Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 23, pp. 66-78, January 2012

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THE CHARACTERISTIC SET WITH RESPECT TO K-MAXIMAL VECTORS OF A TREE*

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Abstract. Let T be a tree on n vertices and $L(T)$ be its Laplacian matrix. The eigenvalues and eigenvectors of T are respectively referred to those of $L(T)$. With respect to a given eigenvector Y of T, a vertex u of T is called a *characteristic vertex* if $Y[u] = 0$ and there is a vertex w adjacent to u with $Y[w] \neq 0$; an edge $e = (u, w)$ of T is called a *characteristic edge* if $Y[u]Y[w] < 0$. $\mathcal{C}(T, Y)$ denotes the *characteristic set* of T with respect to the vector Y, which is defined as the collection of all characteristic vertices and characteristic edges of T corresponding to Y.

Let $\lambda_1(T) \leq \lambda_2(T) \leq \cdots \leq \lambda_n(T)$ be the eigenvalues of a tree T on n vertices. An eigenvector is called a k-vector ($k \geq 2$) of T if the eigenvalue $\lambda_k(T)$ associated by this eigenvector satisfies $\lambda_k(T) > \lambda_{k-1}(T)$. The k-vector Y of T is called *k-maximal* if $\mathcal{C}(T, Y)$ has maximum cardinality among all k-vectors of T. In this paper, the characteristic set with respect to any k-maximal vector of a tree is investigated by exploiting the relationship between the cardinality of the characteristic set and the structure of this tree. With respect to any k-maximal vector Y of a tree T, the structure of the trees T satisfying $|\mathcal{C}(T, Y)| = k - 1 - t$ for any t ($0 \leq t \leq k - 2$) are characterized.

Key words. Laplacian matrix, Characteristic set, k-Vector, k-Maximal vector.

AMS subject classifications. 05C50, 15A15.

1. Introduction.

Let $G = (V, E)$ be a simple graph with vertex set $V = V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E = E(G)$. The *Laplacian matrix* of G is defined as $L = L(G) = D(G) - A(G)$, where $A(G)$ is the *adjacency matrix* of G and $D(G) = \mathrm{diag}\{d(v_1), d(v_2), \ldots, d(v_n)\}$, the diagonal degree matrix of G. Since $L(G)$ is positive semi-definite, its eigenvalues can be arranged as

$$0 = \lambda_1(G) \leq \lambda_2(G) \leq \cdots \leq \lambda_n(G).$$

Henceforth $\lambda_i(G)$ denotes the ith smallest eigenvalue of G. The kth smallest eigenvalue of G will be written as ${}^k\lambda(G)$ if $\lambda_k(G) > \lambda_{k-1}(G)$, and the corresponding eigenvectors will be called *k-vectors* of G.

For an eigenvector Y of a given graph G, a vertex v is called a *characteristic vertex* with respect to Y if $Y[v] = 0$ and there is a vertex w adjacent to v, such that $Y[w] \neq 0$;

*Received by the editors on September 9, 2010. Accepted for publication on December 18, 2011. Handling Editor: Bryan L. Shader.

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Supported by NSF of China (11171373, 10871230), NSF of Department of Education of Anhui (KJ2010A092), and NSF of Zhejiang (Y7080364, Y607480).

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 23, pp. 66-78, January 2012

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an edge $e = (u, w)$ is called a *characteristic edge* of G with respect to Y if $Y[u]Y[w] < 0$. We denote by $\mathcal{C}(G, Y)$ the *characteristic set* of G with respect to the vector Y, which is defined as the collection of all characteristic vertices and characteristic edges of G corresponding to Y. For convenience we relax the requirement that Y be an eigenvector of G in the definition of $\mathcal{C}(G, Y)$, and allow Y to be an arbitrary vector defined on the vertex set of G.

For a graph G, an eigenvector corresponding to the second smallest eigenvalue is called a *Fiedler vector* of G. It is known that $\lambda_2(G) > \lambda_1(G) = 0$ if and only if G is connected [5]. Thus, each Fiedler vector of a connected graph is a 2-vector. Fiedler's remarkable result [5, Theorem 3.14] on the structure of Fiedler vectors (i.e., 2-vectors) of a connected graph motivated a lot of work on the structure of eigenvectors; see, e.g., [1, 2, 7, 8, 9, 10, 11, 12, 13, 14].

Merris introduced the notion of a characteristic set and showed that $|\mathcal{C}(T, Y)| = 1$. In [11], Merris also showed that $\mathcal{C}(T, Y)$ is fixed regardless of the choice of Fiedler vectors Y of a given tree T; see [11, Theorem 2]. With respect to any Fiedler vectors Y of a given graph G, Bapat and Pati [1] investigated the cardinality of the characteristic set $\mathcal{C}(G, Y)$. In [14], Pati extended the notation the characteristic set from Fiedler vectors to 3-vectors of trees and gave a complete description of 3-vectors of a given tree. Then Fan and Gong [2] further extended the concept of characteristic set to any k-vector of a tree.

