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INCIDENCE MATRIX AND COVER MATRIX OF NESTED INTERVAL ORDERS*

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Abstract. For any poset P, its incidence matrix $\mathbb{n}$ and its cover matrix $\mathfrak{C}$ are the $P \times P$ $(0,1)$ matrices such that $\mathbb{n}(x,y) = 1$ if any only if x is less than y in P and $\mathfrak{C}(x,y) = 1$ if any only if x is covered by y in P. It is shown that $\mathbb{n}$ and $\mathfrak{C}$ are conjugate to each other in the incidence algebra of P over a field of characteristic 0 provided P is the nested interval order. In particular, when P is the Bruhat order of a dihedral group, which consists of a special family of nested intervals, $\mathbb{n}$ and $\mathfrak{C}$ turn out to be conjugate in the incidence algebra over every field. Moreover, $\mathbb{n}$ and $\mathfrak{C}$ are proved to be conjugate in the incidence algebra over every field when P is the weak order of a dihedral group. Many relevant problems and observations are also presented in this note.

Key words. Hierarchy, Jordan canonical form, Rank, Strict incidence algebra.

AMS subject classifications. 06A07, 06A11, 15A21, 15A24, 20F55.

1. Poset and its incidence algebra. A (finite) *partially ordered set*, also known as a (finite) *poset* [33, p. 97], is a finite set P together with a binary relation $\leq_P$, which is often denoted $\leq$ if there is no confusion, such that:

- for all $x \in P, x \leq x$ (reflexivity);
- if $x \leq y$ and $y \leq x$, then $x = y$ (antisymmetry);
- if $x \leq y$ and $y \leq z$, then $x \leq z$ (transitivity).

We use the obvious notation $x < y$ to mean $x \leq y$ and $x \neq y$. Similarly, $x \geq y$ and $x > y$ stand for $y \leq x$ and $y < x$, respectively. For any $x, y \in P$, the *interval* $[x,y] = [x,y]_P$ is the set (subposet) $\{z \in P : \ x \leq z \leq y\}$. We say that y *covers* x provided $|[x,y]| = 2$ and denote this by $x \lessdot y$. For any $x \in P$, let $x^{\uparrow_P} = \{y \in P : \ y > x\}$. A *linear extension* of a poset is a listing of its elements as $x_1, \ldots, x_r$ such that if $x_i \leq x_j$ then $i \leq j$. It is well-known that each poset admits a linear extension.

As the uniquely determined minimal transitive reduction of the poset P, its *Hasse diagram* is the digraph $\Gamma(P)$ with P as the vertex set and there is an arc from y to

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x if and only if $x \lessdot y$. We say that a poset P has the *unique path property*, or is a UPP *poset*, if for any two elements $x \leq y$ from P there exists a unique (directed) path in $\Gamma(P)$ from y to x; in other words, P is a UPP poset if each interval $[x, y]$ of P is a chain containing $|[x, y]|$ elements. We mention that the underlying graph of the Hasse diagram of a UPP poset might have cycles.

An *ideal* P' of a poset P is a subposet of P such that $x \in P'$ implies $x^{\uparrow} \subseteq P'$. The subposet induced by the complement of an ideal is a *filter*. A *simplicial complex* K is a set of sets such that $A \in K$ and $B \subseteq A$ implies $B \in K$. Under the set inclusion relationship, each simplicial complex is naturally a poset. A *relative simplicial complex* is an ideal of a simplicial complex.

The *incidence algebra* $Inc_F(P)$ [6, 24] of a (locally finite) poset P over a field F (in many contexts, we can assume F to be merely a commutative ring having a multiplicative unit) is the algebra of functions (matrices) $f\colon P \times P \to F$ such that $f(x, y) = 0$ unless $x \leq y$ in P with pointwise addition and convolution (matrix multiplication) $(fg)(x, y) = \sum_{x \leq z \leq y} f(x, z)g(z, y)$. The *reduced incidence algebra* is the subalgebra of the incidence algebra consisting of those elements of $Inc_F(P)$ which take constant values on isomorphic intervals. The *strict incidence algebra* of P consists of those elements of $Inc_F(P)$ which are nilpotent, i.e., those elements which take value 0 on the diagonal $\{(x, x) : x \in P\}$. Note that $Inc_F(P)$ naturally acts on F^P (viewed as space of column vectors) from the left as linear operators and $Inc_F(P)$ is a subalgebra of the corresponding full matrix algebra. The incidence algebras of posets are important computational devices for many enumeration problem on posets and their algebraic properties have been intensively studied [33]. In particular, Stanley [31, 36] shows that the poset P can be uniquely recovered from $Inc_F(P)$. We adhere to the convention that P represents a (finite) poset and F a field throughout the paper.

Recall that the *Kronecker delta*, denoted δ, is a function of two variables, which is 1 if they are equal, and 0 otherwise. If the two variables are restricted to be from a poset P, we call it the *Kronecker delta function (identity matrix) on* P and use the notation δ_P to signify this. It is noteworthy that the function δ_P is the multiplicative unit of $Inc_F(P)$. For any two elements $x < y$ from the poset P and any number h from the given field F, the *transvection* $T_{xy}(h)$ is the matrix obtained from the identity matrix δ_P by putting h in the (x, y)-position. Observe that $T_{xy}(h)^{-1} = T_{xy}(-h)$ and $T_{xy}(h) \in Inc_F(P)$.

2. Incidence matrix and cover matrix. We are interested in two special but basic matrices lying in both the reduced incidence algebra and the strict incidence algebra of P, the *incidence function* (matrix) $\mathbb{n}_P$ and the *cover function* (matrix) $\mathfrak{C}_P$, which encode full information about the poset P and are the indicator function of $<$

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and that of $\lessdot$, respectively, and can be described more explicitly as follows:

$$\mathfrak{n}_P(x,y) = \begin{cases} 1, & \text{if } x < y; \\ 0, & \text{otherwise;} \end{cases} \quad \mathfrak{C}_P(x,y) = \begin{cases} 1, & \text{if } x \lessdot y; \\ 0, & \text{otherwise.} \end{cases}$$

In some sense, $\mathfrak{n}_P$ is a global view of P and $\mathfrak{C}_P$ is a local view of P. Note that $\mathfrak{n}_P - \mathfrak{C}_P$ is a $(0,1)$-matrix, and hence, $\mathfrak{C}_P$ is more sparse than $\mathfrak{n}_p$. We also remark that $\Gamma(P)$ has $\mathfrak{C}_P^\top$ as an adjacency matrix.

Both $\mathfrak{n}_P$ and $\mathfrak{C}_P$ come into play in various important situations. The zeta function (integral operator) of P is $\zeta_P = \mathfrak{n}_P + \delta_P$, which is the indicator function of the partial order $\leq_P$. The key to the Möbius Inversion Theorem is the determination of the Möbius function (differential operator) of P, $\mu_P = \zeta_P^{-1}$, which is, considering that $\mathfrak{n}_P$ is nilpotent, $\delta_P - \mathfrak{n}_P + \mathfrak{n}_P^2 - \mathfrak{n}_P^3 + \mathfrak{n}_P^4 - \cdots$. Let d_P be the dimension of the kernel of $\mathfrak{n}_P$. This parameter d_P turns out to be a lower bound of the number of incomparable adjacent pairs in any linear extension of P [14] and an upper bound of the width of P [13, 26], and for almost all posets in the uniform random poset model it is exactly the width of P [10]. The operator $\mathfrak{C}_P$ may be regarded as an instance of the finite Radon transform in general [16, 32] and becomes the usual boundary operator for relative simplicial homology when P is a relative simplicial complex. It might be interesting to see if there is some connection between $\mathfrak{n}_P$ and $\mathfrak{C}_P$ to offer a bridge between different research topics.

3. Stanley's problem. In the following, we say that two elements A and B of $Inc_F(P)$ are *conjugate* if there is $\alpha \in Inc_F(P)$ such that $A = \alpha B \alpha^{-1}$ and we say that A and B are *similar* if when viewing them as matrices over the complex field we can find a complex matrix α, which is not necessarily a member of $Inc_{\mathbb{C}}(P)$, such that $A = \alpha B \alpha^{-1}$. Surely, the statement that A and B are similar just means that they have the same Jordan canonical form over $\mathbb{C}$ or the same Jordan invariants (the sizes of its Jordan blocks over $\mathbb{C}$ for the same eigenvalue). For a nilpotent matrix, say a cover matrix or an incidence matrix, its Jordan invariant is necessarily the sizes of its Jordan blocks for eigenvalue 0. We say that A and B are *equivalent* (in $Inc_F(P)$) if there are nonsingular matrices α_1 and α_2 (in $Inc_F(P)$) such that $A = \alpha_1 B \alpha_2$.

Example 3.1. [35] Let

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ and } \; C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

It is easy to check that both B and C are the Jordan canonical forms of A in the full complex matrix algebra but there is no upper triangular matrix α such that $\alpha A \alpha^{-1} \in \{B, C\}$. Note that the set of $n \times n$ upper triangular matrices can be identified with the incidence algebra of the linear order on n elements.

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The next example is basic to our studies. It not only suggests possible directions of generalizations, but also acts as a key preliminary fact for Corollaries 6.2 and 6.4.

EXAMPLE 3.2. Let P be a linear order, namely a poset of dimension 1, on n elements, say $1 < 2 < \cdots < n$. Then,

$$\mathfrak{C}_P = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ & & & & 0 \end{bmatrix}_{n\times n} \quad \text{and} \quad \mathfrak{n}_P = \begin{bmatrix} 0 & 1 & 1 & \cdots & 1 \\ & 0 & 1 & \cdots & 1 \\ & & \ddots & \ddots & \vdots \\ & & & 0 & 1 \\ & & & & 0 \end{bmatrix}_{n\times n}.$$

The matrix $\mathfrak{C}_P$ is clearly the Jordan canonical form of $\mathfrak{n}_P$ (also see Example 4.1). Moreover, for $1 \le i < j \le n-2$, it holds

$$\binom{n-1-i}{n-1-j} = \sum_{t=i+1}^{j+1} \binom{n-1-t}{n-1-(j+1)},$$

and so we find that

$$T(2)^{-1}\cdots T(n-2)^{-1}T(n-1)^{-1}\mathfrak{n}_P T(n-1)T(n-2)\cdots T(2) = \mathfrak{C}_P$$

where

$$T(j) = \prod_{i=1}^{j-1} T_{ij}\Big(\binom{n-1-i}{n-1-j}\Big),\ j = 2,\ldots,n-1. \tag{3.1}$$

This means that $\mathfrak{n}_P$ and $\mathfrak{C}_P$ are even conjugate in $Inc_F(P)$ for any field F.

Prompted by the theory of Jordan canonical form in the full matrix algebra over an algebraically closed field, Stanley asks if there is any reasonable criterion for determining when two elements of the incidence algebra of a poset are conjugate [33, p. 159, Exercise 29 (e)]. Marenich [17, 18] finds some interesting results in her effort to tackling the problem of Stanley and she proposes to view $\lambda\delta_{P'} + \mathfrak{C}_{P'}$, where P' is a subposet of P. as a "Jordan block" for $Inc_F(P)$. Once this problem of Stanley has come up, it is natural to push on to a broader formulation and also ask if two matrices from $Inc_F(P)$ are similar or equivalent or have the same Smith normal form, etc. Weyr's Theorem [30] says that the two nilpotent matrices $\mathfrak{n}_P$ and $\mathfrak{C}_P$ have the same Jordan invariants if and only if $\text{rank}_{\mathbb{C}}(\mathfrak{n}_P^k) = \text{rank}_{\mathbb{C}}(\mathfrak{C}_P^k)$ for every k. So, it is interesting to test for which ℓ we have $\text{rank}(\mathfrak{n}_P^k) = \text{rank}(\mathfrak{C}_P^k)$ for every $k \le \ell$. We refer to [2, 3, 4, 11, 13, 19, 23, 25, 26, 27, 29] for some work on Jordan canonical forms determined by combinatorial patterns and refer to [12, 22] for some work on ranks of matrix powers determined by combinatorial patterns.

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The remainder of the paper is devoted to Stanley's problem and its variants restricted on the two special matrices $\mathfrak{n}_P$ and $\mathfrak{C}_P$. We first collect an assortment of observations in Section 4 for the purpose of inviting readers develop them further into potential theories. In Section 5, we report how the appearance of the hierarchy structure helps to bridge the incidence matrix and the cover matrix. Especially, we provide a sufficient condition for the incidence matrix and the cover matrix to have the same row space (Theorem 5.6) and we take Example 3.2 one step further to nested interval orders (Theorem 5.2). There is already a nice theory on determining the Jordan canonical form of the tensor product of two matrices in the full matrix algebra; see [21] and the references therein. In the spirit of this line of work, we establish in Section 6 two simple lemmas (Lemmas 6.1 and 6.3) on some constructions similar to tensor products and further use them to show that $\mathfrak{n}_P$ and $\mathfrak{C}_P$ are conjugate to each other over any field when P is either the Bruhat order or the weak order of a dihedral group.

4. Examples. For any nilpotent complex matrix A and any analytic function $f(x)$ such that $f(0) = 0$ and $\frac{df(x)}{dx}|_{x=0} \neq 0$, it is clear that A and $f(A)$ have the same Jordan canonical form. This suggests to investigate those posets P for which $\mathfrak{n}_P$ is a function of $\mathfrak{C}_P$ and hence also those P for which $\mathfrak{n}_P$ and $\mathfrak{C}_P$ commute.

EXAMPLE 4.1. [28] If P has the unique path property, then $\mathfrak{n}_P = \mathfrak{C}_P + \mathfrak{C}_P^2 + \cdots = \mathfrak{C}_P(\delta_P - \mathfrak{C}_P)^{-1}$. Since $\mathfrak{C}_P$ is nilpotent, it follows that $\mathfrak{n}_P$ and $\mathfrak{C}_P$ are similar in the full complex matrix algebra.

EXAMPLE 4.2. Let P be a poset whose intervals are always Boolean algebras (it is called a *simplicial* poset if it also contains a smallest element [33, p. 135]), say being a relative simplicial complex. Viewed as matrices over integers, it is easy to see that

$$\mathfrak{n}_P = \mathfrak{C}_P + \frac{\mathfrak{C}_P^2}{2\downarrow} + \frac{\mathfrak{C}_P^3}{3\downarrow} + \cdots = \exp(\mathfrak{C}_P) - \delta_P.$$

This tells us that $\mathfrak{n}_P$ and $\mathfrak{C}_P$ have the same Jordan canonical form.

EXAMPLE 4.3. Suppose that $\mathfrak{n}_P$ and $\mathfrak{C}_P$ are conjugate in $Inc_F(P)$. Let P' be an ideal of P. Writing the matrix representation of elements of $Inc_F(P)$ in such a way that the lines corresponding to elements of $P \setminus P'$ appear before the other lines, we can assume that $T^{-1}\mathfrak{n}_P T = \mathfrak{C}_p$ where

$$T = \begin{bmatrix} T_{11} & T_{12} \\ \mathbf{0} & T_{22} \end{bmatrix},\ \mathfrak{n}_P = \begin{bmatrix} \mathfrak{n}_{P\setminus P'} & \mathfrak{n}_{12} \\ \mathbf{0} & \mathfrak{n}_{P'} \end{bmatrix},\ \mathfrak{C}_P = \begin{bmatrix} \mathfrak{C}_{P\setminus P'} & \mathfrak{C}_{12} \\ \mathbf{0} & \mathfrak{C}_{P'} \end{bmatrix}.$$

It then follows that $\mathfrak{n}_{P'}$ and $\mathfrak{C}_{P'}$ are conjugate in $Inc_F(P')$ and $\mathfrak{n}_{P\setminus P'}$ and $\mathfrak{C}_{P\setminus P'}$ are conjugate in $Inc_F(P \setminus P')$. Similar reasoning shows that if $\mathfrak{C}_P$ and $\mathfrak{n}_P$ are equivalent

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in $Inc_F(P)$ then $\mathbb{n}_{P'}$ and $\mathfrak{C}_{P'}$ are equivalent in $Inc_F(P')$ and $\mathbb{n}_{P\setminus P'}$ and $\mathfrak{C}_{P\setminus P'}$ are equivalent in $Inc_F(P \setminus P')$.

EXAMPLE 4.4. [38] Let P be a poset whose Hasse diagram is weakly connected. If $f\mathfrak{C}_P = \mathbb{n}_P f$ for some $f \in Inc_F(P)$, then f has a constant main diagonal. Indeed, it is enough to prove $f(x,x) = f(y,y)$ on the condition that $x \lessdot y$. But this is a result of $f\mathfrak{C}_P(x,y) = \mathbb{n}_P f(x,y)$.

EXAMPLE 4.5. Dress and Wu [8] show that $\mathbb{n}_P$ and $\mathfrak{C}_P$ are conjugate in $Inc_F(P)$ when P is a relative simplicial complex and F is a field of characteristic 2. To see this, by Example 4.3, we can now assume that P is a simplicial complex. Fix a linear order $\prec$ on $\cup_{S\in P}S$. For any $S = \{x_1 \prec x_2 \prec \cdots \prec x_t\} \in P$, set $E(S) = \{x_2, x_4, \ldots, x_{2\lfloor \frac{t}{2} \rfloor}\}$. Specify $\Omega \in Inc_F(P)$ by letting $\Omega(R,S) = 1$ if $E(S) \subseteq R \subseteq S$ and $\Omega(R,S) = 0$ otherwise. It is not difficult to check that

$$\Omega\mathfrak{C}_P\Omega^{-1} = \mathbb{n}_P. \tag{4.1}$$

Note that for each ordering of $\cup_{S\in P}S$ the above construction gives a solution Ω to Eq. (4.1). To investigate if there is any other solution to Eq. (4.1), we are led to another problem of Stanley [33, p. 159, Exercise 29 (e)], namely determining the dimension of the centralizer algebra of $\mathfrak{C}_P$ in $Inc_F(P)$.

EXAMPLE 4.6. Let P be a simplicial complex of dimension at least 2. Then, we can suppose that P contains a 2-dimensional face $\{1,2,3\}$ and hence the three 1-dimensional faces $\{1,2\}, \{2,3\}$, and $\{3,1\}$. Over $\mathbb{F}_2$ we have

$$\mathbb{n}_P(\cdot, \{1,2\}) + \mathbb{n}_P(\cdot, \{2,3\}) + \mathbb{n}_P(\cdot, \{3,1\}) = \mathbb{n}_P(\cdot, \{1\}),$$

while

$$\mathfrak{C}_P(\cdot, \{1,2\}) + \mathfrak{C}_P(\cdot, \{2,3\}) + \mathfrak{C}_P(\cdot, \{3,1\}) \neq \mathfrak{C}_P(\cdot, \{1\}).$$

This means that $\mathbb{n}_P$ and $\mathfrak{C}_P$ have different row spaces over $\mathbb{F}_2$.

EXAMPLE 4.7. Let P be the poset with the Hasse diagram as shown in Fig. 4.1. Note that P is a poset of dimension 2. Simple calculation leads to

$$\mathrm{rank}_{\mathbb{Z}}(\mathbb{n}_P) = \mathrm{rank}_{\mathbb{F}_2}(\mathbb{n}_P) = 5 > 4 = \mathrm{rank}_{\mathbb{Z}}(\mathfrak{C}_P) = \mathrm{rank}_{\mathbb{F}_2}(\mathfrak{C}_P).$$

A poset P is *graded* if there is a rank function ρ from P to integers such that if y covers x then $\rho(y) = \rho(x) + 1$. This graded poset P is *homogenous* provided for any $n \leq k \leq \ell$, any $x \in \rho^{-1}(n)$ and $y \in \rho^{-1}(\ell)$ satisfying $x \leq y$, the set $[x,y] \cap \rho^{-1}(k)$ has a size $t_{n,k,\ell}$, which is totally determined by n, k, ℓ and is independent of the choice

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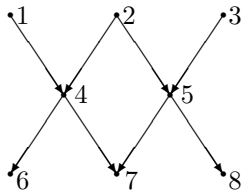

FIG. 4.1. *Hasse diagram of the poset P in Example 4.7.*

of x and y. Homogenous posets include the lattices of linear (affine) subspaces of a finite vector space and the posets of relative simplicial complexes.

EXAMPLE 4.8. Let P be a homogenous poset as defined above for which $t_{n,n+1,\ell} \neq 0$ for all $n < \ell$. Suppose that $\{\rho(x) : x \in P\} = \{0, 1, 2, \ldots, m\}$. Put $T_a = \{(i,j) : 0 \leq i \leq j \leq m, j - i = a\}$ and $T = \cup_{a=0}^{m} T_a$. Let g be a map from T to F satisfying $g^{-1}(0) \supseteq T_0$ and $g^{-1}(0) \cap T_1 = \emptyset$. Let $f \in Inc_F(P)$ be a function such that $f(x,y) = g(\rho(x), \rho(y))$ for any $x \leq_P y$. Note that when g takes constant value 1, f is nothing but $\mathfrak{n}_P$. Marenich [18, Theorem 6] finds that f is always conjugate to $\mathfrak{C}_P$ in $Inc_F(P)$, namely there exists an invertible element $\alpha \in Inc_F(P)$ such that

$$\alpha f = \mathfrak{C}_P \alpha. \tag{4.2}$$

To solve Eq. (4.2), Marenich suggests to consider a function h from T to F satisfying $h^{-1}(0) \cap T_0 = \emptyset$ and

$$\sum_{k=n}^{\ell-1} h(n,k) g(k,\ell) t_{n,k,l} = t_{n,n+1,\ell} h(n+1,\ell),\ 0 \leq n < \ell \leq m. \tag{4.3}$$

If such a function h exists, we choose $\alpha \in Inc_F(P)$ by requiring $\alpha(x,y) = h(\rho(x), \rho(y))$ for any $x \leq_P y$ and it is easy to see that α is invertible and satisfies Eq. (4.2). We follow Marenich to indicate briefly why the required h exists. Setting $\ell = n+1$ in Eq. (4.3), we see that by taking any nonzero initial value for $h(0,0)$, $h(i,i)$ can be determined recursively to be $h(0,0) \prod_{t=0}^{i-1} g(t, t+1) \neq 0$. Suppose the values of h on $\cup_{a=0}^{s-1} T_a$ have been determined by Eq. (4.3) for $\ell - n \leq a$ and by any valuation of $h(0,0), \ldots, h(0, s-1)$. Appealing to (4.3) for $\ell - n = s+1$ and noticing $t_{n,n+1,\ell} \neq 0$, we see that $h(n+1, \ell)$ is determined by $h(n, \ell-1)$ and those known values of h on $\cup_{a=0}^{s-1} T_a$. This says that Eq. (4.3) is solvable and the solution is determined uniquely by the value of $h(0,i), i = 0, \ldots, m$, and as long as $h(0,0)$ is chosen to be nonzero, we will have $h^{-1}(0) \cap T_0 = \emptyset$.

Given an $n \times n$ matrix A, its digraph $\Gamma(A)$ has vertex set $\{v_1, v_2, \ldots, v_n\}$ and there is an arc from v_i to v_j if and only if $A(i,j) \neq 0$. A *k-path* of a digraph D is a subset of $V(D)$ which can be partitioned into k (possibly empty) sets $X_1, \ldots, X_k$

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such that each X_i is the set of vertices of a path of D. The largest number of vertices in a k-path of D is the *k-path number* of D and is denoted by $p_k(D)$.

EXAMPLE 4.9. For any nonnegative matrix A belonging to the strict incidence algebra of P, the Rothblum Index Theorem [11, 25] implies that the size of the largest Jordan block of A is equal to $p_1(\Gamma(A))$. For any poset P, it is clear that a longest path in $\Gamma(\mathfrak{n}_P)$ must also be a longest path in $\Gamma(\mathfrak{C}_P)$, and hence, we know that the largest Jordan blocks of $\mathfrak{n}_P$ and $\mathfrak{C}_P$ have equal size.

EXAMPLE 4.10. For any generic nilpotent matrix A of order n, its Jordan invariants are $p_1(\Gamma(A))$. $p_2(\Gamma(A)) - p_1(\Gamma(A))$, $p_3(\Gamma(A)) - p_2(\Gamma(A))$, ..., $p_s(\Gamma(A)) - p_{s-1}(\Gamma(A))$, where $s = n - \mathrm{rank}(A)$ [4, 13, 27]. This suggests to investigate those posets P for which the equality $p_i(\Gamma(\mathfrak{n}_P)) = p_i(\Gamma(\mathfrak{C}_P))$ holds for every positive integer i.

For any $f \in Inc_F(P)$ and $U, V \subseteq P$, we adopt the notation $f(U, V)$ for the $U \times V$ matrix that is the restriction of f on $U \times V$. We often write f_U for $f(U, U)$.

EXAMPLE 4.11. Let P, Q and R be three posets such that both P and Q have R as an ideal. Suppose that

$$\mathfrak{n}_P = \begin{bmatrix} \mathfrak{n}_{P\setminus R} & \mathfrak{n}(P \setminus R, R) \\ \mathbf{0} & \mathfrak{n}_R \end{bmatrix}, \ \mathfrak{C}_P = \begin{bmatrix} \mathfrak{C}_{P\setminus R} & \mathfrak{C}(P \setminus R, R) \\ \mathbf{0} & \mathfrak{C}_R \end{bmatrix};$$

$$\mathfrak{n}_Q = \begin{bmatrix} \mathfrak{n}_{Q\setminus R} & \mathfrak{n}(Q \setminus R, R) \\ \mathbf{0} & \mathfrak{n}_R \end{bmatrix}, \ \mathfrak{C}_Q = \begin{bmatrix} \mathfrak{C}_{Q\setminus R} & \mathfrak{C}(Q \setminus R, R) \\ \mathbf{0} & \mathfrak{C}_R \end{bmatrix}.$$

The *wedge sum of P and Q based on R*, denoted $P \vee_R Q$, is the poset which is the disjoint union of $P \setminus R, Q \setminus R$ and R and has the partial order to be specified by

$$\mathfrak{n}_{P\vee_R Q} = \begin{bmatrix} \mathfrak{n}_{P\setminus R} & \mathbf{0} & \mathfrak{n}(P \setminus R, R) \\ \mathbf{0} & \mathfrak{n}_{Q\setminus R} & \mathfrak{n}(Q \setminus R, R) \\ \mathbf{0} & \mathbf{0} & \mathfrak{n}_R \end{bmatrix},$$

$$\mathfrak{C}_{P\vee_R Q} = \begin{bmatrix} \mathfrak{C}_{P\setminus R} & \mathbf{0} & \mathfrak{C}(P \setminus R, R) \\ \mathbf{0} & \mathfrak{C}_{Q\setminus R} & \mathfrak{C}(Q \setminus R, R) \\ \mathbf{0} & \mathbf{0} & \mathfrak{C}_R \end{bmatrix}.$$

Suppose that $\mathfrak{n}_R$ and $\mathfrak{C}_R$ are conjugate in $Inc_F(R)$ and this conjugacy can be lifted to both Q and R, that is to say, there exists $\alpha, \alpha^{-1} \in Inc_F(R)$ such that $\alpha \mathfrak{n}_R \alpha^{-1} = \mathfrak{C}_R$ and two extensions

$$\alpha_P = \begin{bmatrix} \beta & \gamma \\ \mathbf{0} & \alpha \end{bmatrix} \in Inc_F(P), \quad \alpha_Q = \begin{bmatrix} \epsilon & \zeta \\ \mathbf{0} & \alpha \end{bmatrix} \in Inc_F(Q)$$

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such that $\alpha_P \mathbb{n}_P \alpha_P^{-1} = \mathfrak{C}_P$, $\alpha_Q \mathbb{n}_Q \alpha_Q^{-1} = \mathfrak{C}_Q$. Let

$$\theta = \begin{bmatrix} \beta & \mathbf{0} & \gamma \\ \mathbf{0} & \epsilon & \zeta \\ \mathbf{0} & \mathbf{0} & \alpha \end{bmatrix} \in Inc_F(P \vee_R Q).$$

It is straightforward to check that $\theta \mathbb{n}_{P\vee_R Q} \theta^{-1} = \mathfrak{C}_{P\vee_R Q}$ and so $\mathbb{n}_{P\vee_R Q}$ is conjugate to $\mathfrak{C}_{P\vee_R Q}$ in $Inc_F(P \vee_R Q)$. Note that similar result holds when both P and Q have R as a common filter.

When referring to a Coxeter group, we will have in mind the group and a specific set of Coxeter generators tacitly understood. The Bruhat orders and the weak orders of Coxeter groups, as well as some other posets associated with Coxeter groups, are of much interest in algebraic combinatorics [1]. Coxeter groups are classified into several types [1, Appendix A1]. For instance, the symmetry group of an n-simplex is of type A_n, the symmetry group of an n-cube is of type B_n and the symmetry group of the regular m-gon is of type $I_2(m)$. Accordingly, it is natural to talk about the type of a Bruhat order or a weak order.

EXAMPLE 4.12. We examine several Bruhat orders and weak orders P and find that $\mathbb{n}_P$ and $\mathfrak{C}_P$ have the same Jordan invariants in all these cases. In the following two tables, we report the common Jordan invariant for each of these posets. The third columns of the tables indicate the place where a Hasse diagram of the corresponding poset can be located; if an empty cell is found, it means that the calculation of the relevant Hasse diagram is based on our own programming work but the diagram is too large to include in this note. When recording the Jordan invariant of the incidence/cover matrix of a poset, we employ the standard notation $s_i^{m_i}$ to mean that there are m_i Jordan blocks of size s_i and the supscript m_i will be omitted when $m_i = 1$. It is clear that $\sum_i m_i s_i$ is just the size of the considered poset.

Type of Bruhat order	Jordan invariant	Hasse diagram
B_2	5, 1^3	[1, Fig. 2.1]
B_3	10, 8^2, 6^2, 4^2, 2	Fig. 4.2
A_2	4, 1^2	[1, Fig. 2.3]
A_3	7, 5, 4^2, 3, 1	[1, Fig. 2.4]
A_4	11, 9, 8^4, 7^3, 5^5, $4^2, 3^2, 2^2, 1^4$	
A_5	$16, 14^2, 13^4, 12^5, 11^4, 10^{12}, 9^2,$ $8^{16}, 7^6, 6^{18}, 5^2, 4^{15}, 3^4, 2^8, 1^6$	
$S_6^{(3)}$	10, 6, 4	[1, Fig. 2.7]
E_6 modulo D_5	17, 9, 1	[1, Fig. 2.8]

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Type of weak order	Jordan invariant	Hasse diagram
$I_2(4)$	5, 3	[1, Fig. 3.1]
A_3	$7, 5^2, 3^2, 1$	[1, Fig. 3.2]
H_3	$16, 14^2, 12^2, 10^2, 8^2, 6^2, 4$	[1, Fig. 3.3]

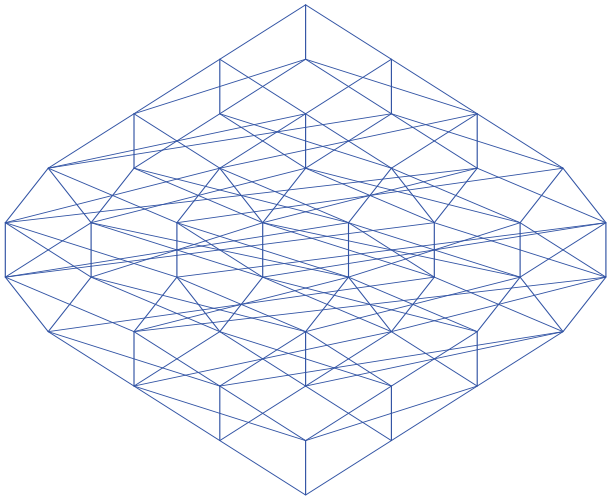

FIG. 4.2. *Hasse diagram of the Bruhat order for the symmetry group of the 3-cube.*

REMARK 4.13. The group of type A_n is the usual symmetry group S_{n+1}. The group of type B_n, sometimes called a hyperoctahedral group with parameter n, can be expressed as the wreath product $S_2 \wr S_n$ of S_2 with S_n and is thus identified with the group of signed permutation matrices of degree n. One can notice the central symmetry of Fig. 4.2, which comes from the sign flipping involution on the set of signed permutation matrices. Note that a diagram depicted in [1, Fig. 2.2] is also asserted to be the Hasse diagram of the Bruhat order of type B_3. It seems that one edge is missing in [1, Fig. 2.2] and this destroys the above-mentioned central symmetry. Anyway, even for the poset P with [1, Fig. 2.2] as its Hasse diagram, $\mathfrak{n}_P$ and $\mathfrak{C}_P$ have the same Jordan invariant and this invariant coincides with the one arising from Fig. 4.2.

5. Hierarchy. A *hierarchy*, also referred to as a *laminar family*, is a set system (hypergraph) $\mathcal{H}$ such that $A \cap B \in \{\emptyset, A, B\}$ for any $A, B \in \mathcal{H}$. This concept is naturally related to tree-like structures and is hence important in phylogenetic combinatorics [7] and algorithmic graph theory [34]. This section aims to highlight some role of hierarchy in connecting the incidence matrix and the cover matrix.

Let Q be a poset. A poset P is an *interval poset* for Q provided to each element

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v of P we can assign a nonempty interval $I_v = [a_v, b_v]_Q$ of Q such that $v <_P w$ if and only if $a_v \leq_Q b_v <_Q a_w \leq_Q b_w$. It is worth noting that $Inc_F(P)$ naturally acts on $F^P = Inc_F(Q)$. Also notice that when talking about an interval poset P for Q we often regard that an interval representation I of P has been given and so each element of P is already identified with an interval of Q. From the next simple result we can tell that neither the poset in Example 4.6 nor that in Example 4.7 can be an interval poset for some UPP poset.

THEOREM 5.1. *If P is the interval poset for a* UPP *poset, then $\mathbb{n}_P$ and $\mathfrak{C}_P$ have the same row space and* $\mathrm{rank}_{\mathbb{Z}}(\mathbb{n}_P) = \mathrm{rank}_{\mathbb{Z}}(\mathfrak{C}_P)$.

An *interval order* [9] is the interval poset for a linear order. If the interval order can be realized as a set of intervals of unit length on the real line, it is called a *unit interval order* or a *semiorder*.

A family of intervals of a poset is *nested* if it is a hierarchy. The interval poset P is a *nested interval poset* if it has an interval representation I such that $\{I_v : v \in P\}$ can be chosen to be nested. Especially, a nested interval poset for a linear order is called a *nested interval order*. It is known that nested interval graphs, i.e., incomparability graphs of nested interval orders, have very interesting combinatorial properties [5, 20]. The next result, which generalizes Example 3.2, is the chief goal of this note.

THEOREM 5.2. *Let P be a nested interval order and let F be a field of characteristic 0. Then $\mathbb{n}_P$ and $\mathfrak{C}_P$ are conjugate in $Inc_F(P)$ and hence have the same Jordan canonical form when F is even an algebraically closed field.*

EXAMPLE 5.3. Take $m \geq 3$. It is known [1, p. 28] that the Bruhat order Q of the dihedral group of order $2m$ (the Coxeter group of type $I_2(m)$) is isomorphic to the nested interval poset for the real line consisting of the following intervals:

$$[1,1],[2,2],[2,2],[3,3],[3,3],\ldots,[m,m],[m,m],[m+1,m+1].$$

Therefore, Theorem 5.2 applies to say that $\mathbb{n}_Q$ and $\mathfrak{C}_Q$ are conjugate to each other in $Inc_F(Q)$ for any field F of characteristic 0. We remind the reader that $I_2(3) = A_2$, $I_2(4) = B_2$, and $I_2(6) = G_2$ [1, Appendix A1].

In the remaining part of this section, we will propose some former definitions so that we can establish Theorems 5.1 and 5.2. We will also give some pertinent examples and discussions.

Define an equivalence relation $\sim$ on a poset P such that $x \sim y$ if and only if $x^{\uparrow} = y^{\uparrow}$. We use the notation $\langle x \rangle = \langle x \rangle_P$ for the equivalence class $\{y \in P : x \sim y\}$. We define the *quotient poset* of P, denoted by $\overline{P}$, as the one with the $\sim$-equivalence classes of P as elements and $\langle x \rangle >_{\overline{P}} \langle y \rangle$ if and only if $\langle x \rangle \cap y^{\uparrow} \neq \emptyset$. A poset P is *green*

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if the set system $\{y^{\uparrow_P} : y \in P, x <_P y\}$ is a hierarchy for every $x \in P$.

EXAMPLE 5.4. If P is a green poset, then its quotient poset $\overline{P}$ and any interval poset for it are also green.

EXAMPLE 5.5. A UPP poset is clearly green and hence so is any interval poset for a UPP poset.

THEOREM 5.6. *Let P be a green poset. