Φ is isomorphic to Φ^*.*

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Lemma 74. *Let A, A^* denote a self-dual Leonard pair on V. Then, there exists an associated Leonard system Φ that is self-dual.*

Proof. Let σ denote an isomorphism of Leonard pairs from A, A^* to A^*, A. Let $\{E_i\}_{i=0}^d$ denote a standard ordering of the primitive idempotents of A. For $0 \leq i \leq d$, define $E_i^* = E_i^\sigma$. Then, $\Phi = (A : \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ is a Leonard system that is associated with A, A^*. By construction $\Phi^\sigma = \Phi^*$, so Φ is isomorphic to Φ^*. Thus, Φ is self-dual. ∎

Lemma 75. (See [6, Lemma 7.4].) *Recall the Leonard system Φ from (5.3). Then, Φ is self-dual if and only if*

$$\theta_i = \theta_i^* \qquad (0 \leq i \leq d).$$

In this case,

$$\phi_i = \phi_{d-i+1} \qquad (1 \leq i \leq d).$$

Lemma 76. *Recall the Leonard system Φ from (5.3). If Φ is self-dual, then Φ has one of the following types:*

q-Racah, affine q-Krawtchouk, Racah, Krawtchouk, Bannai/Ito, Orphan.

For each of the above types, Φ is self-dual if and only if $\theta_0 = \theta_0^*$ and the following condition holds:

Type	*Condition*
q-Racah	$h = h^*$ *and* $s = s^*$
affine q-Krawtchouk	$h = h^*$
Racah	$h = h^*$ *and* $s = s^*$
Krawtchouk	$s = s^*$
Bannai/Ito	$h = h^*$ *and* $s = s^*$
Orphan	$h = h^*$ *and* $s = s^*$

Proof. Routine verification using Lemma 75 and the data in Section 7. ∎

14. Spin Leonard pairs.

In this section, we recall the notion of a spin Leonard pair. We then characterize the spin Leonard pairs using the zero diagonal space.

Definition 77. (See [2, Definition 1.2].) *Let A, A^* denote a Leonard pair on V. This Leonard pair is said to have* spin *whenever there exist invertible $W, W^* \in \mathrm{End}(V)$ such that*

$$WA = AW,$$
$$W^*A^* = A^*W^*,$$
$$WA^*W^{-1} = (W^*)^{-1}AW^*.$$

Lemma 78. (See [2, Theorem 1.6], [3, Lemma 9.2(iii)].) *Let A, A^* denote a spin Leonard pair on V. Then, A, A^* is self-dual.*

Proposition 79. (See [2, Lemmas 1.7–1.11, 1.13].) *Recall the Leonard system Φ from (5.3) and assume that Φ is self-dual. If A, A^* has spin, then Φ has one of the following types:*

q-Racah, Racah, Krawtchouk, Bannai/Ito.

If Φ has Krawtchouk type, then A, A^* has spin. For the remaining types, A, A^* has spin if and only if the following condition holds:

Type	*Condition*
q-Racah	$s = r_1^2$ *or* $s = r_2^2$
Racah	$s = 2r_1$ *or* $s = 2r_2$
Bannai/Ito	$s = -2r_1$ *or* $s = -2r_2$

NOTE 80. *In [2, Lemma 1.11], there is a typo. In the formula for b_i (i odd), the minus sign should be removed.*

THEOREM 81. *Let A, A^* denote a Leonard pair on V. Then, the following are equivalent:*

(i) *A, A^* has spin;*
(ii) *A, A^* is self-dual and $\mathcal{Z}(A, A^*) \neq 0$.*

Proof. (i) ⇒ (ii) By Lemma 78, the Leonard pair A, A^* is self-dual. By this and Lemma 74, there exists a self-dual Leonard system Φ associated with A, A^*. By Section 9, Lemma 76, and Proposition 79, we get $\mathcal{Z}(\Phi) \neq 0$. By this and Definition 32, $\mathcal{Z}(A, A^*) \neq 0$.

(ii) ⇒ (i) By Lemma 74, there exists a self-dual Leonard system Φ associated with A, A^*. By $\mathcal{Z}(A, A^*) \neq 0$ and Definition 32, $\mathcal{Z}(\Phi) \neq 0$. By this and Section 9, Lemma 76, and Proposition 79, we find that A, A^* has spin. □

COROLLARY 82. *Let $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a self-dual Leonard system on V. Then, the following* (i)–(iii) *are equivalent:*

(i) *the Leonard pair A, A^* has spin;*
(ii) *there exist scalars f_0, f_1, f_2, f_3 (not all zero) such that*

$$f_0 + f_1\theta_i^* + f_2 a_i + f_3 a_i \theta_i^* = 0 \qquad (0 \leq i \leq d);$$