Recall that, for any 2-vector Y of a tree T, $|\mathcal{C}(T, Y)| = 1$ and $\mathcal{C}(T, Y)$ is fixed regardless of the choice of 2-vectors Y, even though the eigenspace for ${}^2\lambda(T)$ (well known as the algebraic connectivity of T [3]) is large (see [11], Theorem 2).

However, for $k \geq 3$, the characteristic set $\mathcal{C}(T, Y)$ may depend upon the choice of the k-vectors. For example, consider the tree T in Figure 1.1 (or see Figure 3.2 in [14]). One can find that Y_1, Y_2 and Y_3 are all 3-vectors of T, where
$Y_1 = [\,0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ .2638\ .4754\ .5929\ 0\ 0\ 0\ -.2638\ -.4754\ -.5929\,]^T$,
$Y_2 = [\,.5929\ .4754\ .2638\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ -.5929\ -.4754\ -.2638\ 0\ 0\ 0\ 0\,]^T$, and
$Y_3 = [\,.9098\ .7296\ .4049\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ -.0490\ -.0884\ -.1102\ -.4049\ -.7296\ -.9098\ .0490\ .0884\ .1102\,]^T$.

But one can verify that $\mathcal{C}(T, Y_1) = \{11\}$, $\mathcal{C}(T, Y_2) = \{4\}$, and $|\mathcal{C}(T, Y_3)| = \{4, 11\}$.

For a given tree T and a k-vector $\overline{Y}$ of T, $\overline{Y}$ is called *k-maximal* if $\mathcal{C}(T, \overline{Y})$ has maximum cardinality over all k-vectors of T, i.e.,

$$|\mathcal{C}(T, \overline{Y})| = \max_{Y} |\mathcal{C}(T, Y)|,$$

where the maximum is taken over all k-vectors of T (see [6]).

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A publication of the International Linear Algebra Society
Volume 23, pp. 66-78, January 2012

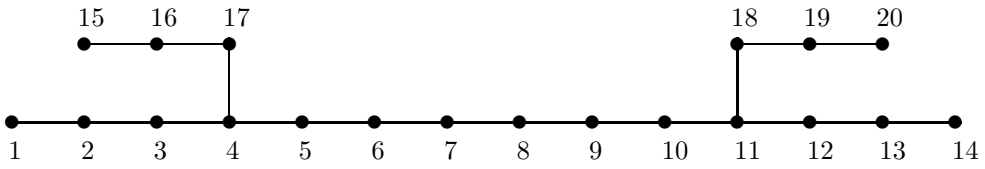

FIG. 1.1. *A tree T on 20 vertices.*

For any k-maximal vector of a tree, the following result is interesting.

PROPOSITION 1.1. ([6], Theorem 3.2) *Let T be a tree on n vertices. Suppose that both Y_1 and Y_2 are the $k(\geq 2)$-maximal vectors of T. Then*

$$\mathcal{C}(T, Y_1) = \mathcal{C}(T, Y_2).$$

Proposition 1.1 implies that for any k with $2 \leq k \leq n$, the characteristic set $\mathcal{C}(T, Y)$ is fixed regardless the choice of the k-maximal vector Y, i.e., the characteristic set is determined by the tree structure and independent of the k-maximal vectors, which is consistent with Merris' result (see [11], Theorem 2). Henceforth, to exploit the relationship between the cardinality of the characteristic set and the tree structure, we focus on studying the k-maximal vectors of trees.

With respect to any k-(maximal) vector Y of a given tree T, Fan et al. showed that [2, Corollary 2.5]

$$1 \leq |\mathcal{C}(T, Y)| \leq k - 1. \tag{1.1}$$

In particular, they also gave a characterization for trees whose characteristic set $\mathcal{C}(T, Y)$ with respect to its any k-vector Y contains exactly one element, i.e., the k-simple trees; see [2, Theorem 2.11]. Naturally, the following problem is posed:

> For a general tree T on n vertices and an arbitrary integer k $(\leq n)$, can we exploit the relationship between the cardinality of the characteristic set $\mathcal{C}(T, Y)$ with respect to its any k-maximal vector Y and the structure of such a tree T?

In this paper, we investigate the characteristic set with respect to any k-maximal vector of a given tree and consider the problem above. The rest paper is organized as follows. In Section 2, we first list several preliminary results. Then, for any k-(maximal) vector Y of a given tree T, we establish some lemmas that relate characteristic vertex and the structure for the subvector of Y. In Section 3, we study the cardinality of the characteristic set $\mathcal{C}(T, Y)$ with respect to any k-maximal vector Y

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A publication of the International Linear Algebra Society
Volume 23, pp. 66-78, January 2012

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of a tree T, and determine the structure of the trees T satisfying $|\mathcal{C}(T,Y)| = k-1-t$, where $0 \leq t \leq k-2$. In addition, examples that illustrate the occurrence of each of the case described in our theorems are given.

2. Preliminary results. Let G be a connected graph on n vertices, L, its Laplacian matrix, and Y, a vector defined on the vertex set of G. We will use following notation. For $U \subseteq V(G), W \subseteq V(G)$, denote by $L[U,W]$ the submatrix of L with rows corresponding to the vertices of U and columns corresponding to the vertices of W, if $U = W$, $L[U,W]$ is simply written as $L[U]$; and similarly, denote by $Y[U]$ the subvector of Y corresponding to the vertices of U. For convenience, we usually write $L[G_1,G_2]$ and $Y[G_1]$ instead of $L[V(G_1),V(G_2)]$ and $Y[V(G_1)]$ for subgraphs G_1, G_2 of G, respectively.

With respect to a vector Y which gives a valuation of vertices of G, a vertex v is called a *zero* (*nonzero*) *vertex* if $Y[v] = 0$ ($Y[v] \neq 0$), a component containing a nonzero vertex is called a *nonzero component*. Denote by $S(G)$ and $m_G(\lambda)$ the spectrum and the multiplicity of the eigenvalue λ of the Laplacian matrix of a graph G, respectively.

Let L be the Laplacian matrix of a graph $G = (V,E)$ and Y, an eigenvector of L corresponding to the eigenvalue λ. Then the *eigencondition* at the vertex v is the equation

$$\sum_{(i,v)\in E} L[i,v]Y[i] = (\lambda - L[v,v])Y[v].$$

An $n \times n$ matrix A will be called *acyclic* if it is symmetric and if for any mutually distinct indices $k_1, k_2, \ldots, k_s$ ($s \geq 3$) in $\{1,2,\ldots,n\}$, the equality

$$A[k_1,k_2]A[k_2,k_3]\cdots A[k_s,k_1] = 0$$

is fulfilled. Then the Laplacian matrix of a tree is acyclic. Denote by $m_A^+(\lambda)$ (respectively, $m_A^-(\lambda)$) the number of eigenvalues of the matrix A greater than (respectively, less than) λ, and let $m_A(\lambda)$ the multiplicity of λ. The following results are known from the work of Fiedler.

LEMMA 2.1. ([4], Lemma 1.12) *Let*

$$A = \begin{bmatrix} B & C \\ C^T & d \end{bmatrix}$$

be a partitioned symmetric real matrix, where C is a vector. If there exists a vector U such that $BU = 0$ and $C^TU \neq 0$. Then

$$m_A^-(0) = m_B^-(0) + 1 \quad \textit{and} \quad m_A^+(0) = m_B^+(0) + 1.$$

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LEMMA 2.2. ([4], Theorem 2.3) *Let A be an $n \times n$ acyclic matrix. Let Y be an eigenvector of A corresponding to an eigenvalue λ.*

Let there first be no "isolated" zero coordinate of Y, that is coordinate $Y[k] = 0$ such that $A[k, j]Y[j] = 0$ for all j. Then

$$m_A^+(\lambda) = a^+ + r, \quad m_A^-(\lambda) = a^- + r,$$

where r is the number of zero coordinates of Y, a^+ is the number of those unordered pairs (i, k) for which

$$A[i, k]Y[i]Y[k] < 0$$

and a^- is the number of those unordered pairs (i, k) $(i \neq k)$, for which

$$A[i, k]Y[i]Y[k] > 0.$$

If there are isolated zero coordinates of Y, M is the set of indices corresponding to such coordinates and A' the matrix obtained from A by deleting all rows and columns with indices from M, then the numbers $m_A^+(\lambda)$, $m_A^-(\lambda)$ and $m_A(\lambda)$ satisfy

$$m_A^+(\lambda) = m_{A'}^+(\lambda) + c_1, \quad m_A^-(\lambda) = m_{A'}^-(\lambda) + c_2, \quad m_A(\lambda) = m_{A'}(\lambda) + c_0,$$

where c_1, c_2 and c_0 are nonnegative integers such that

$$c_1 + c_2 + c_0 = |M|,$$

the number of elements in M.

LEMMA 2.3. ([4], Corollary 2.5) *Let A be an $n \times n$ irreducible acyclic matrix with eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. If λ_r corresponding to an eigenvector Y with all coordinates different from zero, then λ_r is simple and there are exactly $r - 1$ (unordered) pairs $(i, k), i \neq k$, for which*

$$A[i, k]Y[i]Y[k] > 0.$$

Denote by $\mathcal{C}_{\mathcal{V}}(T, Y)$ the collection of all characteristic vertices in $\mathcal{C}(T, Y)$ (or briefly $\mathcal{C}_{\mathcal{V}}$). From Lemma 2.3, for any k-vector Y of a given tree T on n vertices, either $|\mathcal{C}(T, Y)| = k - 1$ or $\mathcal{C}_{\mathcal{V}}(T, Y)$ contains characteristic vertices. Thus, as a consequence of Lemma 2.3 and (1.1), we have:

COROLLARY 2.4. *Let T be a tree on n vertices and Y, a k-vector of T with $2 \leq k \leq n$. If $|\mathcal{C}(T, Y)| \leq k - 2$, then*

$$|\mathcal{C}_{\mathcal{V}}(T, Y)| \geq 1.$$

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A publication of the International Linear Algebra Society
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In addition, the following two lemmas are needed for our discussion.

LEMMA 2.5. ([6], Lemma 2.2) *Let T be a tree on n vertices with Laplacian matrix L. Suppose that $\lambda \in S(T)$ and $v \in V(T)$. Let also Y be an eigenvector of L corresponding to λ. If $v \in \mathcal{C}(T, Y)$, then $m_{L[T-v]}(\lambda) = m_L(\lambda) + 1$.*

LEMMA 2.6. ([6], Lemma 2.3) *Let T be a tree on n vertices with Laplacian matrix L. Let also $\lambda \in S(T)$ and $v \in V(T)$. If $m_{L[T-v]}(\lambda) = m_L(\lambda) + 1$, then $Y[v] = 0$ for any eigenvector Y of L corresponding to λ.*

Let $T = (V, E)$ be a tree on n vertices with Laplacian matrix L and let λ be an nonzero eigenvalue of L. Suppose W is a subset of V and $T - W$ denotes the graph obtained from T by deleting the vertices W together with all edges incident to them. Suppose also that $M = \{T_i : i = 1, 2, \ldots, m\}$ is the collection of all components of $T - W$. According to whether or not the eigenvalue λ is contained in $S(L[T_i])$, we partition M as follows:

(a) $M_1(W; \lambda) = \{T_i : \lambda < \lambda_1(L[T_i]); T_i \in M\}$,
(b) $M_2(W; \lambda) = \{T_i : \lambda = \lambda_1(L[T_i]); T_i \in M\}$,
(c) $M_3(W; \lambda) = \{T_i : \lambda > \lambda_1(L[T_i])$ *and* $\lambda \in S(L[T_i]); T_i \in M\}$, and
(d) $M_4(W; \lambda) = \{T_i : \lambda > \lambda_1(L[T_i])$ *and* $\lambda \notin S(L[T_i]); T_i \in M\}$.

Let Y be a k-vector of a tree T, $v \in \mathcal{C}(T, Y)$ and T' a component of $T - v$. According to whether the component T' is of $M_1(v;^k \lambda)$, $M_2(v;^k \lambda)$, $M_3(v;^k \lambda)$, or $M_4(v;^k \lambda)$, we establish the following structural property for $Y[T']$.

LEMMA 2.7. *Let T be a tree with Laplacian matrix L and let Y be a k-vector of T. Suppose $v \in \mathcal{C}(T, Y)$ and T' is a component of T at v. Then*
(a) $Y[T'] = 0$ *if* $T' \in M_1(v;^k \lambda)$ *or* $T' \in M_4(v;^k \lambda)$,
(b) $Y[T']$ *is either zero, or positive, or negative if* $T' \in M_2(v;^k \lambda)$, *and*
(c) $Y[T']$ *is either zero or non-zero containing both positive entries and negative entries if* $T' \in M_3(v;^k \lambda)$.

Proof. From Lemma 2.5, we have $m_{L[T-v]}({}^k\lambda) = m_L({}^k\lambda) + 1$ as $v \in \mathcal{C}(T, Y)$. Thus, $Y[v] = 0$ by Lemma 2.6. Combining with the equation $(L - {}^k\lambda I)Y = 0$, we have

$$(L[T'] - {}^k\lambda I)Y[T'] = 0.$$

Then part (a) holds, since in that case $\det(L[T'] - {}^k\lambda I) \neq 0$. Note that $L[T']$ is an M-matrix, then the eigenvector corresponding to its least eigenvalue is either positive or negative (see, for instance, [1], Lemma 1). Consequently, part (b) follows. Part

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A publication of the International Linear Algebra Society
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(c) follows from the Perron-Frobenius theorem and the fact that the eigenvectors corresponding to the least eigenvalue are orthogonal to the eigenvectors corresponding to each other eigenvalue. □

Furthermore, if the k-vector Y is restricted to k-maximal, then Lemma 2.7(c) can be strengthened as follows:

LEMMA 2.8. *Let T be a tree with Laplacian matrix L and let Y be a k-maximal vector of T. Suppose that $v \in \mathcal{C}(T,Y)$ and T' is a component of T at v. If $T' \in M_3(v;^k\lambda)$, then $Y[T']$ has at least one positive and at least one negative entry.*

Proof. Assume to the contrary that $Y[T'] = 0$ by Lemma 2.7. By the definition of the k-maximal vector, it is sufficient to construct a k-vector W of L such that $|\mathcal{C}(T,W)| \geq |\mathcal{C}(T,Y)| + 1$.

Firstly, we have

$$\mathcal{C}(T,Y) = \mathcal{C}(T-T', Y[T-T']) + \mathcal{C}(T', Y[T']) = \mathcal{C}(T-T', Y[T-T']), \tag{2.1}$$

the last equation holds from $\mathcal{C}(T', Y[T']) = 0$ as $Y[T'] = 0$ by assumption. Note that if $^k\lambda$ is an eigenvalue of the principle submatrix $L[T']$, then there exists a nonzero vector, say Y', such that $L[T']Y' =^k \lambda Y'$. Since $T' \in M_3(v;^k\lambda)$, $^k\lambda$ is not the least eigenvalue of $L[T']$. Applying the Perron-Frobenius theorem again, we see that the vector Y' contains both positive entries and negative entries. Therefore, $|\mathcal{C}(T', Y')| \geq 1$.

Let $v' \in T'$ adjacent to v. Since $v \in \mathcal{C}(T,Y)$, there exists a vertex, say v_1, adjacent to v such that $Y[v_1] \neq 0$. Without loss of generality, suppose $Y[v_1] \neq Y'[v']$. (otherwise, we can replace Y' by $\alpha Y'$ for some nonzero scalar α ($\neq 1$).) Let T_1 be the component of T at v containing v_1 and W be the vector obtained from Y by replacing $Y[T']$ and $Y[T_1]$ by Y' and $tY[T_1]$, respectively, in which $t = (Y[v_1] - Y'[v'])/Y[v_1]$. We can readily verify that, corresponding to the vector W, the vertex v satisfies the eigencondition. Then W is also a k-vector of T and $|\mathcal{C}(T_1, Y[T_1])| = |\mathcal{C}(T_1, tY[T_1])|$ as $t \neq 0$. Henceforth, with respect to the k-vector W, we have

$$\begin{aligned} |\mathcal{C}(T,W)| &= |\mathcal{C}(T-T', W[T-T'])| + |\mathcal{C}(T', Y[T'])| \\ &\geq |\mathcal{C}(T-T', W[T-T'])| + 1 \\ &= |\mathcal{C}(T-T', Y[T-T'])| + 1 \\ &= |\mathcal{C}(T,Y)| + 1. \end{aligned}$$

The third equality follows from (2.1). □

3. The cardinality of the characteristic set with respect to k-maximal vectors of a tree. In this section, we investigate $|\mathcal{C}(T,Y)|$ for a k-maximal vector Y and characterize the structure of trees T with $|\mathcal{C}(T,Y)| = k-1-t$ for some t with $0 \leq t \leq k-2$.

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A publication of the International Linear Algebra Society
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Denote by $A \oplus B$ the direct sum of matrices A and B. We begin our discussion with the result which reveals the secret why the upper bound in (1.1) for the cardinality of the characteristic set $|\mathcal{C}(T,Y)|$ with respect to k-maximal vector Y of a given tree T is sometimes not sharp.

LEMMA 3.1. *Let T be a tree with Laplacian matrix L and let Y be a k-maximal vector of T. Suppose that $v \in \mathcal{C}(T,Y)$ and $M_4(v;{}^k\lambda) = \{T_i : i = 1,2,\ldots,p; p \geq 1\}$. Let $t = \bigoplus_{i=1}^{p} m^-_{L[T_i]}({}^k\lambda)$. Then*

$$|\mathcal{C}(T,Y)| \leq k - 1 - t.$$

Proof. For convenience, let $T' = \cup_{i=1}^{p} T_i$. Firstly, we have

$$\begin{aligned} \mathcal{C}(T,Y) &= \mathcal{C}(T-T', Y[T-T']) + \mathcal{C}(T', Y[T']) \\ &= \mathcal{C}(T-T', Y[T-T']), \end{aligned} \tag{3.1}$$

as $Y[v] = 0$ by Lemma 2.6 and $Y[T'] = 0$ by Lemma 2.7. From Lemma 2.5, we have $m_{L[T-v]}({}^k\lambda) = m_L({}^k\lambda) + 1$. Therefore,

$$m^-_{L[T-v]}({}^k\lambda) = m^-_L({}^k\lambda) - 1,$$

which implies that

$$\lambda_{k-2}(L[T-v]) < \lambda_{k-1}(L[T-v]) =^k \lambda.$$

Note that $S(L[T-v]) = S(L[T-T'-v]) \cup S(L[T'])$. Combining this with $t = \bigoplus_{i=1}^{p} m^-_{L[T_i]}({}^k\lambda)$, we have

$$\begin{aligned} \lambda_{k-2-t}(L[T-T'-v]) &< \lambda_{k-1-t}(L[T-T'-v]) \\ &= {}^k\lambda \\ &= \lambda_{k-t}(L[T-T'-v]). \end{aligned}$$

Applying Lemma 2.1 to the matrix $L[T-T']$, its principal submatrix $L[T-T'-v]$ and the vector $L[T-T', v]$, we have

$$\lambda_{k-1-t}(L[T-T']) <^k \lambda = \lambda_{k-t}(L[T-T']),$$

i.e.,

$$m^-_{L[T-T']}{}^k\lambda = k - 1 - t.$$

Furthermore, applying Lemma 2.2 to $L[T-T']$ and $Y[T-T']$, we have $m^-_L({}^k\lambda) = a^- + r$. One can find that a^- and r are exactly the number of characteristic edges and the characteristic vertices in $\mathcal{C}(T-T', Y[T-T'])$, respectively. Then

$$|\mathcal{C}(T-T', Y[T-T'])| = m^-_L({}^k\lambda) = a^- + r \leq k - 1 - t,$$

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where the last inequality follows from the fact that $L[T-T']$ has exactly

$$k-1-t = k-1-\bigoplus_{i=1}^{p} m^{-}_{L[T_i]}({}^{k}\lambda)$$

eigenvalues less than ${}^{k}\lambda$. Thus, $|\mathcal{C}(T,Y)| \leq k-1-t$ by (3.1), and the result follows. □

Applying the method above repeatedly to every element of $\mathcal{C}_{\mathcal{V}}(T,Y)$, the following result can be obtained immediately.

THEOREM 3.2. *Let T be a tree with its Laplacian matrix L and let Y be a k-maximal vector of T. Suppose that $M_4(\mathcal{C}_{\mathcal{V}};{}^{k}\lambda) = \{T_i : i = 1,2,\ldots,p; p \geq 1\}$. Let $t = \bigoplus_{i=1}^{p} m^{-}_{L[T_i]}({}^{k}\lambda)$. Then*

$$|\mathcal{C}(T,Y)| \leq k-1-t.$$

Let T be a tree on n vertices with Laplacian matrix L, and let Y be a k-vector of T. Suppose that $v \in \mathcal{C}(T,Y)$ and $T' \in M_3(v;{}^{k}\lambda)$. From Lemma 2.8, the maximum of the vector Y ensures that the subvector $Y[T']$ is nonzero. In fact, as we will see in the lemma below that such a maximum even preserves the cardinality of the characteristic set $\mathcal{C}(T',Y[T'])$ with respect to any k-maximal vector $Y[T']$ of T'; this gives the sharp upper bound in (1.1).

LEMMA 3.3. *Let T be a tree with Laplacian matrix L and let Y be a k-maximal vector of T. Suppose that $\mathcal{C}_{\mathcal{V}}(T,Y) \neq \emptyset$ and $T' \in M_3(\mathcal{C}_{\mathcal{V}};{}^{k}\lambda)$. Then*

$$|\mathcal{C}(T',Y[T'])| = m^{-}_{L[T']}({}^{k}\lambda).$$

Proof. From Lemma 2.3, it is sufficient to show that all coordinates of the subvector $Y[T']$ are nonzero. Otherwise, assume that $Y[T']$ contains zero entries, then there exists a zero vertex, say v, adjacent to some nonzero vertex. Then, with respect to the k-vector Y, such zero vertex v forms a characteristic vertex, which is a contradiction to the hypothesis that v lies in the component T' belonging to $T - \mathcal{C}_{\mathcal{V}}(T,Y)$. □

Next we show that the upper bound in Theorem 3.2 is indeed the cardinality of the characteristic set with respect to any k-maximal vector described as above.

THEOREM 3.4. *Let T be a tree with Laplacian matrix L and let Y be an arbitrary k-maximal vector of T. Suppose that $\mathcal{C}_{\mathcal{V}}(T,Y) \neq \emptyset$ and $M_4(\mathcal{C}_{\mathcal{V}};{}^{k}\lambda) = \{T_i : i = 1,2,\ldots,p; p \geq 0\}$. Let $t = \bigoplus_{i=1}^{p} m^{-}_{L[T_i]}({}^{k}\lambda)$. Then*

$$|\mathcal{C}(T,Y)| = k-1-t.$$

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Proof. Let $\mathcal{C}_{\mathcal{V}}(T,Y) = \{v_1, v_2, \ldots, v_m\}$ and $\{T_i : i = 1, 2, \ldots, p, p+1, p+2, \ldots, p+l\}$ be all components of $T - \mathcal{C}_{\mathcal{V}}$. Obviously $1 \leq m \leq |\mathcal{C}(T,Y)|$. Note that T contains no cycles. Thus each vertex $v_i \in \mathcal{C}_{\mathcal{V}}(T,Y)$ is adjacent to at least two nonzero components, and each pair of characteristic vertices is adjacent to at most one common nonzero component. Thus, $l \geq m+1$. Hence, we can take m mutually distinct nonzero components, say $T_1', T_2', \ldots, T_m'$, such that each component contains a vertex, labeling as $v_1', v_2', \ldots, v_m'$, respectively, such that, for each i, $Y[v_i'] \neq 0$ and $v_i' \in T_i'$ adjacent to v_i.

For each i $(1 \leq i \leq m)$, from Lemmas 2.5 and 2.6 we have $Y[v_i] = 0$. Thus,

$$L[T_i']Y[T_i'] =^k \lambda Y[T_i'], \text{ for each } i.$$

Write $T - \mathcal{C}_{\mathcal{V}}(T,Y)$ as T' for simplicity. Let $Y_1 = [Y[T_1']^T \ 0 \ 0 \ \cdots \ 0]^T$, where the zeros are appended so that $(L[T'] -^k \lambda I)Y_1 = 0$. One can readily verify that $L[v_1, T']Y_1 = L[v_1, T_1']Y[T_1'] = L[v_1, v_1']Y_1[v_1'] \neq 0$, since the vector $L[v_1, T_1']$ has exactly one nonzero coordinate $L[v_1, v_1']$ and $Y_1[v_1'] \neq 0$. Thus, applying Lemma 2.1,

$$m^-_{L[T' \cup \{v_1\}]}({}^k\lambda) = m^-_{L[T']}({}^k\lambda) + 1.$$

Further, let $Y_2 = [Y[T_2']^T \ 0 \ 0 \ \cdots \ 0]^T$, where the zeros are appended so that $(L[T' \cup \{v_1\}] -^k \lambda I)Y_2 = 0$. Thus, by a similar discussion, $L[v_2, T' \cup \{v_1\}]Y_2 = L[v_1, T_2']Y[T_2'] = L[v_2, v_2']Y_2[v_2'] \neq 0$. Applying Lemma 2.1 again, we have

$$m^-_{L[T' \cup \{v_1, v_2\}]}({}^k\lambda) = m^-_{L[T' \cup \{v_1\}]}({}^k\lambda) + 1 = m^-_{T'}({}^k\lambda) + 2.$$

Using the above operation repeatedly, we have

$$\begin{aligned} m^-_L({}^k\lambda) &= m^-_{L[T' \cup \mathcal{C}_{\mathcal{V}}]}({}^k\lambda) \\ &= m^-_{L[T' \cup \{v_1, \ldots, v_{m-1}\}]}({}^k\lambda) + 1 \\ &\ \ \vdots \\ &= m^-_{L[T' \cup \{v_1\}]}({}^k\lambda) + m - 1 \\ &= m^-_{L[T']}({}^k\lambda) + m. \end{aligned}$$

Thus,

$$m^-_{L[T']}({}^k\lambda) = m^-_L({}^k\lambda) - m.$$

Consequently,

$$\lambda_{k-1-m}(L[T']) <^k \lambda \ \text{ and } \ \lambda_{k-m}(L[T']) =^k \lambda,$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
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since ${}^k\lambda$ is an eigenvalue of $L[T']$ by Lemma 2.5.

Without loss of generality, suppose $M_3(\mathcal{C}_\mathcal{V};{}^k\lambda) = \{T_1^*, T_2^*, \ldots, T_q^*\}$. For each $i (i = 1, 2, \ldots, q)$, let $t_i = m^-_{L[T_i^*]}({}^k\lambda)$. From Lemma 3.3, $|\mathcal{C}(T_i^*, Y[T_i^*])| = m^-_{L[T_i^*]}({}^k\lambda) = t_i$ holds for each i. Then

$$\sum_{i=1}^{q} |\mathcal{C}(T_i^*, Y[T_i^*])| = \sum_{i=1}^{q} t_i =: t^*.$$

Hence,

$$\begin{aligned} |\mathcal{C}(T,Y)| &= |\mathcal{C}(T - \mathcal{C}_\mathcal{V}, Y[T - \mathcal{C}_\mathcal{V}])| + |\mathcal{C}_\mathcal{V}| \\ &= \textstyle\sum_{i=1}^{l} |\mathcal{C}(T_i, Y[T_i])| + m \\ &= \textstyle\sum_{i=1}^{q} |\mathcal{C}(T_i^*, Y[T_i^*])| + m \\ &= t^* + m. \end{aligned}$$

On the other hand, note that $S(T') = \bigcup_{i=1}^{p+l} S(T_i)$ and each eigenvalue corresponding to the component being of $M_1(\mathcal{C}_\mathcal{V};{}^k\lambda)$ or $M_2(\mathcal{C}_\mathcal{V};{}^k\lambda)$ is no less than ${}^k\lambda$, then $t + t^* = k - 1 - m$. Hence,

$$|\mathcal{C}(T,Y)| = t^* + m = k - 1 - t. \quad \square$$

Putting Theorem 3.4 together with Lemma 2.3, we can give the characterization for the structure of the trees with any possible cardinality of the characteristic set with respect to its k-maximal vector.

THEOREM 3.5. *Let T be a tree with Laplacian matrix L, and let Y be an arbitrary k-maximal vector of T. Then $|\mathcal{C}(T,Y)| = k - 1$ if and only if either every coordinate of Y different from zero, or $M_4(v;{}^k\lambda) = \emptyset$ holds for each $v \in \mathcal{C}(T,Y)$.*

THEOREM 3.6. *Let T be a tree with Laplacian matrix L, and let Y be an arbitrary k-maximal vector of T. Suppose t is an arbitrary integer with $1 \le t \le k-2$. Then $|\mathcal{C}(T,Y)| = k - 1 - t$ if and only if $\mathcal{C}_\mathcal{V}(T,Y) \neq \emptyset$ and $M_4(\mathcal{C}_\mathcal{V};{}^k\lambda) = \{T_i : i = 1, 2, \ldots, p (p \ge 1)\}$, where $t = \bigoplus_{i=1}^{p} m^-_{L[T_i]}({}^k\lambda)$.*

Below we give an example to show the occurrence of each of the case described in the above theorems.

EXAMPLE 3.7. Let T be a tree on $n = 2m + 4p + q + 1$ $(p + q \ge 2)$ vertices obtained from a star on $m + p + q + 1$ vertices by appending m pendent edges to m pendent vertices and p paths with length 3 to other p pendent vertices, respectively, see Figure 3.1.

By a little calculation, we have ${}^k\lambda(T) = 1$ with multiplicity $p+q-1$, $M_1(u;1) = \emptyset$, $M_2(u;1) = \{T[v_{4p+2m+i}] : i = 1, 2, \ldots, q\}$, $M_3(u;1) = \{T[v_i, v_{p+i}, v_{2p+i}, v_{3p+i}] : i =$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 23, pp. 66-78, January 2012

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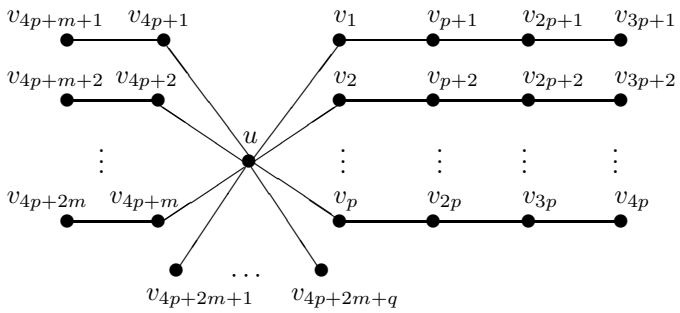

FIG. 3.1. *A tree on $2m + 4p + q + 1$ vertices with $p + q \geq 2$.*

$1, 2, \ldots, p\}$, and $M_4(u; 1) = \{T[v_{4p+i}, v_{4p+m+i}] : i = 1, 2, \ldots, m\}$, where $k = m+p+2$ and $T[S]$ is the subgraph of T induced by its vertex subset S. Then the k-vector Y has the following partitioned form:

$$Y = [t_1 X_1^T \ t_2 X_2^T \cdots \ t_p X_p^T \ \ W_1^T \ \cdots W_m^T \quad s_1 \cdots s_q \ \ 0],$$

where $X_i = [-1 \ -1 \ 0 \ 1]^T$ for $i = 1, 2, \ldots, p$, $W_l = [0 \ 0]^T$ for $l = 1, 2, \ldots, m$, and each t_i (or s_j) is real such that the vertex u satisfies eigencondition.

One can see that, for each k-vector Y, $t_i X_i$ is either zero or non-zero containing both positive entries and negative entries for each i, s_j is either zero, or positive, or negative for each j, and $W_l = 0$ for each l, which is consistent with Lemma 2.7. On the other hand, from the partitioned form of Y, $u \in \mathcal{C}(T, Y)$ if some t_i (or s_j) is nonzero. Moreover, we have $|\mathcal{C}(T[v_i, v_{p+i}, v_{2p+i}, v_{3p+i}], Y[v_i, v_{p+i}, v_{2p+i}, \quad v_{3p+i}])| = |\mathcal{C}(T[v_i, v_{p+i}, v_{2p+i}, v_{3p+i}], t_i X_i)|$ is either 1 or 0 according to the real t_i is nonzero or not. Thus,

$$\begin{array}{rcl} |\mathcal{C}(T, Y)| & = & 1 + \sum_{i=1}^{p} |\mathcal{C}(L[T[v_i, v_{p+i}, v_{2p+i}, v_{3p+i}]], Y[v_i, v_{p+i}, v_{2p+i}, v_{3p+i}])| \\ & \leq & 1 + p, \end{array}$$

from which we have that Y is k-maximal if $t_i \neq 0$ for each i, which is consistent with Lemma 2.8. One can also find that, with respect to any k-maximal vector Y, $|\mathcal{C}(T, Y)| = 1 + p$ regardless of the choice of the integer m. Hence, $|\mathcal{C}(T, Y)| = 1 + p = k - 1$ if $m = 0$ (in such a case $M_4(u; 1) = \emptyset$), and $|\mathcal{C}(T, Y)| = 1 + p \leq k - 2$ otherwise, which is consistent with Lemma 3.1.

Moreover, if Y is k-maximal, then $\mathcal{C}_{\mathcal{V}}(T, Y) = \{u\} \cup \{v_{2p+i} : i = 1, 2, \ldots, p\}$, and $M_1(\mathcal{C}_{\mathcal{V}}; 1) = M_3(\mathcal{C}_{\mathcal{V}}; 1) = \emptyset$, $M_2(\mathcal{C}_{\mathcal{V}}; 1) = \{T[v_{4p+2m+i}] : i = 1, 2, \ldots, q\} \cup \{T[v_{3p+j}] : j = 1, 2, \ldots, p\} \cup \{T[v_l, v_{p+l}] : l = 1, 2, \ldots, p\}$, $M_4(\mathcal{C}_{\mathcal{V}}; 1) = \{T[v_{4p+i}, v_{4p+m+i}] : i = 1, 2, \ldots, m\}$, which is consistent with Theorems 3.4, 3.5 and 3.6.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 23, pp. 66-78, January 2012

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