



ON LINEAR EQUATIONS OVER SPLIT-OCTONIONS*

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Abstract. Over an algebraically closed field, we describe the affine varieties of solutions to the linear equations $(ax)b = c$ and $a(bx) = c$ over the split-octonions. We also determine the dimensions of the solution sets of arbitrary linear monomial equations in the split-octonions. Moreover, we show that if a linear monomial equation over the split-octonions with nonzero constant term has at least two solutions, then it necessarily possesses an invertible solution.

Key words. Octonions, Split-octonions, Linear equation, Positive characteristic.

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1. Introduction. Unless stated otherwise, let $\mathbb{F}$ be a field of arbitrary characteristic $p = \text{char } \mathbb{F} \geq 0$. All vector spaces and algebras are assumed to be defined over $\mathbb{F}$.

1.1. Equations over octonions. The problem of solving polynomial equations has historically been regarded as one of the central problems in mathematics, and its study contributed to the development of Algebraic Geometry and several other mathematical disciplines. Polynomial equations have been investigated not only over fields but also over matrix algebras, as well as over the algebras of quaternions, octonions, and related structures.

Rodríguez-Ordóñez [28] proved that every polynomial equation of positive degree over the algebra $\mathbf{A}_{\mathbb{R}}$ of *Cayley numbers* (i.e., the division algebra of real octonions) containing only a single term of highest degree has at least one solution. Furthermore, Wang, Zhang, and Zhang [36] obtained an explicit algorithm for solving quadratic equations of the form $x^2 + bx + c = 0$ over $\mathbf{A}_{\mathbb{R}}$, together with criteria determining whether such an equation admits one, two, or infinitely many solutions.

In general, an *octonion algebra* (or *Cayley algebra*) $\mathbf{C}$ over the field $\mathbb{F}$ is an 8-dimensional unital nonassociative alternative algebra equipped with a nonsingular quadratic multiplicative form $\mathbf{n} : \mathbf{C} \rightarrow \mathbb{F}$, called the *norm* (see Section 2.3 for details). The norm $\mathbf{n}$ is said to be *isotropic* if $\mathbf{n}(a) = 0$ for some nonzero $a \in \mathbf{C}$ and *anisotropic* otherwise. When $\mathbf{n}$ is anisotropic, the algebra $\mathbf{C}$ is a division algebra. If $\mathbf{n}$ is isotropic, then there exists a unique octonion algebra $\mathbf{O}_{\mathbb{F}}$ over $\mathbb{F}$ with isotropic norm (see Theorem 1.8.1 of [30]); this algebra is referred to as the *split-octonion algebra*. Moreover, if $\mathbb{F}$ is algebraically closed, then every octonion algebra over $\mathbb{F}$ is isomorphic to the split-octonion algebra $\mathbf{O}_{\mathbb{F}}$ (see Lemma 2.2 below).

Since an octonion algebra $\mathbf{C}$ is alternative, it is therefore power associative, i.e., the subalgebra generated by a single element is associative. Hence, given $a \in \mathbf{C}$, the expression a^n may be written without specifying the placement of brackets in the product. Flaut and Shpakivskyi [15] studied the equation $x^n = a$ over real octonion division algebras. For an octonion division algebra $\mathbf{C}$ over an arbitrary field $\mathbb{F}$, Chapman [10]

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presented a complete method for finding the solutions of the polynomial equation $a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 = 0$ over $\mathbf{C}$. Moreover, Chapman and Vishkautsan [12], working over a division algebra $\mathbf{C}$, determined the solutions of the polynomial equation $(a_n c) x^n + (a_{n-1} c) x^{n-1} + \dots + (a_1 c) x + (a_0 c) = 0$ and discussed the solutions of the polynomial equation $(c a_n) x^n + (c a_{n-1}) x^{n-1} + \dots + (c a_1) x + (c a_0) = 0$. Chapman and Levin [11] described a method for finding so-called “alternating roots” of polynomials over an arbitrary division Cayley–Dickson algebra. Chapman and Vishkautsan [13] investigated conditions under which, for a root a of a polynomial $f(x)$ over a general Cayley–Dickson algebra, there exists a polynomial $g(x)$ such that $f(x) = g(x)(x - a)$. Working over the split-octonions over an algebraically closed field, Lopatin and Rybalov [25] solved all polynomial equations in which all coefficients except the constant term are scalar. As a consequence, the n -th roots of a split-octonion were computed.

The split-octonions have numerous applications in physics. For example, the Dirac equation, i.e., the equation of motion of a free spin 1/2 particle, such as an electron or proton, can be expressed in terms of split-octonions (see [16, 17, 20, 21]). Further applications of split-octonions occur in electromagnetic theory (see [7, 8, 9]), geometrodynamics (see [6]), unified quantum theories (see [3, 5, 22]), and special relativity (see [18]).

The case of polynomial equations over an arbitrary algebra was recently considered by Illmer and Netzer in [19], where conditions were established that guarantee the existence of a common solution to n polynomial equations in n variables, together with an application to polynomial equations over $\mathbf{A}_{\mathbb{R}}$.

Linear equations over matrices have been extensively studied in [1, 2, 4, 14, 23, 31], among many other works. The principal questions addressed in these works were

- determining conditions under which a solution to a given linear equation exists;
- describing the general form of the solution.

For particular matrix equations, further information can be found in the following recent surveys:

- for the equation $AXB = C$, see [35];
- for the generalized Sylvester equation $AX - YB = C$, see [32];
- for the system of equations $AX = C$ and $XB = D$, see [33];
- for the system of equations $A_1 X B_1 = C_1$ and $A_2 X B_2 = C_2$, see [34].

1.2. Results. In this paper, we consider the linear equations $ax = c$, $(ax)b = c$, and $a(bx) = c$ over the split-octonion algebra $\mathbf{O}$, assuming that $\mathbb{F}$ is algebraically closed. Note that over a division octonion algebra, these equations can be solved easily, and for nonzero a, b , there exists a unique solution. Over an algebraically closed field, however, the situation is drastically different. Namely, let Ω_d denote the set of all affine subspaces (i.e., flats) in $\mathbf{O} \cong \mathbb{F}^8$ of dimension $1 \leq d \leq 8$, i.e.,

$$\Omega_d = \{ \mathcal{V} + a \mid \mathcal{V} \subset \mathbf{O} \text{ is an } \mathbb{F}\text{-subspace with } \dim \mathcal{V} = d \text{ and } a \in \mathbf{O} \}.$$

Moreover, for convenience, set $\Omega_0 = \{ \{a\} \mid a \in \mathbf{O} \}$ and $\Omega_{-1} = \{\emptyset\}$. Given $Y \in \Omega_r$ for some $r \in \{-1, 0, 1, \dots, 8\}$, we define the dimension of Y to be $\dim Y = r$.

The set X of all solutions of one of the above linear equations with nonzero a, b is either empty, consists of a single element, or satisfies $X \in \Omega_r$, where

- $r = 4$ when we consider the equation $ax = c$ (see Corollary 4.2);
- $r \in \{4, 5, 7\}$ when we consider the equation $(ax)b = c$ (see Corollary 5.2);

- $r \in \{4, 6, 8\}$ when we consider the equation $a(bx) = c$ (see Corollary 6.2).

Moreover, the set X of all solutions of the general linear monomial equation $w(x) = c$, where $w(x)$ is a product of a variable x and coefficients $a_1, \dots, a_m \in \mathbf{O}$ (see Section 7 for the definition), is either

- empty, or
- consists of a single element, or
- lies in Ω_r for some $4 \leq r \leq 7$, or
- coincides with $\mathbf{O}$,

see Corollary 7.1. As a consequence, in Corollary 7.2, we show that if a linear monomial equation over octonions with nonzero constant term has at least two solutions, then it possesses an invertible solution. We also study how the dimension of the solution set changes under degeneration of a linear monomial equation in Corollary 8.1.

To solve a linear equation $ax = c$, we consider the action of an automorphism $g \in \text{Aut}(\mathbf{O})$ on this equation, yielding $(ga)(gx) = gc$, such that ga is a canonical noninvertible octonion (see Definition 3.5). Similarly, to solve the equation $(ax)b = c$ (or $a(bx) = c$, respectively), we consider the action of an automorphism $g \in \text{Aut}(\mathbf{O})$ on the equation so that (ga, gb) is a pair of canonical noninvertible octonions (see Definition 3.5). Hence, our solution method is based on the classification of pairs of octonions obtained by Lopatin and Zubkov in [27] (see Theorem 3.2 below), where the split-octonion algebra $\mathbf{O}$ is referred to as the octonion algebra. Note that the problem of separating $\text{Aut}(\mathbf{O})$ -orbits on pairs of octonions remains open, but closed $\text{Aut}(\mathbf{O})$ -orbits in $\mathbf{O}^n$ can be distinguished by polynomial invariants (see [26], [29] and [37], for details). We provide explicit solutions for the equations $(ax)b = c$ and $a(bx) = c$ with noninvertible canonical (a, b) in Theorems 5.1 and 6.1.

In Sections 2–8, with the exception of Section 2.3, we assume that the field $\mathbb{F}$ is algebraically closed. In Section 2, we explicitly define the octonion algebra $\mathbf{O}$. Additional properties of octonions, together with key notations for certain sets of octonions, are provided in Section 3. The solutions of the equations $ax = c$, $(ax)b = c$, and $a(bx) = c$ are studied in Sections 4, 5, and 6, respectively. Finally, in Section 7, we investigate the solution sets of linear monomial equations, and in Section 8, we examine degenerations of linear equations.

2. Octonions. The definitions in this section are adopted from [26].

2.1. Split-octonions. The *split-octonion algebra* $\mathbf{O} = \mathbf{O}(\mathbb{F})$, also known as the *split Cayley algebra*, is the vector space of all matrices

$$a = \begin{pmatrix} \alpha & \mathbf{u} \\ \mathbf{v} & \beta \end{pmatrix} \text{ with } \alpha, \beta \in \mathbb{F} \text{ and } \mathbf{u}, \mathbf{v} \in \mathbb{F}^3,$$

together with the multiplication given by the following formula:

$$aa' = \begin{pmatrix} \alpha\alpha' + \mathbf{u} \cdot \mathbf{v}' & \alpha\mathbf{u}' + \beta'\mathbf{u} - \mathbf{v} \times \mathbf{v}' \\ \alpha'\mathbf{v} + \beta\mathbf{v}' + \mathbf{u} \times \mathbf{u}' & \beta\beta' + \mathbf{v} \cdot \mathbf{u}' \end{pmatrix}, \text{ where } a' = \begin{pmatrix} \alpha' & \mathbf{u}' \\ \mathbf{v}' & \beta' \end{pmatrix},$$

$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3$ and $\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1)$. For short, we denote $\mathbf{c}_1 = (1, 0, 0)$, $\mathbf{c}_2 = (0, 1, 0)$, $\mathbf{c}_3 = (0, 0, 1)$, $\mathbf{0} = (0, 0, 0)$ from $\mathbb{F}^3$. Consider the following basis of $\mathbf{O}$:

$$e_1 = \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 & \mathbf{0} \\ \mathbf{0} & 1 \end{pmatrix}, \mathbf{u}_i = \begin{pmatrix} 0 & \mathbf{c}_i \\ \mathbf{0} & 0 \end{pmatrix}, \mathbf{v}_i = \begin{pmatrix} 0 & \mathbf{0} \\ \mathbf{c}_i & 0 \end{pmatrix},$$

for $i = 1, 2, 3$. The unity of $\mathbf{O}$ is denoted by $1_{\mathbf{O}} = e_1 + e_2$. We identify octonions

$$\alpha 1_{\mathbf{O}}, \begin{pmatrix} 0 & \mathbf{u} \\ \mathbf{0} & 0 \end{pmatrix}, \begin{pmatrix} 0 & \mathbf{0} \\ \mathbf{v} & 0 \end{pmatrix},$$

with $\alpha \in \mathbb{F}$, $\mathbf{u}, \mathbf{v} \in \mathbb{F}^3$, respectively. Note that $\mathbf{u}_i \mathbf{u}_j = (-1)^{\epsilon_{ij}} \mathbf{v}_k$ and $\mathbf{v}_i \mathbf{v}_j = (-1)^{\epsilon_{ji}} \mathbf{u}_k$, where $\{i, j, k\} = \{1, 2, 3\}$ and ϵ_{ij} is the parity of permutation $\begin{pmatrix} 1 & 2 & 3 \\ k & i & j \end{pmatrix}$.

The algebra $\mathbf{O}$ is endowed with the linear involution

$$\bar{a} = \begin{pmatrix} \beta & -\mathbf{u} \\ -\mathbf{v} & \alpha \end{pmatrix},$$

satisfying $\overline{aa'} = \bar{a'}\bar{a}$, a norm $n(a) = a\bar{a} = \alpha\beta - \mathbf{u} \cdot \mathbf{v}$, and a nondegenerate symmetric bilinear form $q(a, a') = n(a + a') - n(a) - n(a') = \alpha\beta' + \alpha'\beta - \mathbf{u} \cdot \mathbf{v}' - \mathbf{u}' \cdot \mathbf{v}$. Define the linear function *trace* by $\text{tr}(a) = a + \bar{a} = \alpha + \beta$. The subspace of traceless octonions is denoted by $\mathbf{O}_0 = \{a \in \mathbf{O} \mid \text{tr}(a) = 0\}$, and the affine variety of octonions with zero norm is denoted by $\mathbf{O}_{\#} = \{a \in \mathbf{O} \mid n(a) = 0\}$. Notice that

$$(2.1) \quad \text{tr}(aa') = \text{tr}(a'a) \text{ and } n(aa') = n(a)n(a').$$

The next quadratic equation holds:

$$(2.2) \quad a^2 - \text{tr}(a)a + n(a) = 0.$$

The algebra $\mathbf{O}$ is a simple *alternative* algebra, i.e., the following identities hold for $a, b \in \mathbf{O}$:

$$(2.3) \quad a(ab) = (aa)b, \quad (ba)a = b(aa).$$

Moreover,

$$(2.4) \quad \bar{a}(ab) = n(a)b, \quad (ba)\bar{a} = n(a)b.$$

The following remark is well-known and can easily be proven.

REMARK 2.1. Given an $a \in \mathbf{O}$, one of the following cases holds:

- if $n(a) \neq 0$, then there exist a unique $b, c \in \mathbf{O}$ such that $ba = 1_{\mathbf{O}}$ and $ac = 1_{\mathbf{O}}$; moreover, in this case, we have $b = c = \bar{a}/n(a)$, and we denote $a^{-1} = b = c$.
- if $n(a) = 0$, then a does not have a left inverse as well as a right inverse.

Equalities (2.4) imply that for $a \notin \mathbf{O}_{\#}$, we have

$$(2.5) \quad a^{-1}(ab) = b, \quad (ba)a^{-1} = b.$$

2.2. The group G_2 . The group $\text{Aut}(\mathbf{O})$ of all automorphisms of the algebra $\mathbf{O}$ is the exceptional simple group $G_2 = G_2(\mathbb{F})$. The group G_2 contains a Zariski closed subgroup $\text{SL}_3 = \text{SL}_3(\mathbb{F})$. Namely, every $g \in \text{SL}_3$ defines an automorphism of $\mathbf{O}$ as follows:

$$a \rightarrow \begin{pmatrix} \alpha & \mathbf{u}g \\ \mathbf{v}g^{-T} & \beta \end{pmatrix},$$

where g^{-T} stands for $(g^{-1})^T$ and $\mathbf{u}, \mathbf{v} \in \mathbb{F}^3$ are considered as row vectors. For every $\mathbf{u}, \mathbf{v} \in \mathbb{F}^3$ define $\delta_1(\mathbf{u}), \delta_2(\mathbf{v}) \in \text{Aut}(\mathbf{O})$ as follows:

$$\delta_1(\mathbf{u})(a') = \begin{pmatrix} \alpha' - \mathbf{u} \cdot \mathbf{v}' & (\alpha' - \beta' - \mathbf{u} \cdot \mathbf{v}')\mathbf{u} + \mathbf{u}' \\ \mathbf{v}' - \mathbf{u}' \times \mathbf{u} & \beta' + \mathbf{u} \cdot \mathbf{v}' \end{pmatrix},$$

$$\delta_2(\mathbf{v})(a') = \begin{pmatrix} \alpha' + \mathbf{u}' \cdot \mathbf{v} & \mathbf{u}' + \mathbf{v}' \times \mathbf{v} \\ (-\alpha' + \beta' - \mathbf{u}' \cdot \mathbf{v})\mathbf{v} + \mathbf{v}' & \beta' - \mathbf{u}' \cdot \mathbf{v} \end{pmatrix}.$$

The group G_2 is generated by SL_3 and $\delta_1(t\mathbf{u}_i), \delta_2(t\mathbf{v}_i)$ for all $t \in \mathbb{F}$ and $i = 1, 2, 3$. As an example, by straightforward calculations, we can see that

$$(2.6) \quad h : \mathbf{O} \rightarrow \mathbf{O}, \text{ defined by } a \rightarrow \begin{pmatrix} \beta & -\mathbf{v} \\ -\mathbf{u} & \alpha \end{pmatrix},$$

belongs to G_2 .

The action of G_2 on $\mathbf{O}$ satisfies the following properties:

$$\overline{ga} = g\overline{a}, \text{ tr}(ga) = \text{tr}(a), n(ga) = n(a), q(ga, ga') = q(a, a'),$$

for all $g \in \text{Aut}(\mathbf{O})$ and $a, a' \in \mathbf{O}$. Thus, G_2 also acts on $\mathbf{O}_0$ and $\mathbf{O}_\#$. Moreover, the group G_2 acts diagonally on the vector space $\mathbf{O}^n = \mathbf{O} \oplus \cdots \oplus \mathbf{O}$ (n times) by $g(a_1, \dots, a_n) = (ga_1, \dots, ga_n)$ for all $g \in G_2$ and $a_1, \dots, a_n \in \mathbf{O}$.

2.3. General octonions. An *octonion algebra* $\mathbf{C}$ (or a *Cayley algebra*) over an arbitrary field $\mathbb{F}$ is a nonassociative alternative unital algebra of dimension 8, endowed with nonsingular quadratic multiplicative form $n : \mathbf{C} \rightarrow \mathbb{F}$, which is called the *norm*. As above, define over $\mathbf{C}$ the symmetric bilinear form $q(a, a') = n(a + a') - n(a) - n(a')$, the linear involution $\overline{a} = q(a, 1_{\mathbf{C}})1_{\mathbf{C}} - a$, and the linear trace $\text{tr}(a) = q(a, 1_{\mathbf{C}})$. Then, equality (2.2) also holds for every $a \in \mathbf{C}$, $\text{tr}(a)1_{\mathbf{C}} = a + \overline{a}$ and $n(a)1_{\mathbf{C}} = a\overline{a} = \overline{a}a$. The following lemma is well-known.

LEMMA 2.2. Assume that the field $\mathbb{F}$ is algebraically closed. Then, any octonion algebra $\mathbf{C}$ over $\mathbb{F}$ is isomorphic to the split-octonion algebra $\mathbf{O}_{\mathbb{F}}$.

Proof. If the norm n on $\mathbf{C}$ is isotropic, then $\mathbf{C}$ is isomorphic to $\mathbf{O}_{\mathbb{F}}$.

Assume that $n(a) \neq 0$ for every nonzero $a \in \mathbf{C}$. Consider some nonzero $a \in \mathbf{C}$ such that $a \neq \lambda 1_{\mathbf{C}}$ for every $\lambda \in \mathbb{F}$. Denote $\alpha = n(a) \neq 0$. Then,

$$n(a - \lambda 1_{\mathbf{C}})1_{\mathbf{C}} = (a - \lambda 1_{\mathbf{C}})(\overline{a} - \lambda 1_{\mathbf{C}}) = a\overline{a} - \lambda \text{tr}(a)1_{\mathbf{C}} + \lambda^2 1_{\mathbf{C}} = (\lambda^2 - \lambda \text{tr}(a) + \alpha)1_{\mathbf{C}}.$$

Since $\mathbb{F}$ is algebraically closed, then there exists $\lambda \in \mathbb{F}$ such that $n(a - \lambda 1_{\mathbf{C}}) = 0$. Since $a - \lambda 1_{\mathbf{C}}$ is nonzero, we obtain a contradiction. $\square$

2.4. Notations. We fix a binary relation $<$ on the field $\mathbb{F}$ such that

- for each pair $\alpha, \beta \in \mathbb{F}$ with $\alpha \neq \beta$ exactly one of $\alpha < \beta$ or $\beta < \alpha$ holds;
- $\alpha < 0$ for every nonzero $\alpha \in \mathbb{F}$.

Note that we do not assume that $<$ is transitive, and we do not assume compatibility with the field operations. Denote by $\mathbb{F}^\times$ the set of all nonzero elements of $\mathbb{F}$. We write $a = (\alpha_1, \dots, \alpha_8)$, where $\alpha_i \in \mathbb{F}$, for an octonion

$$a = \begin{pmatrix} \alpha_1 & (\alpha_2, \alpha_3, \alpha_4) \\ (\alpha_5, \alpha_6, \alpha_7) & \alpha_8 \end{pmatrix} \in \mathbf{O}.$$

3. Classification of octonions.

3.1. Sets of octonions. Introduce the following sets of diagonal octonions:

$$(D) \begin{pmatrix} \alpha_1 & \mathbf{0} \\ \mathbf{0} & \alpha_8 \end{pmatrix}, \quad (E) \alpha_1 1_{\mathbf{O}}, \quad (F) \begin{pmatrix} \alpha_1 & \mathbf{0} \\ \mathbf{0} & \alpha_8 \end{pmatrix} \text{ with } \alpha_1 \neq \alpha_8,$$

where $\alpha_1, \alpha_8 \in \mathbb{F}$. Note that set (D) is the union of sets (E) and (F). Given $a \in \mathbf{O}$, denote by $a^\top$ the transpose octonion of a , i.e.

$$a^\top = \begin{pmatrix} \alpha & \mathbf{v} \\ \mathbf{u} & \beta \end{pmatrix} \text{ for } a = \begin{pmatrix} \alpha & \mathbf{u} \\ \mathbf{v} & \beta \end{pmatrix}.$$

Consider the following sets of non-diagonal octonions:

$$\begin{aligned} (K) & \begin{pmatrix} \alpha_1 & (1, 0, 0) \\ \mathbf{0} & \alpha_8 \end{pmatrix}, \\ (L) & \begin{pmatrix} \alpha_1 & (\alpha_2, 0, 0) \\ \mathbf{0} & \alpha_8 \end{pmatrix} \text{ with } \alpha_2 \neq 0, \\ (M) & \begin{pmatrix} \alpha_1 & (0, 1, 0) \\ \mathbf{0} & \alpha_8 \end{pmatrix}, \\ (N) & \begin{pmatrix} \alpha_1 & (1, 0, 0) \\ (\alpha_5, 0, 0) & \alpha_8 \end{pmatrix} \text{ with } \alpha_5 \neq 0, \\ (P) & \begin{pmatrix} \alpha_1 & (1, 0, 0) \\ (0, 1, 0) & \alpha_8 \end{pmatrix}, \end{aligned}$$

for $\alpha_1, \alpha_2, \alpha_5, \alpha_8 \in \mathbb{F}$. Given some set (A) of octonions, we denote by

- (A₀) the set of octonions $a \in \mathbf{O}_0$ of set (A);
- ($\bar{A}$) the set of octonions $a \in \mathbf{O}$ of set (A) with $\alpha_1 \leq \alpha_8$;
- (A₁) the set of octonions $a \in \mathbf{O}$ of set (A) with $\alpha_1 = \alpha_8$;
- (A[⊤]) the set of octonions $a^\top$ such that a has set (A);
- (A₀[⊤]) the set of octonions $a^\top$ such that $a \in \mathbf{O}_0$ has set (A);
- (A₁[⊤]) the set of octonions $a^\top$ such that a has set (A) with $\alpha_1 = \alpha_8$.

Note that in case $\text{char } \mathbb{F} = 2$ sets (A₀) and (A₁) are the same.

PROPOSITION 3.1 (Proposition 3.3 from [27]). *The following set is a minimal set of representatives of G_2 -orbits on $\mathbf{O}$:*

$$(E) \alpha_1 1_{\mathbf{O}},$$

$$\begin{aligned} &(\bar{F}) \begin{pmatrix} \alpha_1 & \mathbf{0} \\ \mathbf{0} & \alpha_8 \end{pmatrix} \text{ with } \alpha_1 < \alpha_8, \\ &(K_1) \begin{pmatrix} \alpha_1 & (1, 0, 0) \\ \mathbf{0} & \alpha_1 \end{pmatrix}, \end{aligned}$$

where $\alpha_1, \alpha_8 \in \mathbb{F}$.

THEOREM 3.2 (Theorem 4.4 from [27]). *The following set is a minimal set of representatives of G_2 -orbits on $\mathbf{O}^2$:*

$$\begin{aligned} &(\text{EE}) (\alpha_1 \mathbf{1}_{\mathbf{O}}, \beta_1 \mathbf{1}_{\mathbf{O}}), \\ &(\text{EF}) \left(\alpha_1 \mathbf{1}_{\mathbf{O}}, \begin{pmatrix} \beta_1 & \mathbf{0} \\ \mathbf{0} & \beta_8 \end{pmatrix} \right) \text{ with } \beta_1 < \beta_8, \\ &(\text{EK}_1) \left(\alpha_1 \mathbf{1}_{\mathbf{O}}, \begin{pmatrix} \beta_1 & (1, 0, 0) \\ \mathbf{0} & \beta_1 \end{pmatrix} \right), \\ &(\bar{\text{F}}\text{D}) \left(\begin{pmatrix} \alpha_1 & \mathbf{0} \\ \mathbf{0} & \alpha_8 \end{pmatrix}, \begin{pmatrix} \beta_1 & \mathbf{0} \\ \mathbf{0} & \beta_8 \end{pmatrix} \right) \text{ with } \alpha_1 < \alpha_8, \\ &(\bar{\text{F}}\text{K}) \left(\begin{pmatrix} \alpha_1 & \mathbf{0} \\ \mathbf{0} & \alpha_8 \end{pmatrix}, \begin{pmatrix} \beta_1 & (1, 0, 0) \\ \mathbf{0} & \beta_8 \end{pmatrix} \right) \text{ with } \alpha_1 < \alpha_8, \\ &(\bar{\text{F}}\text{K}^\top) \left(\begin{pmatrix} \alpha_1 & \mathbf{0} \\ \mathbf{0} & \alpha_8 \end{pmatrix}, \begin{pmatrix} \beta_1 & \mathbf{0} \\ (1, 0, 0) & \beta_8 \end{pmatrix} \right) \text{ with } \alpha_1 < \alpha_8, \\ &(\bar{\text{F}}\text{N}) \left(\begin{pmatrix} \alpha_1 & \mathbf{0} \\ \mathbf{0} & \alpha_8 \end{pmatrix}, \begin{pmatrix} \beta_1 & (1, 0, 0) \\ (\beta_5, 0, 0) & \beta_8 \end{pmatrix} \right) \text{ with } \alpha_1 < \alpha_8 \text{ and } \beta_5 \neq 0, \\ &(\bar{\text{F}}\text{P}) \left(\begin{pmatrix} \alpha_1 & \mathbf{0} \\ \mathbf{0} & \alpha_8 \end{pmatrix}, \begin{pmatrix} \beta_1 & (1, 0, 0) \\ (0, 1, 0) & \beta_8 \end{pmatrix} \right) \text{ with } \alpha_1 < \alpha_8, \\ &(\text{K}_1\text{E}) \left(\begin{pmatrix} \alpha_1 & (1, 0, 0) \\ \mathbf{0} & \alpha_1 \end{pmatrix}, \beta_1 \mathbf{1}_{\mathbf{O}} \right), \\ &(\text{K}_1\text{F}) \left(\begin{pmatrix} \alpha_1 & (1, 0, 0) \\ \mathbf{0} & \alpha_1 \end{pmatrix}, \begin{pmatrix} \beta_1 & \mathbf{0} \\ \mathbf{0} & \beta_8 \end{pmatrix} \right) \text{ with } \beta_1 \neq \beta_8, \\ &(\text{K}_1\text{L}_1) \left(\begin{pmatrix} \alpha_1 & (1, 0, 0) \\ \mathbf{0} & \alpha_1 \end{pmatrix}, \begin{pmatrix} \beta_1 & (\beta_2, 0, 0) \\ \mathbf{0} & \beta_1 \end{pmatrix} \right) \text{ with } \beta_2 \neq 0, \\ &(\text{K}_1\bar{\text{L}}^\top) \left(\begin{pmatrix} \alpha_1 & (1, 0, 0) \\ \mathbf{0} & \alpha_1 \end{pmatrix}, \begin{pmatrix} \beta_1 & \mathbf{0} \\ (\beta_5, 0, 0) & \beta_8 \end{pmatrix} \right) \text{ with } \beta_1 \leq \beta_8 \text{ and } \beta_5 \neq 0, \\ &(\text{K}_1\bar{\text{M}}) \left(\begin{pmatrix} \alpha_1 & (1, 0, 0) \\ \mathbf{0} & \alpha_1 \end{pmatrix}, \begin{pmatrix} \beta_1 & (0, 1, 0) \\ \mathbf{0} & \beta_8 \end{pmatrix} \right) \text{ with } \beta_1 \leq \beta_8, \\ &(\text{K}_1\bar{\text{M}}_1^\top) \left(\begin{pmatrix} \alpha_1 & (1, 0, 0) \\ \mathbf{0} & \alpha_1 \end{pmatrix}, \begin{pmatrix} \beta_1 & \mathbf{0} \\ (0, 1, 0) & \beta_1 \end{pmatrix} \right), \end{aligned}$$

where $\alpha_1, \alpha_8, \beta_1, \beta_2, \beta_5, \beta_8 \in \mathbb{F}$.

3.2. Pairs of octonions with zero norm. The following remark is a consequence of Proposition 3.1.

REMARK 3.3. The following set is a minimal set of representatives of G_2 -orbits on $\mathbf{O}_\#$:

$$(3.1) \quad \{0, \alpha_1 e_1, \mathbf{u}_1 \mid \alpha_1 \in \mathbb{F}^\times\}.$$

PROPOSITION 3.4. *The following set is a minimal set of representatives of G_2 -orbits on pairs of octonions with zero norm $\mathbf{O}_\#^2$:*

- (0) $(0, 0)$,
- (I) $(0, \beta_1 e_1)$ with $\beta_1 \in \mathbb{F}^\times$,
- (II) $(0, \mathbf{u}_1)$,
- (III) $(\alpha_1 e_1, 0)$,
- (IV) $(\mathbf{u}_1, 0)$,
- (V) $\left(\alpha_1 e_1, \begin{pmatrix} \beta_1 & \mathbf{0} \\ \mathbf{0} & \beta_8 \end{pmatrix} \right)$ with $\beta_1 \beta_8 = 0$ and $(\beta_1, \beta_8) \neq (0, 0)$,
- (VI) $\left(\alpha_1 e_1, \begin{pmatrix} \beta_1 & (1, 0, 0) \\ \mathbf{0} & \beta_8 \end{pmatrix} \right)$ with $\beta_1 \beta_8 = 0$,
- (VII) $\left(\alpha_1 e_1, \begin{pmatrix} \beta_1 & \mathbf{0} \\ (1, 0, 0) & \beta_8 \end{pmatrix} \right)$ with $\beta_1 \beta_8 = 0$,
- (VIII) $\left(\alpha_1 e_1, \begin{pmatrix} \beta_1 & (1, 0, 0) \\ (\beta_1 \beta_8, 0, 0) & \beta_8 \end{pmatrix} \right)$ with $\beta_1, \beta_8 \neq 0$,
- (IX) $\left(\alpha_1 e_1, \begin{pmatrix} \beta_1 & (1, 0, 0) \\ (0, 1, 0) & \beta_8 \end{pmatrix} \right)$ with $\beta_1 \beta_8 = 0$,
- (X) $\left(\mathbf{u}_1, \begin{pmatrix} \beta_1 & \mathbf{0} \\ \mathbf{0} & \beta_8 \end{pmatrix} \right)$ with $\beta_1 \beta_8 = 0$ and $(\beta_1, \beta_8) \neq (0, 0)$,
- (XI) $(\mathbf{u}_1, \beta_2 \mathbf{u}_1)$ with $\beta_2 \neq 0$,
- (XII) $\left(\mathbf{u}_1, \begin{pmatrix} \beta_1 & \mathbf{0} \\ (\beta_5, 0, 0) & 0 \end{pmatrix} \right)$ with $\beta_5 \neq 0$,
- (XIII) $\left(\mathbf{u}_1, \begin{pmatrix} \beta_1 & (0, 1, 0) \\ \mathbf{0} & 0 \end{pmatrix} \right)$,
- (XIV) $(\mathbf{u}_1, \mathbf{v}_2)$,

where $\alpha_1 \in \mathbb{F}^\times$ and $\beta_1, \beta_2, \beta_5, \beta_8 \in \mathbb{F}$.

Proof. The claim follows from Theorem 3.2 as the result of case by case consideration. $\square$

DEFINITION 3.5. 1. The elements from set (3.1) are called *canonical noninvertible octonions*.
 2. The pairs of elements from the formulation of Proposition 3.4 are called *pairs of canonical noninvertible octonions*.

4. Linear equation $ax = c$. In this section, we consider the linear equation

$$(4.1) \quad ax = c,$$

where $a, c \in \mathbf{O}$ are given constants, and $x \in \mathbf{O}$ is a variable. Clearly, if $a \notin \mathbf{O}_\#$, then equation (4.1) has the unique solution $x = a^{-1}c$. The case $a \in \mathbf{O}_\#$ is addressed in the following proposition.

PROPOSITION 4.1. *Let $a \in \mathbf{O}_\#$ be nonzero and $c \in \mathbf{O}$. By acting with G_2 on the equation $ax = c$, we may assume that a is a canonical noninvertible octonion, as in Remark 3.3. Denote by X the solution set of the equation $ax = c$. Then, X is nonempty if and only if one of the following cases occurs for some $\gamma_i \in \mathbb{F}$:*

1. If $a = \alpha_1 e_1$ with $\alpha_1 \in \mathbb{F}^\times$ and $c = \begin{pmatrix} \gamma_1 & (\gamma_2, \gamma_3, \gamma_4) \\ (0, 0, 0) & 0 \end{pmatrix}$, the set X consists of all

$$x = \begin{pmatrix} \gamma_1/\alpha_1 & (\gamma_2/\alpha_1, \gamma_3/\alpha_1, \gamma_4/\alpha_1) \\ (x_5, x_6, x_7) & x_8 \end{pmatrix} \quad \text{for any } x_i \in \mathbb{F}.$$

2. If $a = \mathbf{u}_1$ and $c = \begin{pmatrix} \gamma_1 & (\gamma_2, 0, 0) \\ (0, \gamma_6, \gamma_7) & 0 \end{pmatrix}$, the set X consists of all

$$x = \begin{pmatrix} x_1 & (x_2, \gamma_7, -\gamma_6) \\ (\gamma_1, x_6, x_7) & \gamma_2 \end{pmatrix} \quad \text{for any } x_i \in \mathbb{F}.$$

Moreover, the equation $ax = c$ has a solution if and only if $\bar{a}c = 0$.

Proof. In case x is a solution for $ax = c$, equation (2.4) implies that $0 = n(a)x = \bar{a}(ax) = \bar{a}c$.

Assume that $\bar{a}c = 0$. We will show that in this case there is a solution for $ax = c$ and case 1 or 2 from the formulation of the proposition holds. Denote octonions $c = (\gamma_1, \dots, \gamma_8)$ and $x = (x_1, \dots, x_8)$, where $\gamma_j, x_j \in \mathbb{F}$.

1. Assume that $a = \alpha_1 e_1$ with $\alpha_1 \in \mathbb{F}^\times$. Then, equality $\bar{a}c = 0$ implies that $\gamma_5 = \dots = \gamma_8 = 0$. Hence, x is a solution for $ax = c$ if and only if $x_i = \gamma_i/\alpha_1$ for $i = 1, \dots, 4$.

2. Assume that $a = \mathbf{u}_1$. Then, equality $\bar{a}c = 0$ implies that $\gamma_3 = \gamma_4 = \gamma_5 = \gamma_8 = 0$. Hence, x is a solution for $ax = c$ if and only if $x_3 = \gamma_7$, $x_4 = -\gamma_6$, $x_5 = \gamma_1$, $x_8 = \gamma_2$. $\square$

COROLLARY 4.2. Let $a \in \mathbf{O}$ be nonzero and $c \in \mathbf{O}$, and let $X \subset \mathbf{O}$ denote the set of all solutions of the equation $ax = c$. Then,

- (a) $|X| = 1$ if and only if $a \notin \mathbf{O}_\#$;
- (b) $X \in \Omega_4$ if and only if $a \in \mathbf{O}_\#$ and $\bar{a}c = 0$;
- (c) X is empty if and only if $a \in \mathbf{O}_\#$ and $\bar{a}c \neq 0$.

Proof. If $a \notin \mathbf{O}_\#$, then a is invertible, and $x = a^{-1}c$ is the unique solution of $ax = c$ by formula (2.5).

Assume $a \in \mathbf{O}_\#$. Proposition 4.1 implies that either $X \in \Omega_4$ or X is empty. Since X is nonempty if and only if $\bar{a}c = 0$, the proof is complete. $\square$

5. Linear equation $(ax)b = c$. In this section, we consider the linear equation

$$(5.1) \quad (ax)b = c,$$

where $a, b, c \in \mathbf{O}$ are given constants and $x \in \mathbf{O}$ is a variable. Clearly, if $a \notin \mathbf{O}_\#$ or $b \notin \mathbf{O}_\#$, then solving equation (5.1) can be reduced to solving an equation of the form $a'x = b'$ for some $a', b' \in \mathbf{O}$ (see Corollary 5.2 below for details). The case $a, b \in \mathbf{O}_\#$ is addressed in the following theorem.

THEOREM 5.1. Let $a, b \in \mathbf{O}_\#$ be nonzero and $c \in \mathbf{O}$. By acting with G_2 on the equation $(ax)b = c$, we may assume that (a, b) is a canonical pair of noninvertible octonions as in Proposition 3.4. Denote by X the solution set of the equation $(ax)b = c$. Then, X is nonempty if and only if one of the following cases occurs for some $\alpha_1 \in \mathbb{F}^\times$ and $\beta_i, \gamma_i \in \mathbb{F}$:

(V) 1. For $(a, b) = (\alpha_1 e_1, \beta_8 e_2)$, $c = \gamma_2 \mathbf{u}_1 + \gamma_3 \mathbf{u}_2 + \gamma_4 \mathbf{u}_3$ with $\beta_8 \neq 0$, the set X consists of all

$$x = \begin{pmatrix} x_1 & (\frac{\gamma_2}{\alpha_1 \beta_8}, \frac{\gamma_3}{\alpha_1 \beta_8}, \frac{\gamma_4}{\alpha_1 \beta_8}) \\ (x_5, x_6, x_7) & x_8 \end{pmatrix} \text{ for any } x_i \in \mathbb{F}.$$

2. For $(a, b) = (\alpha_1 e_1, \beta_1 e_1)$, $c = \gamma_1 e_1$ with $\beta_1 \neq 0$, the set X consists of all

$$x = \begin{pmatrix} \frac{\gamma_1}{\alpha_1 \beta_1} & (x_2, x_3, x_4) \\ (x_5, x_6, x_7) & x_8 \end{pmatrix} \text{ for any } x_i \in \mathbb{F}.$$

(VI) 1. For $(a, b) = (\alpha_1 e_1, \beta_8 e_2 + \mathbf{u}_1)$ and

$$c = \begin{pmatrix} 0 & (\gamma_2, -\beta_8 \gamma_7, \beta_8 \gamma_6) \\ (0, \gamma_6, \gamma_7) & 0 \end{pmatrix},$$

with $\beta_8 \neq 0$, the set X consists of all

$$x = \begin{pmatrix} \frac{\gamma_2}{\alpha_1} - \beta_8 x_2 & (x_2, -\frac{\gamma_7}{\alpha_1}, \frac{\gamma_6}{\alpha_1}) \\ (x_5, x_6, x_7) & x_8 \end{pmatrix} \text{ for any } x_i \in \mathbb{F}.$$

2. For $(a, b) = (\alpha_1 e_1, \beta_1 e_1 + \mathbf{u}_1)$ and

$$c = \begin{pmatrix} \beta_1 \gamma_2 & (\gamma_2, 0, 0) \\ (0, \gamma_6, \gamma_7) & 0 \end{pmatrix},$$

the set X consists of all

$$x = \begin{pmatrix} \frac{\gamma_2}{\alpha_1} & (x_2, -\frac{\gamma_7}{\alpha_1}, \frac{\gamma_6}{\alpha_1}) \\ (x_5, x_6, x_7) & x_8 \end{pmatrix} \text{ for any } x_i \in \mathbb{F}.$$

(VII) 1. For $(a, b) = (\alpha_1 e_1, \beta_8 e_2 + \mathbf{v}_1)$ and

$$c = \begin{pmatrix} \gamma_1 & (\beta_8 \gamma_1, \gamma_3, \gamma_4) \\ (0, 0, 0) & 0 \end{pmatrix},$$

with $\beta_8 \neq 0$, the set X consists of all

$$x = \begin{pmatrix} x_1 & (\frac{\gamma_1}{\alpha_1}, \frac{\gamma_3}{\alpha_1 \beta_8}, \frac{\gamma_4}{\alpha_1 \beta_8}) \\ (x_5, x_6, x_7) & x_8 \end{pmatrix} \text{ for any } x_i \in \mathbb{F}.$$

2. For $(a, b) = (\alpha_1 e_1, \beta_1 e_1 + \mathbf{v}_1)$, $c = \gamma_1 e_1$, the set X consists of all

$$x = \begin{pmatrix} x_1 & (\frac{\gamma_1}{\alpha_1} - \beta_1 x_1, x_3, x_4) \\ (x_5, x_6, x_7) & x_8 \end{pmatrix} \text{ for any } x_i \in \mathbb{F}.$$

(VIII) For $(a, b) = (\alpha_1 e_1, \beta_1 e_1 + \beta_8 e_2 + \mathbf{u}_1 + \beta_1 \beta_8 \mathbf{v}_1)$ and

$$c = \begin{pmatrix} \beta_1 \gamma_2 & (\gamma_2, -\beta_8 \gamma_7, \beta_8 \gamma_6) \\ (0, \gamma_6, \gamma_7) & 0 \end{pmatrix},$$

with $\beta_1, \beta_8 \neq 0$, the set X consists of all

$$x = \begin{pmatrix} \frac{\gamma_2}{\alpha_1} - \beta_8 x_2 & (x_2, -\frac{\gamma_7}{\alpha_1}, \frac{\gamma_6}{\alpha_1}) \\ (x_5, x_6, x_7) & x_8 \end{pmatrix} \text{ for any } x_i \in \mathbb{F}.$$

(IX) 1. For $(a, b) = (\alpha_1 e_1, \beta_8 e_2 + \mathbf{u}_1 + \mathbf{v}_2)$ and

$$c = \begin{pmatrix} \gamma_1 & (\gamma_2, \beta_8 \gamma_1, \beta_8 \gamma_6) \\ (0, \gamma_6, -\gamma_1) & 0 \end{pmatrix},$$

with $\beta_8 \neq 0$, the set X consists of all

$$x = \begin{pmatrix} \frac{\gamma_2}{\alpha_1} - \beta_8 x_2 & (x_2, \frac{\gamma_1}{\alpha_1}, \frac{\gamma_6}{\alpha_1}) \\ (x_5, x_6, x_7) & x_8 \end{pmatrix} \text{ for any } x_i \in \mathbb{F}.$$

2. For $(a, b) = (\alpha_1 e_1, \beta_1 e_1 + \mathbf{u}_1 + \mathbf{v}_2)$ and

$$c = \begin{pmatrix} \gamma_1 & (\gamma_2, 0, 0) \\ (0, \gamma_6, -\gamma_1 + \beta_1 \gamma_2) & 0 \end{pmatrix},$$

the set X consists of all

$$x = \begin{pmatrix} \frac{\gamma_2}{\alpha_1} & (x_2, \frac{\gamma_1 - \beta_1 \gamma_2}{\alpha_1}, \frac{\gamma_6}{\alpha_1}) \\ (x_5, x_6, x_7) & x_8 \end{pmatrix} \text{ for any } x_i \in \mathbb{F}.$$

(X) 1. For $(a, b) = (\mathbf{u}_1, \beta_8 e_2)$, $c = \gamma_2 \mathbf{u}_1$ with $\beta_8 \neq 0$, the set X consists of all

$$x = \begin{pmatrix} x_1 & (x_2, x_3, x_4) \\ (x_5, x_6, x_7) & \frac{\gamma_2}{\beta_8} \end{pmatrix} \text{ for any } x_i \in \mathbb{F}.$$

2. For $(a, b) = (\mathbf{u}_1, \beta_1 e_1)$, $c = \gamma_1 e_1 + \gamma_6 \mathbf{v}_2 + \gamma_7 \mathbf{v}_3$ with $\beta_1 \neq 0$, the set X consists of all

$$x = \begin{pmatrix} x_1 & (x_2, \frac{\gamma_7}{\beta_1}, -\frac{\gamma_6}{\beta_1}) \\ (\frac{\gamma_1}{\beta_1}, x_6, x_7) & x_8 \end{pmatrix} \text{ for any } x_i \in \mathbb{F}.$$

(XI) For $(a, b) = (\mathbf{u}_1, \beta_2 \mathbf{u}_1)$, $c = \gamma_2 \mathbf{u}_1$ with $\beta_2 \neq 0$, the set X consists of all

$$x = \begin{pmatrix} x_1 & (x_2, x_3, x_4) \\ (\frac{\gamma_2}{\beta_2}, x_6, x_7) & x_8 \end{pmatrix} \text{ for any } x_i \in \mathbb{F}.$$

(XII) For $(a, b) = (\mathbf{u}_1, \beta_1 e_1 + \beta_5 \mathbf{v}_1)$ and

$$c = \begin{pmatrix} \gamma_1 & (0, \gamma_3, \gamma_4) \\ (0, \frac{\beta_1 \gamma_4}{\beta_5}, -\frac{\beta_1 \gamma_3}{\beta_5}) & 0 \end{pmatrix},$$

with $\beta_5 \neq 0$, the set X consists of all

$$x = \begin{pmatrix} x_1 & (x_2, -\frac{\gamma_3}{\beta_5}, -\frac{\gamma_4}{\beta_5}) \\ (x_5, x_6, x_7) & \frac{\gamma_1 - \beta_1 x_5}{\beta_5} \end{pmatrix} \text{ for any } x_i \in \mathbb{F}.$$

(XIII) For $(a, b) = (\mathbf{u}_1, \beta_1 e_1 + \mathbf{u}_2)$ and

$$c = \begin{pmatrix} \beta_1 \gamma_3 & (0, \gamma_3, 0) \\ (0, \beta_1 \gamma_8, \gamma_7) & \gamma_8 \end{pmatrix},$$

the set X consists of all

$$x = \begin{pmatrix} x_1 & (x_2, x_3, -\gamma_8) \\ (\gamma_3, x_6, x_7) & \gamma_7 - \beta_1 x_3 \end{pmatrix} \text{ for any } x_i \in \mathbb{F}.$$

(XIV) For $(a, b) = (\mathbf{u}_1, \mathbf{v}_2)$, $c = \gamma_2 \mathbf{u}_1$, we have

$$x = \begin{pmatrix} x_1 & (x_2, \gamma_2, x_4) \\ (x_5, x_6, x_7) & x_8 \end{pmatrix} \text{ for any } x_i \in \mathbb{F}.$$

Proof. Denote octonions $c = (\gamma_1, \dots, \gamma_8)$ and $x = (x_1, \dots, x_8)$, where $\gamma_j, x_j \in \mathbb{F}$. Since (a, b) is a canonical pair of noninvertible nonzero octonions from Proposition 3.4, then one of the following ten cases holds, where $\alpha_1 \in \mathbb{F}$ is always nonzero.

- (V) We have $(a, b) = (\alpha_1 e_1, \beta_1 e_1 + \beta_8 e_2)$ with $\beta_1 \beta_8 = 0$ and $(\beta_1, \beta_8) \neq (0, 0)$. Then, equation (5.1) implies that $\gamma_5 = \dots = \gamma_8 = 0$. Since $\beta_1 \beta_8 = 0$, we have that one of the next two cases holds.
1. Assume $\beta_1 = 0$ and $\beta_8 \neq 0$. Hence, it follows from equation (5.1) that $\gamma_1 = 0$, $x_2 = \gamma_2 / \alpha_1 \beta_8$, $x_3 = \gamma_3 / \alpha_1 \beta_8$, $x_4 = \gamma_4 / \alpha_1 \beta_8$.
 2. Assume that $\beta_1 \neq 0$ and $\beta_8 = 0$. Hence, it follows from equation (5.1) that $\gamma_2 = 0$, $\gamma_3 = 0$, $\gamma_4 = 0$, $x_1 = \gamma_1 / \alpha_1 \beta_1$.
- (VI) We have $(a, b) = (\alpha_1 e_1, \beta_1 e_1 + \beta_8 e_2 + \mathbf{u}_1)$ with $\beta_1 \beta_8 = 0$. Since $\beta_1 \beta_8 = 0$, we have that one of the next two cases holds.
1. Assume $\beta_1 = 0$ and $\beta_8 \neq 0$. Then, equation (5.1) implies that $\gamma_1 = 0$, $\gamma_5 = 0$, $\gamma_8 = 0$. Hence, it follows from equation (5.1) that $x_1 = (\gamma_2 / \alpha_1) - \beta_8 x_2$, $x_3 = \gamma_3 / \alpha_1 \beta_8$, $x_4 = \gamma_4 / \alpha_1 \beta_8$. Finally, equation (5.1) implies that $\gamma_4 = \beta_8 \gamma_6$ and $\gamma_3 = -\beta_8 \gamma_7$.
 2. Assume that $\beta_8 = 0$. Then, equation (5.1) implies that $\gamma_3 = 0$, $\gamma_4 = 0$, $\gamma_5 = 0$, $\gamma_8 = 0$. Hence, it follows from equation (5.1), that is, $x_1 = \gamma_2 / \alpha_1$, $x_3 = -\gamma_7 / \alpha_1$, $x_4 = \gamma_6 / \alpha_1$. Finally, equation (5.1) implies that $\gamma_1 = \beta_1 \gamma_2$.
- (VII) We have $(a, b) = (\alpha_1 e_1, \beta_1 e_1 + \beta_8 e_2 + \mathbf{v}_1)$ with $\beta_1 \beta_8 = 0$. Then, equation (5.1) implies that $\gamma_5 = 0$, $\gamma_6 = 0$, $\gamma_7 = 0$, $\gamma_8 = 0$. Since $\beta_1 \beta_8 = 0$, we have that one of the next two cases holds.
1. Assume $\beta_1 = 0$ and $\beta_8 \neq 0$. Then, equation (5.1) implies that $x_2 = \gamma_2 / \alpha_1 \beta_8$, $x_3 = \gamma_3 / \alpha_1 \beta_8$, $x_4 = \gamma_4 / \alpha_1 \beta_8$. Hence, it follows from equation (5.1) that $\gamma_2 = \beta_8 \gamma_1$.
 2. Assume that $\beta_8 = 0$. Hence, it follows from equation (5.1) that $\gamma_2 = 0$, $\gamma_3 = 0$, $\gamma_4 = 0$. Then, equation (5.1) implies that $x_2 = (\gamma_1 / \alpha_1) - \beta_1 x_1$.
- (VIII) We have $(a, b) = \left(\alpha_1 e_1, \begin{pmatrix} \beta_1 & (1, 0, 0) \\ (\beta_1 \beta_8, 0, 0) & \beta_8 \end{pmatrix} \right)$ with $\beta_1, \beta_8 \neq 0$. Then, equation (5.1) implies that $\gamma_5 = 0$ and $\gamma_8 = 0$. Hence, it follows from equation (5.1) that $x_1 = (\gamma_2 / \alpha_1) - \beta_8 x_2$, $x_3 = -\gamma_7 / \alpha_1$, $x_4 = \gamma_6 / \alpha_1$. Finally, it follows from equation (5.1) that $\gamma_1 = \beta_1 \gamma_2$, $\gamma_3 = -\beta_8 \gamma_7$, $\gamma_4 = \beta_8 \gamma_6$.
- (IX) We have $(a, b) = \left(\alpha_1 e_1, \begin{pmatrix} \beta_1 & (1, 0, 0) \\ (0, 1, 0) & \beta_8 \end{pmatrix} \right)$ with $\beta_1 \beta_8 = 0$. Then, equation (5.1) implies that $\gamma_5 = 0$ and $\gamma_8 = 0$. Since $\beta_1 \beta_8 = 0$, we have that one of the next two cases holds.
1. Assume $\beta_1 = 0$ and $\beta_8 \neq 0$. Then, equation (5.1) implies that $x_1 = (\gamma_2 / \alpha_1) - \beta_8 x_2$, $x_3 = \gamma_1 / \alpha_1$, $x_4 = \gamma_6 / \alpha_1$. Hence, it follows from equation (5.1) that $\gamma_3 = \beta_8 \gamma_1$, $\gamma_4 = \beta_8 \gamma_6$, $\gamma_7 = -\gamma_1$.
 2. Assume that $\beta_8 = 0$. Hence, it follows from equation (5.1) that $\gamma_3 = 0$ and $\gamma_4 = 0$. Then, equation (5.1) implies that $x_1 = \gamma_2 / \alpha_1$, $x_3 = -\gamma_7 / \alpha_1$ and $x_4 = \gamma_6 / \alpha_1$. Finally, equation (5.1) implies that $\gamma_7 = -\gamma_1 + \beta_1 \gamma_2$.
- (X) We have $(a, b) = (\mathbf{u}_1, \beta_1 e_1 + \beta_8 e_2)$ with $\beta_1 \beta_8 = 0$ and $(\beta_1, \beta_8) \neq (0, 0)$. Then, equation (5.1) implies that $\gamma_3 = 0$, $\gamma_4 = 0$, $\gamma_5 = 0$, $\gamma_8 = 0$. Since $\beta_1 \beta_8 = 0$, we have that one of the next two cases holds.
1. Assume $\beta_1 = 0$ and $\beta_8 \neq 0$. Hence, it follows from equation (5.1) that $\gamma_1 = 0$, $\gamma_6 = 0$, $\gamma_7 = 0$, $x_8 = \gamma_2 / \beta_8$.

2. Assume $\beta_1 \neq 0$ and $\beta_8 = 0$. Hence, it follows from equation (5.1) that $\gamma_2 = 0$, $x_3 = \gamma_7/\beta_1$, $x_4 = -\gamma_6/\beta_1$, $x_5 = \gamma_1/\beta_1$.

(XI) We have $(a, b) = (\mathbf{u}_1, \beta_2 \mathbf{u}_1)$ with $\beta_2 \neq 0$. Then, equation (5.1) implies that $\gamma_1 = 0$, $\gamma_3 = \dots = \gamma_8 = 0$, $x_5 = \gamma_2/\beta_2$.

(XII) We have $(a, b) = (\mathbf{u}_1, \beta_1 e_1 + \beta_5 \mathbf{v}_1)$ with $\beta_5 \neq 0$. Then, equation (5.1) implies that $\gamma_2 = 0$, $\gamma_5 = 0$, $\gamma_8 = 0$. Hence, it follows from equation (5.1) that $x_3 = -\gamma_3/\beta_5$, $x_4 = -\gamma_4/\beta_5$, $x_8 = (\gamma_1 - \beta_1 x_5)/\beta_5$. Finally, it follows from equation (5.1) that $\gamma_6 = \beta_1 \gamma_4/\beta_5$ and $\gamma_7 = -\beta_1 \gamma_3/\beta_5$.

(XIII) We have $(a, b) = (\mathbf{u}_1, \beta_1 e_1 + \mathbf{u}_2)$. Then, equation (5.1) implies that $\gamma_2 = 0$, $\gamma_4 = 0$, $\gamma_5 = 0$, $x_4 = -\gamma_8$. Hence, it follows from equation (5.1) that $x_5 = \gamma_3$, $x_8 = \gamma_7 - \beta_1 x_3$. Finally, it follows from equation (5.1) that $\gamma_1 = \beta_1 \gamma_3$ and $\gamma_6 = \beta_1 \gamma_8$.

(XIV) We have $(a, b) = (\mathbf{u}_1, \mathbf{v}_2)$. Then, equation (5.1) implies that $\gamma_1 = 0$, $x_3 = \gamma_2$, $\gamma_3 = \dots = \gamma_8 = 0$. $\square$

COROLLARY 5.2. *Given nonzero $a, b \in \mathbf{O}$ and $c \in \mathbf{O}$, let $X \subset \mathbf{O}$ be the set of all solutions of the equation $(ax)b = c$. Then, X satisfies one of the following conditions:*

- (a) X is empty;
- (b) $|X| = 1$;
- (c) $X \in \Omega_r$ for some $r \in \{4, 5, 7\}$.

Moreover,

- each of the above cases occurs for some nonzero $a, b \in \mathbf{O}$ and $c \in \mathbf{O}$;
- $|X| = 1$ if and only if $a, b \notin \mathbf{O}_\#$;
- $X \in \Omega_4$ if and only if either $a \notin \mathbf{O}_\#$, $b \in \mathbf{O}_\#$, $b\bar{c} = 0$, or $a \in \mathbf{O}_\#$, $b \notin \mathbf{O}_\#$, $\bar{a}(cb^{-1}) = 0$;
- $X \in \Omega_r$ for $r \in \{5, 7\}$ if and only if $a, b \in \mathbf{O}_\#$ and X is nonempty.

Proof. Assume that $a, b \notin \mathbf{O}_\#$. Then by (2.5), we have that x is a solution of (5.1) if and only if $x = a^{-1}(cb^{-1})$.

Assume that $a \notin \mathbf{O}_\#$ and $b \in \mathbf{O}_\#$. Then, x is a solution of (5.1) if and only if ax is a solution of the equation $yb = c$, or equivalently, $\bar{a}x$ is a solution of $\bar{b}z = \bar{c}$, where z is a variable. By Corollary 4.2, the set $Z \subset \mathbf{O}$ of all solutions of $\bar{b}z = \bar{c}$ is either empty or $Z \in \Omega_4$. In the first case, X is empty. In the second case, we consider a linear invertible map $f : \mathbf{O} \rightarrow \mathbf{O}$, $z \rightarrow a^{-1}\bar{z}$; therefore, $X = f(Z) \in \Omega_4$. Moreover, by Corollary 4.2, the second case holds if and only if $b\bar{c} = 0$. Thus, we have proven that

$$(5.2) \quad \text{if } b\bar{c} = 0, \text{ then } X \in \Omega_4; \text{ otherwise, } X \text{ is empty.}$$

Assume that $a \in \mathbf{O}_\#$ and $b \notin \mathbf{O}_\#$. Then, x is a solution of (5.1) if and only if $ax = cb^{-1}$. Corollary 4.2 implies that

$$(5.3) \quad \text{if } \bar{a}(cb^{-1}) = 0, \text{ then } X \in \Omega_4; \text{ otherwise, } X \text{ is empty.}$$

If $a, b \in \mathbf{O}_\#$, then Theorem 5.1 implies that X is empty or $X \in \Omega_r$ for $r \in \{5, 7\}$. The required statement is proven. $\square$

REMARK 5.3. If the equation $(ax)b = c$ has a solution for some $a, b \in \mathbf{O}_\#$, $c \in \mathbf{O}$, then necessarily $c\bar{b} = 0$. However, the converse does not hold in general.

Proof. If $(ax)b = c$, then multiplying both sides on the right by $\bar{b}$ gives

$$c\bar{b} = ((ax)b)\bar{b} = n(b)ax = 0,$$

using equalities (2.4). On the other hand, for $a = e_1$, $b = \mathbf{u}_1$, and $c = e_2$, we have $c\bar{b} = 0$, but the equation $(ax)b = c$ has no solution. $\square$

6. Linear equation $a(bx) = c$. In this section, we consider the linear equation

$$(6.1) \quad a(bx) = c,$$

where $a, b, c \in \mathbf{O}$ are given constants and $x \in \mathbf{O}$ is a variable. Clearly, if $a \notin \mathbf{O}_\#$ or $b \notin \mathbf{O}_\#$, then solving equation (6.1) can be reduced to solving an equation of the form $a'x = b'$ for some $a', b' \in \mathbf{O}$ (see Corollary 6.2 below for details). The case $a, b \in \mathbf{O}_\#$ is addressed in the following theorem.

THEOREM 6.1. *Given a nonzero $a, b \in \mathbf{O}_\#$ and $c \in \mathbf{O}$, acting by G_2 on the equation $a(bx) = c$ we can assume that (a, b) is a canonical pair of noninvertible octonions from Proposition 3.4. Denote by X the solution set of the equation $a(bx) = c$. Then, X is nonempty if and only if one of the following cases holds for some $\alpha_1 \in \mathbb{F}^\times$, $\beta_i, \gamma_i \in \mathbb{F}$:*

(V) 1. For $(a, b) = (\alpha_1 e_1, \beta_8 e_2)$, $c = 0$ with $\beta_8 \neq 0$, the set X consists of all

$$x = \begin{pmatrix} x_1 & (x_2, x_3, x_4) \\ (x_5, x_6, x_7) & x_8 \end{pmatrix} \text{ for any } x_i \in \mathbb{F}.$$

2. For $(a, b) = (\alpha_1 e_1, \beta_1 e_1)$ and

$$c = \begin{pmatrix} \gamma_1 & (\gamma_2, \gamma_3, \gamma_4) \\ (0, 0, 0) & 0 \end{pmatrix},$$

with $\beta_1 \neq 0$, the set X consists of all

$$x = \begin{pmatrix} \frac{\gamma_1}{\alpha_1 \beta_1} & (\frac{\gamma_2}{\alpha_1 \beta_1}, \frac{\gamma_3}{\alpha_1 \beta_1}, \frac{\gamma_4}{\alpha_1 \beta_1}) \\ (x_5, x_6, x_7) & x_8 \end{pmatrix} \text{ for any } x_i \in \mathbb{F}.$$

(VI) 1. For $(a, b) = (\alpha_1 e_1, \beta_8 e_2 + \mathbf{u}_1)$, $c = \gamma_1 e_1 + \gamma_2 \mathbf{u}_1$, the set X consists of all

$$x = \begin{pmatrix} x_1 & (x_2, x_3, x_4) \\ (\frac{\gamma_1}{\alpha_1}, x_6, x_7) & \frac{\gamma_2}{\alpha_1} \end{pmatrix} \text{ for any } x_i \in \mathbb{F}.$$

2. For $(a, b) = (\alpha_1 e_1, \beta_1 e_1 + \mathbf{u}_1)$ and

$$c = \begin{pmatrix} \gamma_1 & (\gamma_2, \gamma_3, \gamma_4) \\ (0, 0, 0) & 0 \end{pmatrix},$$

with $\beta_1 \neq 0$, the set X consists of all

$$x = \begin{pmatrix} x_1 & (x_2, \frac{\gamma_3}{\alpha_1 \beta_1}, \frac{\gamma_4}{\alpha_1 \beta_1}) \\ (\frac{\gamma_1}{\alpha_1} - \beta_1 x_1, x_6, x_7) & \frac{\gamma_2}{\alpha_1} - \beta_1 x_2 \end{pmatrix} \text{ for any } x_i \in \mathbb{F}.$$

(VII) 1. For $(a, b) = (\alpha_1 e_1, \beta_8 e_2 + \mathbf{v}_1)$, $c = \gamma_3 \mathbf{u}_2 + \gamma_4 \mathbf{u}_3$ the set X consists of all

$$x = \begin{pmatrix} x_1 & (x_2, x_3, x_4) \\ (x_5, -\frac{\gamma_4}{\alpha_1}, \frac{\gamma_3}{\alpha_1}) & x_8 \end{pmatrix} \text{ for any } x_i \in \mathbb{F}.$$

2. For $(a, b) = (\alpha_1 e_1, \beta_1 e_1 + \mathbf{v}_1)$ and

$$c = \begin{pmatrix} \gamma_1 & (\gamma_2, \gamma_3, \gamma_4) \\ (0, 0, 0) & 0 \end{pmatrix},$$

with $\beta_1 \neq 0$, the set X consists of all

$$x = \begin{pmatrix} x_1 & (\frac{\gamma_2}{\alpha_1 \beta_1}, x_3, x_4) \\ (x_5, -\frac{\gamma_4}{\alpha_1} + \beta_1 x_4, \frac{\gamma_3}{\alpha_1} - \beta_1 x_3) & x_8 \end{pmatrix} \text{ for any } x_i \in \mathbb{F}.$$

(VIII) For $(a, b) = (\alpha_1 e_1, \beta_1 e_1 + \beta_8 e_2 + \mathbf{u}_1 + \beta_1 \beta_8 \mathbf{v}_1)$ and

$$c = \begin{pmatrix} \gamma_1 & (\gamma_2, \gamma_3, \gamma_4) \\ (0, 0, 0) & 0 \end{pmatrix},$$

with $\beta_1, \beta_8 \neq 0$, the set X consists of all

$$x = \begin{pmatrix} x_1 & (x_2, \frac{\gamma_3}{\alpha_1 \beta_1} - \beta_8 x_7, \frac{\gamma_4}{\alpha_1 \beta_1} + \beta_8 x_6) \\ (\frac{\gamma_1}{\alpha_1} - \beta_1 x_1, x_6, x_7) & \frac{\gamma_2}{\alpha_1} - \beta_1 x_2 \end{pmatrix} \text{ for any } x_i \in \mathbb{F}.$$

(IX) 1. For $(a, b) = (\alpha_1 e_1, \beta_8 e_2 + \mathbf{u}_1 + \mathbf{v}_2)$ and

$$c = \begin{pmatrix} \gamma_1 & (\gamma_2, 0, \gamma_1) \\ (0, 0, 0) & 0 \end{pmatrix},$$

the set X consists of all

$$x = \begin{pmatrix} x_1 & (x_2, x_3, x_4) \\ (\frac{\gamma_1}{\alpha_1}, x_6, x_7) & \frac{\gamma_2}{\alpha_1} + x_7 \end{pmatrix} \text{ for any } x_i \in \mathbb{F}.$$

2. For $(a, b) = (\alpha_1 e_1, \beta_1 e_1 + \mathbf{u}_1 + \mathbf{v}_2)$ and

$$c = \begin{pmatrix} \gamma_1 & (\gamma_2, \gamma_3, \gamma_4) \\ (0, 0, 0) & 0 \end{pmatrix},$$

with $\beta_1 \neq 0$, the set X consists of all

$$x = \begin{pmatrix} x_1 & (x_2, \frac{\gamma_3}{\alpha_1 \beta_1}, \frac{\gamma_4 - \gamma_1}{\alpha_1 \beta_1} + x_1) \\ (\frac{\gamma_1}{\alpha_1} - \beta_1 x_1, x_6, x_7) & \frac{\gamma_2}{\alpha_1} - \beta_1 x_2 + x_7 \end{pmatrix} \text{ for any } x_i \in \mathbb{F}.$$

(X) 1. For $(a, b) = (\mathbf{u}_1, \beta_8 e_2)$, $c = \gamma_1 e_1 + \gamma_2 \mathbf{u}_1$ with $\beta_8 \neq 0$, the set X consists of all

$$x = \begin{pmatrix} x_1 & (x_2, x_3, x_4) \\ (\frac{\gamma_1}{\beta_8}, x_6, x_7) & \frac{\gamma_2}{\beta_8} \end{pmatrix} \text{ for any } x_i \in \mathbb{F}.$$

2. For $(a, b) = (\mathbf{u}_1, \beta_1 e_1)$, $c = \gamma_6 \mathbf{v}_2 + \gamma_7 \mathbf{v}_3$ with $\beta_1 \neq 0$, the set X consists of all

$$x = \begin{pmatrix} x_1 & (x_2, \frac{\gamma_7}{\beta_1}, -\frac{\gamma_6}{\beta_1}) \\ (x_5, x_6, x_7) & x_8 \end{pmatrix} \text{ for any } x_i \in \mathbb{F}.$$

(XI) For $(a, b) = (\mathbf{u}_1, \beta_2 \mathbf{u}_1)$, $c = 0$ with $\beta_2 \neq 0$, the set X consists of all

$$x = \begin{pmatrix} x_1 & (x_2, x_3, x_4) \\ (x_5, x_6, x_7) & x_8 \end{pmatrix} \text{ for any } x_i \in \mathbb{F}.$$

(XII) For $(a, b) = (\mathbf{u}_1, \beta_1 e_1 + \beta_5 \mathbf{v}_1)$ and

$$c = \begin{pmatrix} \gamma_1 & (\gamma_2, 0, 0) \\ (0, \gamma_6, \gamma_7) & 0 \end{pmatrix},$$

with $\beta_5 \neq 0$, the set X consists of all

$$x = \begin{pmatrix} \frac{\gamma_1}{\beta_5} & (\frac{\gamma_2}{\beta_5}, x_3, x_4) \\ (x_5, \frac{\gamma_6 + \beta_1 x_4}{\beta_5}, \frac{\gamma_7 - \beta_1 x_3}{\beta_5}) & x_8 \end{pmatrix} \text{ for any } x_i \in \mathbb{F}.$$

(XIII) For $(a, b) = (\mathbf{u}_1, \beta_1 e_1 + \mathbf{u}_2)$ and

$$c = \begin{pmatrix} \gamma_1 & (0, 0, 0) \\ (0, -\beta_1 \gamma_1, \gamma_7) & 0 \end{pmatrix},$$

the set X consists of all

$$x = \begin{pmatrix} x_1 & (x_2, x_3, \gamma_1) \\ (x_5, x_6, x_7) & \gamma_7 - \beta_1 x_3 \end{pmatrix} \text{ for any } x_i \in \mathbb{F}.$$

(XIV) For $(a, b) = (\mathbf{u}_1, \mathbf{v}_2)$, $c = \gamma_2 \mathbf{u}_1 + \gamma_6 \mathbf{v}_2$, we have

$$x = \begin{pmatrix} x_1 & (x_2, \gamma_2, x_4) \\ (-\gamma_6, x_6, x_7) & x_8 \end{pmatrix} \text{ for any } x_i \in \mathbb{F}.$$

Proof. Denote octonions $c = (\gamma_1, \dots, \gamma_8)$ and $x = (x_1, \dots, x_8)$, where $\gamma_j, x_j \in \mathbb{F}$. Since (a, b) is a canonical pair of noninvertible nonzero octonions from Proposition 3.4, then one of the following ten cases holds, where $\alpha_1 \in \mathbb{F}$ is always nonzero.

(V) $(a, b) = (\alpha_1 e_1, \beta_1 e_1 + \beta_8 e_2)$ with $\beta_1 \beta_8 = 0$ and $(\beta_1, \beta_8) \neq (0, 0)$. Then, equation (6.1) implies that $\gamma_5 = \dots = \gamma_8 = 0$. Since $\beta_1 \beta_8 = 0$, we have that one of the next two cases holds.

1. Assume $\beta_1 = 0$ and $\beta_8 \neq 0$. Hence, it follows from equation (6.1) that $\gamma_1 = \dots = \gamma_4 = 0$.
2. Assume that $\beta_1 \neq 0$ and $\beta_8 = 0$. Hence, it follows from equation (6.1) that $x_1 = \gamma_1 / \alpha_1 \beta_1$, $x_2 = \gamma_2 / \alpha_1 \beta_1$, $x_3 = \gamma_3 / \alpha_1 \beta_1$, $x_4 = \gamma_4 / \alpha_1 \beta_1$.

(VI) $(a, b) = (\alpha_1 e_1, \beta_1 e_1 + \beta_8 e_2 + \mathbf{u}_1)$ with $\beta_1 \beta_8 = 0$. Then, equation (6.1) implies that $\gamma_5 = \dots = \gamma_8 = 0$. Since $\beta_1 \beta_8 = 0$, we have that one of the next two cases holds.

1. Assume $\beta_1 = 0$. Hence, it follows from equation (6.1) that $x_5 = \gamma_1 / \alpha_1$, $x_8 = \gamma_2 / \alpha_1$, $\gamma_3 = 0$, $\gamma_4 = 0$.
2. Assume that $\beta_1 \neq 0$ and $\beta_8 = 0$. Hence, it follows from equation (6.1) that $x_5 = (\gamma_1 / \alpha_1) - \beta_1 x_1$, $x_8 = (\gamma_2 / \alpha_1) - \beta_1 x_2$, $x_3 = \gamma_3 / \alpha_1 \beta_1$, $x_4 = \gamma_4 / \alpha_1 \beta_1$.

(VII) $(a, b) = (\alpha_1 e_1, \beta_1 e_1 + \beta_8 e_2 + \mathbf{v}_1)$ with $\beta_1 \beta_8 = 0$. Then, equation (6.1) implies that $\gamma_5 = \dots = \gamma_8 = 0$. Since $\beta_1 \beta_8 = 0$, we have that one of the next two cases holds.

1. Assume $\beta_1 = 0$. Hence, it follows from equation (6.1) that $\gamma_1 = 0$, $\gamma_2 = 0$, $x_6 = -\gamma_4 / \alpha_1$, $x_7 = \gamma_3 / \alpha_1$.

2. Assume that $\beta_1 \neq 0$ and $\beta_8 = 0$. Hence, it follows from equation (6.1) that $x_1 = \gamma_1/\alpha_1\beta_1$, $x_2 = \gamma_2/\alpha_1\beta_1$, $x_7 = (\gamma_3/\alpha_1) - \beta_1x_3$, $x_6 = (\alpha_1\beta_1x_4 - \gamma_4)/\alpha_1$.

(VIII) $(a, b) = \left(\alpha_1 e_1, \begin{pmatrix} \beta_1 & (1, 0, 0) \\ (\beta_1\beta_8, 0, 0) & \beta_8 \end{pmatrix} \right)$ with $\beta_1, \beta_8 \neq 0$. Then, equation (6.1) implies that $\gamma_5 = \dots = \gamma_8 = 0$. Hence, it follows from equation (6.1) that $x_3 = (\gamma_3/\alpha_1\beta_1) - \beta_8x_7$, $x_4 = (\gamma_4/\alpha_1\beta_1) + \beta_8x_6$, $x_5 = (\gamma_1/\alpha_1) - \beta_1x_1$, $x_8 = (\gamma_2/\alpha_1) - \beta_1x_2$.

(IX) $(a, b) = \left(\alpha_1 e_1, \begin{pmatrix} \beta_1 & (1, 0, 0) \\ (0, 1, 0) & \beta_8 \end{pmatrix} \right)$ with $\beta_1\beta_8 = 0$. Then, equation (6.1) implies that $\gamma_5 = \dots = \gamma_8 = 0$. Since $\beta_1\beta_8 = 0$, we have that one of the next two cases holds.

1. Assume $\beta_1 = 0$. Hence, it follows from equation (6.1) that $\gamma_3 = 0$, $x_5 = \gamma_1/\alpha_1$, $x_8 = (\gamma_2/\alpha_1) + x_7$. Therefore, $\gamma_4 = \gamma_1$ by equation (6.1).
2. Assume that $\beta_1 \neq 0$ and $\beta_8 = 0$. Hence, it follows from equation (6.1) that $x_3 = \gamma_3/\alpha_1\beta_1$, $x_5 = (\gamma_1/\alpha_1) - \beta_1x_1$, $x_8 = (\gamma_2/\alpha_1) + x_7 - \beta_1x_2$. Therefore, equation (6.1) implies that

$$x_4 = \frac{\gamma_4 - \gamma_1}{\alpha_1\beta_1} + x_1.$$

(X) $(a, b) = (\mathbf{u}_1, \beta_1 e_1 + \beta_8 e_2)$ with $\beta_1\beta_8 = 0$ and $(\beta_1, \beta_8) \neq (0, 0)$. Then, equation (6.1) implies that $\gamma_3 = 0$, $\gamma_4 = 0$, $\gamma_5 = 0$, $\gamma_8 = 0$. Since $\beta_1\beta_8 = 0$, we have that one of the next two cases holds.

1. Assume $\beta_1 = 0$ and $\beta_8 \neq 0$. Hence, it follows from equation (6.1) that $\gamma_6 = 0$, $\gamma_7 = 0$, $x_5 = \gamma_1/\beta_8$, $x_8 = \gamma_2/\beta_8$.
2. Assume that $\beta_1 \neq 0$ and $\beta_8 = 0$. Hence, it follows from equation (6.1) that $\gamma_1 = 0$, $\gamma_2 = 0$, $x_3 = \gamma_7/\beta_1$, $x_4 = -\gamma_6/\beta_1$.

(XI) $(a, b) = (\mathbf{u}_1, \beta_2 \mathbf{u}_1)$ with $\beta_2 \neq 0$. Then, equation (6.1) implies that $c = 0$ and x is an arbitrary.

(XII) $(a, b) = (\mathbf{u}_1, \beta_1 e_1 + \beta_5 \mathbf{v}_1)$ with $\beta_5 \neq 0$. Then, equation (6.1) implies that $\gamma_3 = 0$, $\gamma_4 = 0$, $\gamma_5 = 0$, $\gamma_8 = 0$. Hence, it follows from equation (6.1) that $x_1 = \gamma_1/\beta_5$, $x_2 = \gamma_2/\beta_5$, $x_6 = (\gamma_6 + \beta_1x_4)/\beta_5$, $x_7 = (\gamma_7 - \beta_1x_3)/\beta_5$.

(XIII) $(a, b) = (\mathbf{u}_1, \beta_1 e_1 + \mathbf{u}_2)$. Then, equation (6.1) implies that $\gamma_2 = \dots = \gamma_5 = 0$, $\gamma_8 = 0$. Hence, it follows from equation (6.1) that $x_4 = \gamma_1$, $x_8 = \gamma_7 - \beta_1x_3$. Finally, $\gamma_6 = -\beta_1\gamma_1$ by equation (6.1).

(XIV) $(a, b) = (\mathbf{u}_1, \mathbf{v}_2)$. Then, equation (6.1) implies that $\gamma_1 = 0$, $\gamma_3 = 0$, $\gamma_4 = 0$, $\gamma_5 = 0$, $\gamma_7 = 0$, $\gamma_8 = 0$, $x_3 = \gamma_2$, $x_5 = -\gamma_6$. $\square$

COROLLARY 6.2. *Given nonzero $a, b \in \mathbf{O}$ and $c \in \mathbf{O}$, let $X \subset \mathbf{O}$ be the set of all solutions of the equation $a(bx) = c$. Then, X satisfies one of the following conditions:*

- (a) X is empty;
- (b) $|X| = 1$;
- (c) $X \in \Omega_r$ for some $r \in \{4, 6\}$;
- (d) $X = \mathbf{O}$.

Moreover,

- each of the above cases occurs for some nonzero $a, b \in \mathbf{O}$ and $c \in \mathbf{O}$;
- $|X| = 1$ if and only if $a, b \notin \mathbf{O}_\#$;
- $X = \mathbf{O}$ if and only if $b = \xi \bar{a} \in \mathbf{O}_\#$ for some $\xi \in \mathbb{F}^\times$ and $c = 0$.

Proof. Assume that $a, b \notin \mathbf{O}_\#$. Then, by (2.5), we have that x is a solution of equation (6.1) if and only if $x = b^{-1}(a^{-1}c)$.

Assume that $a \notin \mathbf{O}_\#$ and $b \in \mathbf{O}_\#$. Then, x is a solution of (6.1) if and only if $bx = a^{-1}c$. Corollary 4.2 implies that either X is empty or $X \in \Omega_4$.

Assume that $a \in \mathbf{O}_\#$ and $b \notin \mathbf{O}_\#$. Then, x is a solution of (6.1) if and only if bx is a solution of the equation $ay = c$. By Corollary 4.2, the set $Y \subset \mathbf{O}$ of all solutions of the equation $ay = c$ is either empty or $Y \in \Omega_4$. In the first case, X is empty. In the second case, $X = b^{-1}Y \in \Omega_4$, since the multiplication by b^{-1} is the invertible linear map $\mathbf{O} \rightarrow \mathbf{O}$.

If $a, b \in \mathbf{O}_\#$, then Theorem 6.1 implies that either X is empty or $X \in \Omega_r$ for $r \in \{4, 6\}$ or $X = \mathbf{O}$. Assume that $X = \mathbf{O}$. Then, it follows from Theorem 6.1 that the G_2 -orbit of (a, b) contains $(\alpha e_1, \beta e_2)$ or $(\mathbf{u}_1, \beta \mathbf{u}_1)$ for some $\alpha, \beta \in \mathbb{F}^\times$ and $c = 0$; in particular, $b = \xi \bar{a}$ for some $\xi \in \mathbb{F}^\times$. Assume that $b = \xi \bar{a} \in \mathbf{O}_\#$ for some $\xi \in \mathbb{F}^\times$ and $c = 0$. Then, for every $x \in \mathbf{O}$, we have $a(bx) = \xi a(\bar{a}x) = \xi n(a)x = 0$ by equalities (2.4), i.e., $X = \mathbf{O}$. The required statement is proven. $\square$

REMARK 6.3. If $a(bx) = c$ has a solution for some $a, b \in \mathbf{O}_\#$ and $c \in \mathbf{O}$, then $\bar{a}c = 0$. However, the converse does not hold in general.

Proof. If $a(bx) = c$, then $\bar{a}c = n(a)bx = 0$ by equalities (2.4). On the other hand, for $a = e_1$, $b = e_2$, and $c = e_1$, we have $\bar{a}c = 0$, but the equation $a(bx) = c$ has no solution. $\square$

7. Linear monomial equations. We write $a_1 \circ \dots \circ a_m$ for an arbitrary nonassociative product of $a_1, \dots, a_m \in \mathbf{O}$. A *multilinear monomial* $w = w(a_1, \dots, a_m)$ is a product of the form $a_{\sigma(1)} \circ \dots \circ a_{\sigma(m)}$ for some permutation $\sigma \in \mathcal{S}_m$. A *linear monomial equation* over the octonions is an equation $w(a_1, \dots, a_m, x) = c$, where $w(a_1, \dots, a_m, x)$ is a multilinear monomial, $x \in \mathbf{O}$ is a variable, and $c \in \mathbf{O}$ is a constant. For example, the equations $(ax)b = c$ and $a(bx) = c$ are linear monomial equations.

COROLLARY 7.1. Assume that $a_1, \dots, a_m \in \mathbf{O}$ are nonzero and $c \in \mathbf{O}$. Let $X \subset \mathbf{O}$ be the set of all solutions of a linear monomial equation

$$w(a_1, \dots, a_m, x) = c.$$

Then, X satisfies one of the following conditions:

- (a) X is empty;
- (b) $|X| = 1$;
- (c) $X \in \Omega_r$ for some $4 \leq r \leq 7$;
- (d) $X = \mathbf{O}$.

Moreover,

- each of the above cases occurs for some linear monomial equation;
- $|X| = 1$ if and only if $a_1, \dots, a_m \notin \mathbf{O}_\#$.

Proof. By Corollaries 5.2 and 6.2, each of the cases (a)–(d) from the formulation of the corollary hold for some $w(a_1, a_2, x) \in \{(a_1x)a_2, a_1(a_2x)\}$. The rest of the proof we split into two parts.

1. If $a_1, \dots, a_m \notin \mathbf{O}_\#$, then applying equalities (2.5) m times, we obtain that there exists a unique solution of the equation $w(a_1, \dots, a_m, x) = c$.

2. Assume that $a_i \in \mathbf{O}_\#$ for some $1 \leq i \leq m$ and X is not empty, i.e., there exists $x_0 \in \mathbf{O}$ such that $w(a_1, \dots, a_m, x_0) = c$.

If $w(a_1, \dots, a_m, x) = \dots \circ (a_i a_j) \circ \dots$ for some $1 \leq i, j \leq m$, then we can denote $a' = a_i a_j$ and consider $w(a_1, \dots, a_m, x)$ as a multilinear monomial in $\{x, a', a_r \mid 1 \leq r \leq m, r \neq i, j\}$; i.e., we diminish m by one. Note that we can assume that $a' \neq 0$, since otherwise $X = \emptyset$ in case $c \neq 0$ and $X = \mathbf{O}$ in case $c = 0$. After repeating this procedure several times, we can assume that

$$(7.1) \quad w(x) \neq \dots \circ (a_i a_j) \circ \dots \quad \text{for all } 1 \leq i, j \leq m.$$

Since condition (7.1) holds, we have

$$(7.2) \quad w(a_1, \dots, a_m, x) = a_{i_1} \circ \dots \circ a_{i_{r-1}} \circ (a_{i_r} v(x)) \circ a_{i_{r+1}} \circ \dots \circ a_{i_s} \text{ or}$$

$$(7.3) \quad w(a_1, \dots, a_m, x) = a_{i_1} \circ \dots \circ a_{i_{r-1}} \circ (v(x) a_{i_r}) \circ a_{i_{r+1}} \circ \dots \circ a_{i_s},$$

where

- $v(x) := v(a_{j_1}, \dots, a_{j_t}, x)$ is a multilinear monomial for some $a_{j_1}, \dots, a_{j_t} \notin \mathbf{O}_\#$ with $t \geq 0$;
- $\{i_1, \dots, i_s, j_1, \dots, j_t\} = \{1, \dots, m\}$ and $s + t = m$, $1 \leq r \leq s$;
- $a_{i_r} \in \mathbf{O}_\#$.

Denote by $X_1 \subset \mathbf{O}$ the set of solutions of the equation $a_{i_r} v(x) = a_{i_r} v(x_0)$ and denote by $X_2 \subset \mathbf{O}$ the set of solutions of the equation $v(x) a_{i_r} = v(x_0) a_{i_r}$. Note that $X_1 \subset X$ or $X_2 \subset X$. Denote by $Y_1 \subset \mathbf{O}$ the set of solutions of the equation $a_{i_r} y = a_{i_r} v(x_0)$ and denote by $Y_2 \subset \mathbf{O}$ the set of solutions of the equation $y a_{i_r} = v(x_0) a_{i_r}$, where y is a variable. Since $a_{j_1}, \dots, a_{j_t}$ are invertible, equalities (2.5) imply that there are linear invertible maps $f_l : \mathbf{O} \rightarrow \mathbf{O}$ such that $f_l(Y_l) = X_l$, where $l = 1, 2$. One of the following two cases holds:

- Assume that equality (7.2) holds. Then, $x_0 \in X_1$ and Y_1 is also not empty. Corollary 4.2 implies that $Y_1 \in \Omega_4$, since $a_{i_r} \in \mathbf{O}_\#$. Applying the invertible linear map f_1 to Y_1 , we can see that $X_1 \in \Omega_4$.
- Assume that equality (7.3) holds. Then, $x_0 \in X_2$ and Y_2 is also not empty. Applying the involution $- : \mathbf{O} \rightarrow \mathbf{O}$ to the equation $y a_{i_r} = v(x_0) a_{i_r}$, we also obtain that $Y_2 \in \Omega_4$. Applying the invertible linear map f_2 to Y_2 , we can see that $X_2 \in \Omega_4$.

Obviously, $X \in \Omega_r$ for some $0 \leq r \leq 8$. Since $X_l \in \Omega_4$ and $X_l \subset X$ for some $l = 1, 2$, we obtain that $r \geq 4$. The required statement is proven. $\square$

COROLLARY 7.2. *If a linear monomial equation $w(a_1, \dots, a_m, x) = c$ with nonzero $a_1, \dots, a_m, c \in \mathbf{O}$ has at least two solutions, then it has an invertible solution.*

Proof. Assume that $X \subset \mathbf{O}$ is the set of all solutions of a linear monomial equation $w(a_1, \dots, a_m, x) = c$. Since $|X| > 1$, there exist $a_i \in \mathbf{O}_\#$ for some $1 \leq i \leq m$ (see Corollary 7.1). Therefore, case 2 of the proof of Corollary 7.1 holds. Hence, we repeat the reasoning from case 2 of the proof of Corollary 7.1 and use notations from it. One of the following two cases holds:

- Assume that equality (7.2) holds. Then, $x_0 \in X_1$ and Y_1 is also nonempty. Since $c \neq 0$, we have that $a_{i_r} v(x_0)$ is also nonzero. Hence, since $a_{i_r} \in \mathbf{O}_\#$, it is easy to see that Proposition 4.1 implies that there exists an invertible $y_0 \in Y_1$. Finally, $f_1(y_0) \in X_1 \subset X$ is invertible.

- Assume that equality (7.3) holds. Then, $x_0 \in X_2$ and Y_2 is also non-empty. Since $c \neq 0$, we have that $v(x_0)a_{i_r}$ is also nonzero. Applying the involution $- : \mathbf{O} \rightarrow \mathbf{O}$ to the equation $ya_{i_r} = v(x_0)a_{i_r}$, as above, we obtain that there exists an invertible $y_0 \in Y_2$, since $a_{i_r} \in \mathbf{O}_\#$. Finally, $f_2(y_0) \in X_2 \subset X$ is invertible. $\square$

REMARK 7.3. In the statement of Corollary 7.2, the assumption that c is nonzero is essential and cannot be omitted.

Indeed, consider the linear equation $ax = 0$ with $a \in \mathbf{O}_\#$ nonzero. By Corollary 4.2, its solution set X is contained in Ω_4 and is therefore infinite. On the other hand, if X contained an invertible element x_0 , then we would have $0 = ax_0 = (ax_0)x_0^{-1} = a$, by equality (2.5), which contradicts the assumption $a \neq 0$.

Recall that an octonion $a \in \mathbf{O}$ is said to be *nilpotent* if $a^n = 0$ for some $n > 0$. Proposition 3.1, Lemma 3.1 of [25], and equality (2.2) together imply that the following three conditions for $a \in \mathbf{O}$ are equivalent:

$$(7.4) \quad a \text{ is nilpotent} \iff \operatorname{tr}(a) = n(a) = 0 \iff a^2 = 0.$$

REMARK 7.4. In the statement of Corollary 7.2, a nilpotent solution cannot be considered in place of an invertible one.

Indeed, consider the linear equation $ax = a$ with $a \in \mathbf{O}_\#$ nonzero. Since $\bar{a}a = \bar{a}\bar{\bar{a}} = n(\bar{a}) = n(a) = 0$, Corollary 4.2 implies that the set of its solutions X is contained in Ω_4 and is therefore infinite. On the other hand, if X contained a nilpotent element x_0 , then we would have $0 = a \cdot x_0^2 = (ax_0)x_0 = ax_0 = a$, by equality (2.3), which contradicts the assumption that $a \neq 0$.

8. Degenerations of linear monomial equations. Given $a_1, \dots, a_m, c \in \mathbf{O}$ and $a'_1, \dots, a'_m, c' \in \mathbf{O}$, the linear monomial equation $w(a'_1, \dots, a'_m, x) = c'$ is called a *degeneration* of the linear monomial equation $w(a_1, \dots, a_m, x) = c$, if the Zariski closure of the G_2 -orbit of $(a_1, \dots, a_m, c) \in \mathbf{O}^{m+1}$ contains $(a'_1, \dots, a'_m, c')$.

COROLLARY 8.1. *Let $w(a'_1, \dots, a'_m, x) = c'$ be a degeneration of $w(a_1, \dots, a_m, x) = c$, where $a_1, \dots, a_m, a'_1, \dots, a'_m \in \mathbf{O}$ are nonzero. Let X' (X , respectively) be the set of all solutions of the first (the second, respectively) equation. Then*

- (a) *the first equation has a unique solution if and only if the second equation has a unique solution;*
- (b) *in case $w(a_1, x) = a_1x$ we have that $\dim X \geq \dim X'$;*
- (c) *in case $w(a_1, a_2, x) = (a_1x)a_2$ we have that if $\dim X = 4$ if and only if $\dim X' = 4$;*
- (d) *in case $w(a_1, a_2, x) = a_1(a_2x)$ we have that if $X = \mathbf{O}$ if and only if $X' = \mathbf{O}$.*

Proof. Since $n(a_i) = n(a'_i)$ for all $1 \leq i \leq m$, then part (a) follows from Corollary 7.1, part (b) follows from Corollary 4.2, part (c) follows from Corollary 5.2, part (d) follows from Corollary 6.2. $\square$

9. Conclusion. In this manuscript, over an algebraically closed field, we described the affine varieties of solutions to the linear equations $(ax)b = c$ and $a(bx) = c$ over the algebra of split-octonions $\mathbf{O}$, which is non-division. Moreover, we showed that if a linear monomial equation over the split-octonions with a nonzero constant term has at least two solutions, then it necessarily possesses an invertible solution.

Possible directions for further research include solving the following equations.

- Special equations over $\mathbf{O}$, such as the generalized Sylvester equation $ax - yb = c$, the equation $(\bar{x}a)x = b$, the quadratic equation $x^2 + ax + b = c$, and the system of equations $ax = c$ and $xb = d$, where $x, y \in \mathbf{O}$ are variables and $a, b, c, d \in \mathbf{O}$ are given constants.
- The general linear equation over $\mathbf{O}$;
- Polynomial equations over split-octonions. The first step in this direction has already been made in [25], where every polynomial equation

$$\alpha_n x^n + \cdots + \alpha_1 x + a = 0,$$

with $\alpha_1, \dots, \alpha_n \in \mathbb{F}$ and $a \in \mathbf{O}$, was solved over split-octonions when $\mathbb{F}$ is algebraically closed. This result was later generalized to the case of an arbitrary field in [24].

REFERENCES

- [1] J.K. Baksalary and R. Kala. The matrix equation $AX - YB = C$. *Linear Algebra Appl.*, 25:41–43, 1979.
- [2] J.K. Baksalary and R. Kala. The matrix equation $AXB + CYD = E$. *Linear Algebra Appl.*, 30:141–147, 1980.
- [3] P.S. Bisht, S. Dangwal, and O.P.S. Negi. Unified split octonion formulation of dyons. *Int. J. Theor. Phys.*, 47(9):2297–2313, 2008.
- [4] H.W. Braden. The equations $A^T X \pm X^T A = B$. *SIAM J. Matrix Anal. Appl.*, 20(2):295–302, 1999.
- [5] C. Castro. On the noncommutative and nonassociative geometry of octonionic space time, modified dispersion relations and grand unification. *J. Math. Phys.*, 48(7): paper no. 073517, 15 pp., 2007.
- [6] B.C. Chanyal. Classical geometrodynamics with Zorn vector-matrix algebra for gravito-dyons. *Rep. Math. Phys.*, 76(1):1–20, 2015.
- [7] B.C. Chanyal. Split octonion reformulation for electromagnetic chiral media of massive dyons. *Commun. Theor. Phys. (Beijing)*, 68(6):701–710, 2017.
- [8] B.C. Chanyal, P.S. Bisht, and O.P.S. Negi. Generalized split-octonion electrodynamics. *Int. J. Theor. Phys.*, 50(6):1919–1926, 2011.
- [9] B.C. Chanyal, P.S. Bisht, and O.P.S. Negi. Octonion and conservation laws for dyons. *Int. J. Mod. Phys. A*, 28(26): paper no. 1350125, 17 pp., 2013.
- [10] A. Chapman. Polynomial equations over octonion algebras. *J. Algebra Appl.*, 19(6):2050102, 2020.
- [11] A. Chapman and I. Levin. Alternating roots of polynomials over Cayley-Dickson algebras. *Commun. Math.*, 32(2):63–70, 2024.
- [12] A. Chapman and S. Vishkautsan. Roots and dynamics of octonion polynomials. *Commun. Math.*, 30(2):25–36, 2022.
- [13] A. Chapman and S. Vishkautsan. Roots and right factors of polynomials and left eigenvalues of matrices over Cayley-Dickson algebras. *Commun. Math.*, 33(3): paper no. 1, 2025.
- [14] D.E. Ferreyra, M. Lattanzi, F.E. Levis, and N. Thome. Parameterized solutions X of the system $AXA = AEA$ and $A^k EAX = XAE A^k$ for a matrix A having index k . *Electron. J. Linear Algebra*, 35:503–510, 2019.
- [15] C. Flaut and V. Shpakivskyi. An efficient method for solving equations in generalized quaternion and octonion algebras. *Adv. Appl. Clifford Algebr.*, 25:337–350, 2015.
- [16] M. Gogberashvili. Octonionic electrodynamics. *J. Phys. A: Math. Gen.*, 39(22):7099–7104, 2006.
- [17] M. Gogberashvili. Octonionic version of Dirac equations. *Int. J. Mod. Phys. A*, 21(17):3513–3523, 2006.
- [18] M. Gogberashvili and O. Sakhelashvili. Geometrical applications of split octonions. *Adv. Math. Phys.*, pages art. ID 196708, 14 pp., 2015.
- [19] M. Illmer and T. Netzer. A note on polynomial equations over algebras. *Proc. Amer. Math. Soc.*, 152:1831–1839, 2024.
- [20] Jens Köpflinger. Dirac equation on hyperbolic octonions. *Appl. Math. Comput.*, 182(1):443–446, 2006. Corrigendum in [21] footnote 2.
- [21] J. Köpflinger. Gravity and electromagnetism on conic sedenions. *Appl. Math. Comput.*, 188(1):948–953, 2007.
- [22] K. Krasnov. Spin(11,3), particles, and octonions. *J. Math. Phys.*, 63(3): paper no. 031701, 20 pp., 2022.
- [23] Y.H. Liu. Ranks of solutions of the linear matrix equation $AX + YB = C$. *Comput. Math. Appl.*, 52(6–7):861–872, 2006.
- [24] A. Lopatin. On polynomial equations over split-octonions: the arbitrary field case. *Commun. Math.*, 34(2): paper no. 12, 2026.
- [25] A. Lopatin and A.N. Rybalov. On polynomial equations over split octonions. *Commun. Math.*, 33(3): paper no. 8, 2025.

- [26] A. Lopatin and A.N. Zubkov. Separating G_2 -invariants of several octonions. *Algebra Number Theory*, 18(12):2157–2177, 2024.
- [27] A. Lopatin and A.N. Zubkov. Classification of G_2 -orbits for pairs of octonions. *J. Pure Appl. Algebra*, 229:107875, 2025.
- [28] H. Rodríguez-Ordóñez. A note on the fundamental theorem of algebra for the octonions. *Expo. Math.*, 25:355–361, 2007.
- [29] G.W. Schwarz. Invariant theory of G_2 and Spin_7 . *Comment. Math. Helvetici*, 63:624–663, 1988.
- [30] T.A. Springer and F.D. Veldkamp. *Octonions, Jordan algebras and exceptional groups*. Springer-Verlag, Berlin, 2000.
- [31] Y. Tian. The solvability of two linear matrix equations. *Linear Multilinear Algebra*, 48(2):123–147, 2000.
- [32] Q.-W. Wang and J. Gao. Comprehensive review on the generalized Sylvester equation $AX - YB = C$. *Symmetry*, 17(10): paper no. 1686, 2025.
- [33] Q.-W. Wang, Z.-H. Gao, and J.-L. Gao. A comprehensive review on solving the system of equations $AX = C$ and $XB = D$. *Symmetry*, 17(4): paper no. 625, 2025.
- [34] Q.-W. Wang, Z.-H. Gao, and Y.-F. Li. Overview of methods for solving the system of equations $A_1XB_1 = C_1$ and $A_2XB_2 = C_2$. *Symmetry*, 17(8): paper no. 1307, 2025.
- [35] Q.-W. Wang, L.-M. Xie, and Z.-H. Gao. A survey on solving the matrix equation $AXB = C$ with applications. *Mathematics*, 13(3): paper no. 450, 2025.
- [36] Q.-W. Wang, X. Zhang, and Y. Zhang. Algorithms for finding the roots of some quadratic octonion equations. *Commun. Algebra*, 42(8):3267–3282, 2014.
- [37] A.N. Zubkov and I. Shestakov. Invariants of G_2 and $\text{Spin}(7)$ in positive characteristic. *Transform. Groups*, 23(2):555–588, 2018.