(iii) *for $0 \leq i, j \leq d$,*

$$(a_i - a_0)(\theta_i^* - \theta_d^*)(a_j - a_d)(\theta_j^* - \theta_0^*) = (a_i - a_d)(\theta_i^* - \theta_0^*)(a_j - a_0)(\theta_j^* - \theta_d^*).$$

Proof. By Definition 32, Lemma 35, Proposition 51, and Theorem 81. □

NOTE 83. *For each spin Leonard pair, the corresponding W, W^* from Definition 77 are given in [2, Lemma 1.17 and Theorem 1.18].*

We finish this paper with some comments. Recall our assumption $d \geq 3$. It turns out that Corollary 82 is false for $d = 2$. To see this, note that under the assumption $d = 2$, the conditions (ii) and (iii) of Corollary 82 are vacuously true. However, under the assumption $d = 2$, the condition (i) of Corollary 82 might not be true, as the following example shows. For this example, we assume that the field $\mathbb{F}$ has characteristic zero. Consider the algebra $\mathrm{Mat}_3(\mathbb{F})$ of 3×3 matrices that have all entries in $\mathbb{F}$. We index the rows and columns by $0, 1, 2$. Consider the following matrices in $\mathrm{Mat}_3(\mathbb{F})$:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 5 \end{pmatrix}, \qquad A^* = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 2 & -9 \\ 0 & 0 & 5 \end{pmatrix}.$$

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Each of A, A^* is multiplicity-free with eigenvalues $1, 2, 5$. Using (2.1), we find that the corresponding primitive idempotents are

$$E_0 = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 1/4 & 0 & 0 \end{pmatrix}, \qquad E_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ -1/3 & -1/3 & 0 \end{pmatrix}, \qquad E_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1/12 & 1/3 & 1 \end{pmatrix},$$

$$E_0^* = \begin{pmatrix} 1 & 1 & 9/4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad E_1^* = \begin{pmatrix} 0 & -1 & -3 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}, \qquad E_2^* = \begin{pmatrix} 0 & 0 & 3/4 \\ 0 & 0 & -3 \\ 0 & 0 & 1 \end{pmatrix}.$$

By matrix multiplication, we obtain

$$E_i^* A E_j^* = \begin{cases} 0 & \text{if } |i-j| > 1, \\ \neq 0 & \text{if } |i-j| = 1 \end{cases} \qquad (0 \le i, j \le 2),$$

$$E_i A^* E_j = \begin{cases} 0 & \text{if } |i-j| > 1, \\ \neq 0 & \text{if } |i-j| = 1 \end{cases} \qquad (0 \le i, j \le 2).$$

Therefore, $\Phi = (A; \{E_i\}_{i=0}^2; A^*; \{E_i^*\}_{i=0}^2)$ is a Leonard system. By construction, Φ is self-dual. We are going to show that the Leonard pair A, A^* does not have spin. To do this, we assume that A, A^* has spin and get a contradiction. Consider the matrices W, W^* from Definition 77. We have $WA = AW$ and $W^*A^* = A^*W^*$ and

$$W^*WA^* = AW^*W. \tag{14.7}$$

The matrix W commutes with A, and A is multiplicity-free. So, W is a polynomial in A. Therefore, $W \in \operatorname{Span}\{E_0, E_1, E_2\}$. Similarly $W^* \in \operatorname{Span}\{E_0^*, E_1^*, E_2^*\}$. There exist scalars g_0, g_1, g_2, g_0^*, g_1^*, g_2^* such that

$$W = \sum_{i=0}^{2} g_i E_i, \qquad W^* = \sum_{i=0}^{2} g_i^* E_i^*.$$

Since W and W^* are invertible, we obtain

$$g_i \neq 0, \qquad g_i^* \neq 0 \qquad (0 \le i \le 2).$$

In (14.7), evaluate the $(2,2)$-entry of each side to get

$$g_1 g_2^* - g_2 g_1^* = 0. \tag{14.8}$$

In (14.7), evaluate the $(2,1)$-entry of each side to get

$$-12 g_2 g_1^* + (-3g_0 + 4g_1 - g_2) g_2^* = 0. \tag{14.9}$$

In (14.7), evaluate the $(2,0)$-entry of each side to get

$$3(g_0 - g_2) g_1^* + (-3g_0 + 4g_1 - g_2) g_2^* = 0. \tag{14.10}$$

In (14.7), evaluate the $(0,1)$-entry of each side to get

$$9(-g_0 + g_2) g_0^* - 4(g_0 + 3g_2) g_1^* + 3(-g_0 + g_2) g_2^* = 0. \tag{14.11}$$

In (14.7), evaluate the $(1, 0)$-entry of each side to get

$$(-9g_0 - 4g_1 - 3g_2)g_0^* + 3(3g_0 - 4g_1 + g_2)g_2^* = 0. \tag{14.12}$$

Combining (14.8)–(14.12), we get $g_0 g_0^* = 0$, a contradiction. Therefore, the Leonard pair A, A^* does not have spin.

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