



THE NORM OF THE SYMMETRIC PRODUCT OF TENSORS AND ITS APPLICATIONS TO PERMANENTS*

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Abstract. Let A be an $np \times np$ positive semidefinite matrix partitioned into $p \times p$ blocks A_{ij} , and P be the $n \times n$ matrix with $p_{ij} = \text{per}(A_{ij})$. Marcus conjectures in 1965 that $\text{per}(A) \geq \text{per}(P)$, which remains open in general. In this paper, we prove a generalized Neuberger identity, which provides an expression for the norm of the symmetric product of tensors. The proof involves a careful analysis of double cosets of symmetric groups with respect to Young subgroups. Then, we leverage it to the Marcus's conjecture for block tridiagonal matrices.

Key words. Double cosets, Neuberger identity, Norm, Permanent.

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1. Introduction. Let $M_n(\mathbb{C})$ be the set of all $n \times n$ complex matrices, and let S_n be the symmetric group of degree n . We assume $n \geq 2$. For an $n \times n$ matrix $A = [a_{ij}]$, the permanent of A , denoted by $\text{per}(A)$, is

$$\sum_{\sigma \in S_n} \prod_{i=1}^n a_{i\sigma(i)}.$$

The permanent function is a significant matrix function, and it is a natural consequence of the examination of the symmetric tensors in multilinear algebra (e.g., [7]). The term was originally introduced in the 1800s (e.g., [9]), and it continues to be an active topic in research.

Several comprehensive surveys on permanents, encompassing numerous unresolved conjectures and problems, include the monograph [9] by Minc, along with subsequent articles [10] and [11]. These are followed by Cheon and Wanless's work [1] and Zhang's paper [21].

One of the open problems involving the permanent is the Marcus per-in-per conjecture [6]. Let A be a positive semidefinite $np \times np$ matrix partitioned as $A = (A_{ij})$ in which each A_{ij} is $p \times p$. Let P be the $n \times n$ matrix whose (i, j) entry is $\text{per}(A_{ij})$. Then, Marcus conjectured that

$$\text{per}(A) \geq \text{per}(P),$$

and the equality occurs if and only if all $A_{ij} = 0$, $i \neq j$.

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The case of $n = 2$ was proved by Lieb [5] in 1966 and Djokovic [2] in 1969. The weaker inequality

$$\text{per}(A) \geq \frac{1}{n!} \text{per}(P),$$

was proved by Pate [14] in real case. Also in real case, Pate [15], in 1982, demonstrated that the Marcus per-in-per conjecture holds true for every n when p is adequately large.

According to [1] and [21], no advancement in this conjecture has been reported since last Pate's result.

This paper seeks to demonstrate the validity of this conjecture for $np \times np$ positive semidefinite matrices with entries in $\mathbb{C}$, the complex field that are partitioned into $p \times p$ blocks, resulting in a block tridiagonal matrix. Our proof is anchored in an extension of an equality of the norm of symmetric product of two tensors derived from the classic Neuberger identity [12]. This paper is structured as follows: The next section presents the terminology and additional results relevant to the subject. The third section will concentrate on the double cosets of S_n associated with the Young subgroups, with the main objective of calculating the number of elements in each double coset. The fourth section focuses on demonstrating an equality for the norm of the symmetric product of tensors. This equality generalizes a well-known identity for the norm of the symmetric product of two tensors, which is derived from the Neuberger identity. The last section uses this new equality to verify the Marcus per-in-per conjecture in this specific case.

From now on, we denote by $\mathcal{H}_m$ the set of all $m \times m$ positive semidefinite matrices with entries in $\mathbb{C}$. Given a finite set X , by $|X|$, we denote the cardinality of X , and if $x \in \mathbb{C}$, $\bar{x}$ denotes the conjugate of x . Let $m, n_1, n_2, \dots, n_k$ be nonnegative integers such that $m = n_1 + \dots + n_k$. We denote by

$$\begin{pmatrix} m \\ n_1 \quad \dots \quad n_k \end{pmatrix},$$

the integer

$$\frac{m!}{n_1! \dots n_k!}.$$

Given a matrix A , A^T denotes the transpose of A .

2. Preliminaries. Let V be a finite dimensional complex vector space with inner product $\langle \cdot, \cdot \rangle$, and let m be a positive integer. We denote by $\otimes^m V$ the m^{th} -tensor power of V and by $\vee^m V$ the m^{th} -symmetric power of V . We know that

$$\vee^m V = T_m(\otimes^m V),$$

where T_m is the orthogonal projection

$$T_m := \frac{1}{m!} \sum_{\sigma \in S_m} \mathcal{P}(\sigma).$$

Here, $\mathcal{P}(\sigma)$ denotes the unique endomorphism of $\otimes^m V$ such that

$$(2.1) \quad \mathcal{P}(\sigma)(v_1 \otimes \dots \otimes v_m) = v_{\sigma^{-1}(1)} \otimes \dots \otimes v_{\sigma^{-1}(m)},$$

for all $v_1, \dots, v_m \in V$.

By $\vee V$, we denote the symmetric algebra of V , that is,

$$\vee V = \oplus_{m \geq 0} \vee^m V.$$

If $z_1 \in \vee^m V$ and $z_2 \in \vee^n V$, we define their product as

$$(2.2) \quad z_1 \vee z_2 = T_{m+n}(z_1 \otimes z_2) \in \vee^{m+n} V.$$

The following theorem states that $\vee^m V$ is spanned by some special elements of $\otimes^m V$. The proof can be found in [7], page 181.

THEOREM 2.1. [7] *Let V be a finite-dimensional vector space over $\mathbb{C}$, and let m be a positive integer. Then, $\vee^m V$ is spanned by tensors of the form $v \otimes v \otimes \dots \otimes v$, $v \in V$.*

It is also well known that the inner product of V induces an inner product on $\otimes^m V$ such that

$$\langle v_1 \otimes \dots \otimes v_m, w_1 \otimes \dots \otimes w_m \rangle = \prod_{i=1}^m \langle v_i, w_i \rangle,$$

for all $v_i, w_i \in V$, $i = 1, \dots, m$.

It is well known that $A = [a_{i,j}] \in \mathcal{H}_m$ if and only if there exist $x_1, \dots, x_m \in V$ such that

$$\langle x_i, x_j \rangle = a_{i,j}.$$

As a consequence,

$$(2.3) \quad \langle T_m(x_1 \otimes \dots \otimes x_m), x_1 \otimes \dots \otimes x_m \rangle = \frac{1}{m!} \text{per}(A).$$

PROPOSITION 2.2. [7] *Let V be a unitary space and $v_1, \dots, v_n \in V$. Then,*

$$T_n(v_1 \otimes \dots \otimes v_n) = 0,$$

if and only if $v_i = 0$ for some i .

By Proposition 2.2 and (2.3), we have

PROPOSITION 2.3. *Let $A \in \mathcal{H}_n$. Then, $\text{per}(A) = 0$ if and only if A has a zero row.*

Let m and k be two nonnegative integers and let $0 \leq r \leq \min\{m, k\}$. The Neuberger map $\otimes_r$, introduced by Neuberger in [12], is the unique bilinear map

$$\otimes_r : (\otimes^m V) \times (\otimes^k V) \rightarrow (\otimes^{m-r} V) \otimes (\otimes^{k-r} V),$$

such that

$$\begin{aligned} \otimes_r(v_1 \otimes \dots \otimes v_m, w_1 \otimes \dots \otimes w_k) &:= (v_1 \otimes \dots \otimes v_m) \otimes_r (w_1 \otimes \dots \otimes w_k) \\ &= \left(\prod_{i=0}^{r-1} \langle v_{m-i}, w_{k-i} \rangle \right) (v_1 \otimes \dots \otimes v_{m-r} \otimes w_1 \otimes \dots \otimes w_{k-r}), \end{aligned}$$

for all $v_i, w_j \in V$, $i = 1, \dots, m$ and $j = 1, \dots, k$.

The Neuberger identity [12] is

$$\binom{m+k}{k} \langle A \vee B, C \vee D \rangle = \sum_{i=0}^{\min\{m,k\}} \binom{m}{i} \binom{k}{i} \langle A \otimes_i D, C \otimes_i B \rangle,$$

for all $A, C \in \vee^m V$ and $B, D \in \vee^k V$.

Setting $A = C$ and $B = D$, we obtain

$$(2.4) \quad \binom{m+k}{k} \|A \vee B\|^2 = \sum_{i=0}^{\min\{m,k\}} \binom{m}{i} \binom{k}{i} \|A \otimes_i B\|^2,$$

which immediately implies Neuberger's inequality

$$\binom{m+k}{k} \|A \vee B\|^2 \geq \|A\|^2 \|B\|^2.$$

The equality (2.4) is fundamental in proving many results involving the permanent (see [13, 16, 17]). One of our primary goals is to extend the identity (2.4) to the symmetric product of more than two tensors. In order to accomplish this, it is necessary to look into the double cosets of S_m that are associated with Young subgroups. Next, we will outline definitions and well-known results on double cosets:

DEFINITION 2.4. *Let H and K be two subgroups of a group G . The subset of G of elements of the form $h x k$, $h \in H, k \in K$, and $x \in G$ is called double coset of G with respect to H , K , and x .*

In the next proposition, we list some elementary properties of double cosets.

PROPOSITION 2.5. [8] *Let H and K be two subgroups of the group G and let g be an element of G .*

1. *There is a unique double coset containing g ; the double coset HgK ;*
2. *Two distinct double coset never have a common element;*
3. *G is the union of disjoint double coset with respect of H and K ;*
4. *A double coset, with respect to H and K , is a union of right cosets of H as well of the left cosets of K ;*
5. *The double coset HgK contains exactly $\frac{|K|}{|K \cap g^{-1}Hg|}$ right cosets of H .* □

Let $(n_1, \dots, n_t)$ be a sequence of nonnegative integers satisfying

$$(2.5) \quad \sum_{i=1}^t n_i = m.$$

A subgroup of S_m that is isomorphic to the direct product

$$S_{n_1} \times \dots \times S_{n_t},$$

is called a *Young subgroup* of S_m .

Some results regarding the double cosets of S_m in relation to Young subgroups H and K are provided in [3] or [4]. A crucial outcome for obtaining our results is that if H and K are the stabilizer subgroups (see the next section) of the partitions $(A_1, \dots, A_t)$ and $(B_1, \dots, B_s)$, of $\{1, \dots, m\}$, respectively, then the double coset $H\sigma K$, $\sigma \in S_m$ is characterized by the $t \times s$ matrices of nonnegative integers given by

$$|A_i \cap \sigma(B_j)|, \quad 1 \leq i \leq t \quad \text{and} \quad 1 \leq j \leq s.$$

Thus, $\pi \in S_m$ belongs to the double coset $H\sigma K$ if and only if

$$|A_i \cap \sigma(B_j)| = |A_i \cap \pi(B_j)|, \quad 1 \leq i \leq t \quad \text{and} \quad 1 \leq j \leq s.$$

Consequently, there is a bijective correspondence between the set of all double cosets, with respect to H and K , and the set of $t \times s$ matrices, over the set of the nonnegative integers, with row sums $(|A_1|, \dots, |A_t|)$ and column sums $(|B_1|, \dots, |B_s|)$. This correspondence looks to be fundamental for our main results.

3. Double cosets related with Young subgroups. The primary objective of this section is to derive a formula for calculating the number of elements in a double coset of S_m of the form $H\gamma H$, where H is a Young subgroup of S_m . This formula is essential for the establishment of our extension of the Neuberger identity. Consequently, the next result is the main result of this section.

THEOREM 3.1. *Let $m, n_1, n_2, \dots, n_t$ be nonnegative integers such that $m = n_1 + \dots + n_t$, and let H be the Young subgroup of S_m , $H = S_{n_1} \times \dots \times S_{n_t}$. Let $\gamma \in S_m$, and assume that $H\gamma H$ is described by the matrix of nonnegative integers $Z = [z_{i,j}]$. Then,*

$$|H\gamma H| = \begin{pmatrix} n_1 & & \\ z_{1,1} & \dots & z_{1,t} \end{pmatrix} \dots \begin{pmatrix} n_t & & \\ z_{t,1} & \dots & z_{t,t} \end{pmatrix} n_1! \dots n_t!.$$

The proof of this theorem is derived from a sequence of lemmas, and the reader may locate it at the end of this section. Now, we introduce the notion of a partition of the set $\{1, \dots, m\}$.

DEFINITION 3.2. *Let m be a positive integer. A partition of the set $\{1, \dots, m\}$ is a sequence $\mathcal{P} = (A_1, \dots, A_t)$ of subsets of $\{1, \dots, m\}$, such that*

1. $A_i \neq \emptyset$, $i = 1, \dots, t$;
2. $A_i \cap A_j = \emptyset$, $i, j = 1, \dots, t$, $i \neq j$;
3. $\bigcup_{i=1}^t A_i = \{1, \dots, m\}$.

If $\mathcal{P} = (A_1, \dots, A_t)$ is a partition of $\{1, \dots, m\}$ such that $|A_i| = n_i$, $i = 1, \dots, t$, then $\mathcal{P}$ is called a $(n_1, \dots, n_t)$ -partition of $\{1, \dots, m\}$.

Let m be a positive integer and let $(n_1, \dots, n_t)$ be a sequence of positive integers satisfying (2.5). It is well known (see [20]) that the number of distinct $(n_1, \dots, n_t)$ -partitions of the set $\{1, \dots, m\}$ is

$$\begin{pmatrix} m & & \\ n_1 & \dots & n_t \end{pmatrix} := \frac{m!}{n_1! \dots n_t!}.$$

Let $\mathcal{P} = (A_1, \dots, A_t)$ be a partition of $\{1, \dots, m\}$. The set

$$\mathcal{R} = \{\sigma \in S_m : \sigma(A_i) = A_i, i = 1, \dots, t\},$$

is a subgroup of S_m called the *stabilizer subgroup* of $\mathcal{P}$.

It is easy to conclude that the stabilizer subgroup of an $(n_1, \dots, n_t)$ -partition of $\{1, \dots, m\}$ is isomorphic to $S_{n_1} \times \dots \times S_{n_t}$. Hence, the stabilizer subgroup of a $(n_1, \dots, n_t)$ -partition of $\{1, \dots, m\}$ is a Young subgroup of S_m .

Let $(n_1, \dots, n_t)$ be a sequence of nonnegative integers satisfying (2.5), and let

$$\mathcal{R} = S_{n_1} \times \dots \times S_{n_t}.$$

From now on, we focus on the double cosets of S_m of the form $\mathcal{R}\gamma\mathcal{R}$, where $\gamma \in S_m$. As previously mentioned, we will derive a formula for the number of members in such double cosets.

Let $\mathcal{P} = (A_1, \dots, A_t)$ be an $(n_1, \dots, n_t)$ -partition and let $\mathcal{R}$ be the stabilizer subgroup of $\mathcal{P}$. Consider the double coset $\mathcal{R}\gamma\mathcal{R}$. We will use the following notations:

$$\mathcal{Z}_{i,j} := A_i \cap \gamma(A_j), \quad \mathcal{Z}_{i,j}^{(-1)} := A_i \cap \gamma^{-1}(A_j), \quad z_{i,j} := |A_i \cap \gamma(A_j)|, \quad z_{i,j}^{(-1)} := |A_i \cap \gamma^{-1}(A_j)|,$$

and $S_{\mathcal{Z}_{i,j}}$ denotes the subgroup of S_m of all permutations that fix every element outside $\mathcal{Z}_{i,j}$, for all $i, j = 1, \dots, t$. The definition of $S_{\mathcal{Z}_{i,j}^{(-1)}}$, $i, j = 1, \dots, t$, is similar.

LEMMA 3.3. *The following statements are true:*

1. $A_i = (A_i \cap \gamma(A_1)) \cup \dots \cup (A_i \cap \gamma(A_t))$, for all $i = 1, \dots, t$.
2. $\gamma(A_i) = (A_1 \cap \gamma(A_i)) \cup \dots \cup (A_t \cap \gamma(A_i))$, for all $i = 1, \dots, t$.
3. $(A_i \cap \gamma(A_r)) \cap (A_i \cap \gamma(A_s)) = \emptyset$, for all $r, s = 1, \dots, t$, $r \neq s$.
4. $S_{\mathcal{Z}_{1,1}} \times S_{\mathcal{Z}_{1,2}} \times \dots \times S_{\mathcal{Z}_{1,t}} \times \dots \times S_{\mathcal{Z}_{t,t}} \subseteq \mathcal{R}$.
5. $S_{\mathcal{Z}_{1,1}^{(-1)}} \times S_{\mathcal{Z}_{1,2}^{(-1)}} \times \dots \times S_{\mathcal{Z}_{1,t}^{(-1)}} \times \dots \times S_{\mathcal{Z}_{t,t}^{(-1)}} \subseteq \mathcal{R}$.

Proof. Easy from the definitions. □

LEMMA 3.4. *Let m be a positive integer, let $(n_1, \dots, n_t)$ be a sequence of positive integers satisfying (2.5), let $\mathcal{P} = (A_1, \dots, A_t)$ be $(n_1, \dots, n_t)$ -partition of $\{1, \dots, m\}$, let $\mathcal{R}$ be the stabilizer subgroup of $\mathcal{P}$ and let $\gamma \in S_m$. The following statements are true:*

1. $\mathcal{Z}_{i,j} \neq \emptyset$ if and only if $\mathcal{Z}_{j,i}^{(-1)} \neq \emptyset$;
2. $\gamma^{-1}(\mathcal{Z}_{i,j}) = \mathcal{Z}_{j,i}^{(-1)}$.

Proof. Assume that $\mathcal{Z}_{i,j} \neq \emptyset$ and let $c \in \mathcal{Z}_{i,j}$. Then,

$$c \in A_i \cap \gamma(A_j),$$

that is

$$c \in A_i \quad \text{and} \quad c \in \gamma(A_j).$$

Therefore,

$$\gamma^{-1}(c) \in \gamma^{-1}(A_i) \quad \text{and} \quad \gamma^{-1}(c) \in A_j,$$

so

$$\gamma^{-1}(c) \in A_j \cap \gamma^{-1}(A_i) = \mathcal{Z}_{j,i}^{(-1)}.$$

Similarly, we prove that if $\mathcal{Z}_{j,i}^{(-1)} \neq \emptyset$ then $\mathcal{Z}_{i,j} \neq \emptyset$, and the proof of the first statement is complete.

To prove the second statement, we consider two cases: $\mathcal{Z}_{i,j} = \emptyset$ and $\mathcal{Z}_{i,j} \neq \emptyset$. If $\mathcal{Z}_{i,j} = \emptyset$, the result comes from the first statement. Assume now that $\mathcal{Z}_{i,j} \neq \emptyset$. Then,

$$\begin{aligned}\gamma^{-1}(\mathcal{Z}_{i,j}) &= \gamma^{-1}(A_i \cap \gamma(A_j)) \\ &= \gamma^{-1}(A_i) \cap \gamma^{-1}\gamma(A_j) \\ &= \mathcal{Z}_{j,i}^{(-1)}.\end{aligned}$$

□

REMARK 3.5. *By this lemma, we conclude that*

$$z_{i,j} = z_{j,i}^{(-1)} \quad i, j = 1, \dots, t.$$

Therefore, if $\sigma \in S_m$ is in a double coset described by a matrix Z , then σ^{-1} is in a double coset characterized by the matrix Z^T , which is the transpose of Z .

LEMMA 3.6. $\mathcal{R} \cap \gamma^{-1}\mathcal{R}\gamma = S_{\mathcal{Z}_{1,1}^{(-1)}} \times S_{\mathcal{Z}_{1,2}^{(-1)}} \times \dots \times S_{\mathcal{Z}_{t,t}^{(-1)}}.$

Proof. Let $\mu \in \mathcal{R} \cap \gamma^{-1}\mathcal{R}\gamma$. Then, $\mu \in \mathcal{R}$ and there exists $\tau \in \mathcal{R}$ such that

$$\mu = \gamma^{-1}\tau\gamma.$$

In order to prove that $\mu \in S_{\mathcal{Z}_{1,1}^{(-1)}} \times S_{\mathcal{Z}_{1,2}^{(-1)}} \times \dots \times S_{\mathcal{Z}_{t,t}^{(-1)}}$, we first show that

$$\tau(\mathcal{Z}_{i,j}) = \mathcal{Z}_{i,j}, \quad i, j = 1, \dots, t.$$

Assume that there are $i, j \in \{1, \dots, t\}$ such that

$$\tau(\mathcal{Z}_{i,j}) \neq \mathcal{Z}_{i,j}.$$

Then, there exists $a \in \mathcal{Z}_{i,j}$ such $\tau(a) \notin \mathcal{Z}_{i,j}$. Since $\mathcal{Z}_{i,j} = A_i \cap \gamma(A_j)$, we have $a \in A_i$ and since $\tau \in \mathcal{R}$, $\tau(a) \in A_i$. Hence, there exists $r \in \{1, \dots, t\}$, $r \neq j$, such

$$\tau(a) \in \mathcal{Z}_{i,r}.$$

This means that

$$\tau(a) \in A_i \cap \gamma(A_r),$$

and so, there exists $c_r \in A_r$ such

$$\gamma(c_r) = \tau(a).$$

By the fact that

$$a \in A_i \cap \gamma(A_j),$$

there is $c_j \in A_j$ such

$$\gamma(c_j) = a.$$

Then, we obtain

$$\mu(c_j) = \gamma^{-1}\tau\gamma(c_j) = \gamma^{-1}\tau(a) = \gamma^{-1}\gamma(c_r) = c_r.$$

But this is impossible because $\mu \in \mathcal{R}$, $c_j \in A_j$, $c_r \in A_r$ and $j \neq r$. Now we are in position to prove that

$$\mu \in S_{\mathcal{Z}_{1,1}}^{(-1)} \times S_{\mathcal{Z}_{1,2}}^{(-1)} \times \dots \times S_{\mathcal{Z}_{t,t}}^{(-1)}.$$

Let $i, j \in \{1, \dots, t\}$. Then,

$$\mu(\mathcal{Z}_{i,j}^{(-1)}) = \gamma^{-1}\tau\gamma(\mathcal{Z}_{i,j}^{(-1)}) = \gamma^{-1}\tau(\mathcal{Z}_{j,i}) = \gamma^{-1}(\mathcal{Z}_{j,i}) = \mathcal{Z}_{i,j}^{(-1)},$$

and so

$$\mathcal{R} \cap \gamma^{-1}\mathcal{R}\gamma \subseteq S_{\mathcal{Z}_{1,1}}^{(-1)} \times S_{\mathcal{Z}_{1,2}}^{(-1)} \times \dots \times S_{\mathcal{Z}_{t,t}}^{(-1)}.$$

Conversely, let

$$\mu \in S_{\mathcal{Z}_{1,1}}^{(-1)} \times S_{\mathcal{Z}_{1,2}}^{(-1)} \times \dots \times S_{\mathcal{Z}_{t,t}}^{(-1)}.$$

By Lemma 3.3, $\mu \in \mathcal{R}$ and so it remains to prove that

$$\mu \in \gamma^{-1}\mathcal{R}\gamma.$$

We have, for all $i, j \in \{1, \dots, t\}$,

$$\gamma\mu\gamma^{-1}(\mathcal{Z}_{i,j}) = \gamma\mu(\mathcal{Z}_{j,i}^{(-1)}) = \gamma(\mathcal{Z}_{j,i}^{(-1)}) = \mathcal{Z}_{i,j},$$

and so

$$\gamma\mu\gamma^{-1} \in S_{\mathcal{Z}_{1,1}} \times S_{\mathcal{Z}_{1,2}} \times \dots \times S_{\mathcal{Z}_{t,t}}.$$

Hence,

$$\gamma\mu\gamma^{-1} \in \mathcal{R},$$

that is

$$\mu \in \gamma^{-1}\mathcal{R}\gamma,$$

and the proof is complete. $\square$

By Proposition 2.5, the number of the right cosets of $\mathcal{R}$ in the double coset $\mathcal{R}\gamma\mathcal{R}$ is

$$\frac{|\mathcal{R}|}{|\mathcal{R} \cap \gamma^{-1}\mathcal{R}\gamma|},$$

which, by Lemma 3.6, is equal to

$$\begin{aligned} \frac{|A_1|! \dots |A_t|!}{z_{11}! \dots z_{1t}! \dots z_{t1}! \dots z_{tt}!} &= \frac{n_1!}{z_{11}! \dots z_{1t}!} \dots \frac{n_t!}{z_{t1}! \dots z_{tt}!} \\ &= \begin{pmatrix} n_1 & & \\ z_{11} & \dots & z_{1t} \end{pmatrix} \dots \begin{pmatrix} n_t & & \\ z_{t1} & \dots & z_{tt} \end{pmatrix}. \end{aligned}$$

Now we can easily prove the main result of this section

Proof of Theorem 3.1. By the previous lemma, the number of right cosets of S_m contained in the double coset $\mathcal{R}\gamma\mathcal{R}$ is

$$\begin{pmatrix} n_1 & & \\ z_{1,1} & \dots & z_{1,t} \end{pmatrix} \cdots \begin{pmatrix} n_t & & \\ z_{t,1} & \dots & z_{t,t} \end{pmatrix}.$$

As $\mathcal{R}\gamma\mathcal{R}$ is a union of right cosets and since each right coset contains $n_1! \dots n_t!$ elements, the result follows. $\square$

4. The norm of symmetric product of tensors. This section presents a generalization of the identity referenced as (2.4), which derives from the Neuberger identity. In the literature, there are several generalizations of this identity (see [18, 19]); however, these generalizations only address the case of two symmetric tensors. Our goal is to extend equality (2.4) to include more than two tensors. To accomplish this, it is necessary to generalize the Neuberger maps, $\otimes_r$. Let $n_1, \dots, n_k$ be positive integers such that $n_1 + \dots + n_k = m$ and $\mathcal{D}$ be the set of all $k \times k$ matrices $Z = [z_{i,j}]$ of nonnegative integers where, for every $i \in \{1, \dots, k\}$, the sums of the entries both in the i -th row and i -th column of Z are n_i .

Let $Z = [z_{i,j}] \in \mathcal{D}$. We define the multilinear map $\otimes_Z$,

$$\otimes_Z : \otimes^{n_1} V \times \dots \times \otimes^{n_k} V \rightarrow (\otimes^{z_{1,1}} V) \otimes \dots \otimes (\otimes^{z_{k,k}} V),$$

by

$$\otimes_Z(x_{1,1} \otimes \dots \otimes x_{1,n_1}, \dots, x_{k,1} \otimes \dots \otimes x_{k,n_k}) = \alpha(x_{1,1} \otimes \dots \otimes x_{1,z_{1,1}}) \otimes \dots \otimes (x_{k,1} \otimes \dots \otimes x_{k,z_{k,k}}),$$

where

$$\begin{aligned} \alpha = & \left(\prod_{j=1}^{z_{1,2}} \langle x_{1,z_{1,1}+j}, x_{2,z_{2,2}+j} \rangle \right) \left(\prod_{j=1}^{z_{1,3}} \langle x_{1,z_{1,1}+z_{1,2}+j}, x_{3,z_{3,3}+j} \rangle \right) \cdots \\ & \left(\prod_{j=1}^{z_{1,k}} \langle x_{1,z_{1,1}+\dots+z_{1,k-1}+j}, x_{k,z_{k,k}+j} \rangle \right) \left(\prod_{j=1}^{z_{2,3}} \langle x_{2,z_{2,1}+z_{2,2}+j}, x_{3,z_{1,3}+z_{3,3}+j} \rangle \right) \cdots \\ & \left(\prod_{j=1}^{z_{2,k}} \langle x_{2,z_{2,1}+z_{2,2}+\dots+z_{2,k-1}+j}, x_{k,z_{1,k}+z_{k,k}+j} \rangle \right) \cdots \\ & \left(\prod_{j=1}^{z_{k-1,k}} \langle x_{k-1,z_{k-1,1}+z_{k-2,1}+\dots+z_{k-1,k-1}+j}, x_{k,z_{1,k}+z_{2,k}+\dots+z_{k-2,k}+z_{k,k}+j} \rangle \right). \end{aligned}$$

Similarly, we define $\otimes_{Z^T}$.

In particular,

$$\otimes_Z(x_1 \otimes \dots \otimes x_1, \dots, x_k \otimes \dots \otimes x_k) = \left(\prod_{\substack{i,j=1 \\ i < j}}^k \langle x_i, x_j \rangle^{z_{i,j}} \right) \underbrace{(x_1 \otimes \dots \otimes x_1)}_{z_{1,1} \text{ times}} \otimes \dots \otimes \underbrace{(x_k \otimes \dots \otimes x_k)}_{z_{k,k} \text{ times}}.$$

REMARK 4.1. Notice that when $k = 2$, the double cosets of S_m that take the form $H\gamma H$, where $H = S_{n_1} \times S_{n_2}$ and $\gamma \in S_m$ are represented by matrices of the following structure:

$$Z = \begin{bmatrix} n_1 - r & r \\ r & n_2 - r \end{bmatrix},$$

where $0 \leq r \leq \min\{n_1, n_2\}$. It is easy to see that $\otimes_Z = \otimes_r$, suggesting that the maps $\otimes_Z$ serve as a generalization of the Neuberger maps.

The following result extends the identity (2.4).

THEOREM 4.2. *Let V be a finite dimensional unitary space and let $m, n_1, n_2, \dots, n_k$ be positive integers such that $m = n_1 + n_2 + \dots + n_k$. Let $\mathcal{D}$ be the set of all $k \times k$ matrices $Z = [z_{i,j}]$ of nonnegative integers where, for every $i = 1, \dots, k$, the sum of the entries in the i^{th} row and the sum of the entries in the i^{th} column of Z is n_i , and for each $i = 1, \dots, k$ let $A_i \in \mathbb{V}^{n_i} V$. Then,*

$$\begin{pmatrix} m \\ n_1 \quad \dots \quad n_k \end{pmatrix} \|A_1 \vee \dots \vee A_k\|^2 = \sum_{Z=[z_{i,j}] \in \mathcal{D}} \begin{pmatrix} n_1 \\ z_{1,1} \quad \dots \quad z_{1,k} \end{pmatrix} \dots \begin{pmatrix} n_k \\ z_{k,1} \quad \dots \quad z_{k,k} \end{pmatrix} \left\langle \otimes_Z (A_1, \dots, A_k), \otimes_{Z^T} (A_1, \dots, A_k) \right\rangle.$$

Proof. By definition,

$$\begin{aligned} \begin{pmatrix} m \\ n_1 \quad \dots \quad n_k \end{pmatrix} \|A_1 \vee \dots \vee A_k\|^2 &= \begin{pmatrix} m \\ n_1 \quad \dots \quad n_k \end{pmatrix} \left\langle A_1 \vee \dots \vee A_k, A_1 \vee \dots \vee A_k \right\rangle \\ &= \begin{pmatrix} m \\ n_1 \quad \dots \quad n_k \end{pmatrix} \left\langle T_m(A_1 \otimes \dots \otimes A_k), A_1 \otimes \dots \otimes A_k \right\rangle \\ &= \frac{m!}{n_1! \dots n_k!} \frac{1}{m!} \left\langle \sum_{\sigma \in S_m} \mathcal{P}(\sigma)(A_1 \otimes \dots \otimes A_k), A_1 \otimes \dots \otimes A_k \right\rangle. \end{aligned}$$

By Theorem 2.1, there is no loss of generality in assuming that for all $i = 1, \dots, k$, $A_i = x_i \otimes \dots \otimes x_i$ (n_i copies of x_i), $x_i \in V$. Consider the double cosets of S_m of the form $H\gamma H$, where $\gamma \in S_m$ and $H = S_{n_1} \times \dots \times S_{n_k}$, and let $\sigma \in S_m$ such σ in the double coset characterized by the matrix $Z = [z_{i,j}]$. Then, by (2.1) and by Remark 3.5, we conclude that $\mathcal{P}(\sigma)(x_1 \otimes \dots \otimes x_1 \otimes \dots \otimes x_k \otimes \dots \otimes x_k)$ is a decomposable tensor that has in the first n_1 vectors, $z_{1,1}$ vectors equal to x_1 , $z_{2,1}$ vectors equal to x_2 , $z_{3,1}$ vectors equal x_3 and so on. In the vectors of the tensor $\mathcal{P}(\sigma)(x_1 \otimes \dots \otimes x_1 \otimes \dots \otimes x_k \otimes \dots \otimes x_k)$ between positions $n_1 + 1$ and $n_1 + n_2$, there are $z_{1,2}$ vectors equal to x_1 , $z_{2,2}$ vectors equal to x_2 , $z_{3,2}$ vectors equal to x_3 and so on. Therefore, $\left\langle \mathcal{P}(\sigma)(x_1 \otimes \dots \otimes x_1 \otimes \dots \otimes x_k \otimes \dots \otimes x_k), x_1 \otimes \dots \otimes x_1 \otimes \dots \otimes x_k \otimes \dots \otimes x_k \right\rangle$ is equal to

$$\left\langle x_1, x_1 \right\rangle^{z_{1,1}} \left\langle x_2, x_1 \right\rangle^{z_{2,1}} \dots \left\langle x_k, x_1 \right\rangle^{z_{k,1}} \dots \left\langle x_k, x_k \right\rangle^{z_{k,k}} = \prod_{i,j=1}^k \left\langle x_i, x_j \right\rangle^{z_{i,j}}.$$

Then, by Theorem 3.1,

$$\begin{aligned} \begin{pmatrix} m \\ n_1 \quad \dots \quad n_k \end{pmatrix} \|A_1 \vee \dots \vee A_k\|^2 \\ = \sum_{Z=[z_{i,j}] \in \mathcal{D}} \begin{pmatrix} n_1 \\ z_{1,1} \quad \dots \quad z_{1,k} \end{pmatrix} \dots \begin{pmatrix} n_k \\ z_{k,1} \quad \dots \quad z_{k,k} \end{pmatrix} \prod_{i,j=1}^k \left\langle x_i, x_j \right\rangle^{z_{i,j}}. \end{aligned}$$

On the other side, using the definition of the maps $\otimes_Z$ and $\otimes_{Z^T}$, $Z = [z_{i,j}] \in \mathcal{D}$, we have

$$\begin{aligned} \left\langle \otimes_Z (A_1, \dots, A_k), \otimes_{Z^T} (A_1, \dots, A_k) \right\rangle &= \prod_{\substack{i,j=1 \\ i < j}}^k \langle x_i, x_j \rangle^{z_{i,j}} \prod_{\substack{i,j=1 \\ i > j}}^k \overline{\langle x_i, x_j \rangle}^{z_{j,i}} \prod_{i=1}^k \langle x_i, x_i \rangle^{z_{i,i}} \\ &= \prod_{\substack{i,j=1 \\ i \neq j}}^k \langle x_i, x_j \rangle^{z_{i,j}} \prod_{i=1}^k \langle x_i, x_i \rangle^{z_{i,i}}, \end{aligned}$$

and the proof is complete. $\square$

By Theorem 4.2, we can easily obtain (2.4), as in the following:

COROLLARY 4.3. Let m, n_1 and n_2 be positive integers such that $m = n_1 + n_2$, and $n_1 \geq n_2$. Define $H = S_{n_1} \times S_{n_2}$, and let $\mathcal{D}$ be the set of all two-by-two matrices of nonnegative integers representing all the double cosets of S_m that take the form $H\gamma H$, $\gamma \in S_m$. Then, for all $A_1 \in \vee^{n_1} V$ and $A_2 \in \vee^{n_2} V$, we have

$$\binom{n_1 + n_2}{n_2} \|A_1 \vee A_2\|^2 = \sum_{i=0}^{n_2} \binom{n_1}{i} \binom{n_2}{i} \|\otimes_i (A_1, A_2)\|^2.$$

Proof. Using the Theorem 4.2, we have

$$\binom{n_1 + n_2}{n_2} \|A_1 \vee A_2\|^2 = \sum_{Z=[z_{i,j}] \in \mathcal{D}} \binom{n_1}{z_{1,1} \ z_{1,2}} \binom{n_2}{z_{2,1} \ z_{2,2}} \left\langle \otimes_Z (A_1, A_2), \otimes_{Z^T} (A_1, A_2) \right\rangle.$$

As

$$\binom{n_1}{z_{1,1} \ z_{1,2}} = \binom{n_1}{z_{1,1}}, \quad \binom{n_2}{z_{2,1} \ z_{2,2}} = \binom{n_2}{z_{2,2}},$$

having in mind the Remark 4.1, and observing that if $Z \in \mathcal{D}$, then Z is symmetric, we derive the result. $\square$

5. Marcus per-in-per conjecture. In this section, we prove Marcus per-in-per conjecture for positive semidefinite matrices $A \in \mathcal{H}_{np}$ that are block partitioned into $p \times p$ blocks, forming a block tridiagonal matrix.

THEOREM 5.1. Let $A = [a_{i,j}] \in \mathcal{H}_m$, where $m = pn$. Assume that A is partitioned into submatrices denoted as $A = [A_{ij}]$, where each A_{ij} is a $p \times p$ matrix, resulting in a block tridiagonal matrix. Consider the matrix P , which is of size $n \times n$, with its (i, j) element defined as $\text{per}(A_{ij})$. Then, it holds that $\text{per}(A) \geq \text{per}(P)$. The equality occurs if and only if either $\text{per}(A) = 0$ or $A_{i,j} = 0$ whenever $i \neq j$.

Proof. Let $\mathcal{D}$ denote the set of all $n \times n$ matrices with nonnegative integer entries, where the sum of the entries in the i^{th} row and the i^{th} column of each matrix in $\mathcal{D}$ equals p for $i = 1, \dots, n$. We will refer to the subset of symmetric matrices in $\mathcal{D}$ as $\mathcal{D}_s$. The subset of $\mathcal{D}$ of the matrices with exactly one nonzero entry in each row and each column is denoted by $\mathcal{D}_1$.

If $Z \in \mathcal{D}_1$, then there exists a permutation σ in the symmetric group S_n such that

$$Z = p P_\sigma,$$

where P_σ denotes the permutation matrix associated with σ . Consequently, the number of matrices in $\mathcal{D}_1$ is $n!$.

Let $A = [a_{i,j}] \in \mathcal{H}_{np}$ be a matrix that is partitioned into submatrices, denoted as $A = [A_{ij}]$, where each A_{ij} is a $p \times p$ matrix. We assume that, with this partitioning, A becomes a block tridiagonal matrix. Therefore, if a permutation $\sigma \in S_{np}$ is such that

$$(5.6) \quad \prod_{i=1}^{np} a_{i,\sigma(i)} \neq 0,$$

then σ belongs to a double coset of S_{np} of the form $H\gamma H$, where $H = S_p \times S_p \times \dots \times S_p$ (with n copies of S_p) defined by a symmetric matrix.

Since $A \in \mathcal{H}_{np}$, there are vectors $x_{1,1}, \dots, x_{1,p}, \dots, x_{n,1}, \dots, x_{n,p}$, such that if $a_{s,t} \in A_{g,h}$ where $g, h = 1, \dots, n$, $s = (g-1)p + u$ and $t = (h-1)p + v$, $u, v = 1, \dots, p-1$, then $a_{s,t} = \langle x_{g,u}, x_{h,v} \rangle$.

Let $A_1 = T_p(x_{1,1} \otimes \dots \otimes x_{1,p})$, $A_2 = T_p(x_{2,1} \otimes \dots \otimes x_{2,p})$, $\dots$, $A_n = T_p(x_{n,1} \otimes \dots \otimes x_{n,p})$.

Then, by Theorem 4.2, and by the previous conclusion on permutations that satisfy (5.6), we have

$$(5.7) \quad \begin{pmatrix} np & & \\ p & \dots & p \end{pmatrix} \|A_1 \vee \dots \vee A_n\|^2 \\ = \sum_{Z=[z_{i,j}] \in \mathcal{D}_s} \begin{pmatrix} p & & \\ z_{1,1} & \dots & z_{1,p} \end{pmatrix} \dots \begin{pmatrix} p & & \\ z_{p,1} & \dots & z_{p,p} \end{pmatrix} \|\otimes_Z (A_1, \dots, A_n)\|^2,$$

and hence

$$(5.8) \quad \begin{pmatrix} np & & \\ p & \dots & p \end{pmatrix} \|A_1 \vee \dots \vee A_n\|^2 \geq \sum_{Z=[z_{i,j}] \in \mathcal{D}_1 \cap \mathcal{D}_s} \begin{pmatrix} p & \\ p & \end{pmatrix}^n \|\otimes_Z (A_1, \dots, A_n)\|^2.$$

First, note that

$$\begin{aligned} \begin{pmatrix} np & & \\ p & \dots & p \end{pmatrix} \|A_1 \vee \dots \vee A_n\|^2 &= \frac{(pn)!}{(p!)^n} \left\langle T_{np}(x_{1,1} \otimes \dots \otimes x_{1,p} \otimes \dots \otimes x_{n,1} \otimes \dots \otimes x_{n,p}), \right. \\ &\quad \left. x_{1,1} \otimes \dots \otimes x_{1,p} \otimes \dots \otimes x_{n,1} \otimes \dots \otimes x_{n,p} \right\rangle \\ &= \frac{1}{(p!)^n} \text{per}(A). \end{aligned}$$

On the other side, if $Z \in \mathcal{D}_s \cap \mathcal{D}_1$, then

$$\otimes_Z (A_1, \dots, A_n) = \left(\prod_{\substack{i=1 \\ \sigma(i) \neq i}}^n \langle A_i, A_{\sigma(i)} \rangle \right) \bigotimes_{\substack{i=1 \\ \sigma(i)=i}}^n A_i,$$

where σ is the element of S_n such that $pP_\sigma = Z$. Since

$$\begin{aligned} \langle A_i, A_{\sigma(i)} \rangle &= \langle x_{i,1} \otimes \dots \otimes x_{i,p}, T_p(x_{\sigma(i),1} \otimes \dots \otimes x_{\sigma(i),p}) \rangle \\ &= \frac{1}{(p!)} \text{per}(A_{i,\sigma(i)}), \end{aligned}$$

we conclude

$$\|\otimes_Z (A_1, \dots, A_n)\|^2 = \frac{1}{(p!)^n} \prod_{i=1}^n \text{per}(A_{i,\sigma(i)}).$$

Therefore,

$$\sum_{Z=[z_{i,j}] \in \mathcal{D}_s \cap \mathcal{D}_1} \|\otimes_Z (A_1, \dots, A_n)\|^2 = \frac{1}{(p!)^n} \left(\sum_{\sigma \in S_n} \prod_{i=1}^n \text{per}(A_{i,\sigma(i)}) \right) = \frac{1}{(p!)^n} \text{per}(P).$$

Having in mind (5.8), we conclude

$$\text{per}(A) \geq \text{per}(P).$$

Now we consider the conditions under which the equality holds. First, assume that $\text{per}(A) = 0$. According to Proposition 2.3, A must have a row of zeros. Consequently, P will also have a row of zeros, leading to the conclusion that

$$\text{per}(A) = \text{per}(P) = 0.$$

Assume now that $\text{per}(A) \neq 0$ and each $A_{i,j} = 0$ whenever $i \neq j$. Then,

$$\prod_{t=1}^{np} a_{t,\sigma(t)} \neq 0,$$

if and only if σ is in the double coset of S_{np} characterized by the matrix $Z = pI_n$. Then, it is easy to conclude

$$\text{per}(A) = \prod_{i=1}^n \text{per}(A_{ii}) = \text{per}(P).$$

Assume there exist $i, j \in \{1, \dots, n\}$ such that $i \neq j$ and $A_{i,j} \neq 0$. Then, there is a permutation $\sigma \in S_{np}$ such that

$$\prod_{t=1}^{np} a_{t,\sigma(t)} \neq 0,$$

and this permutation σ is not in a double coset of S_{np} corresponding to a matrix in $\mathcal{D}_s \cap \mathcal{D}_1$. Therefore, by applying equation (5.8), we obtain

$$\text{per}(A) > \sum_{Z=[z_{i,j}] \in \mathcal{D}_s \cap \mathcal{D}_1} \left(\frac{p}{p} \right)^n \|\otimes_Z (A_1, \dots, A_n)\|^2 = \text{per}(P).$$

This concludes the proof. □

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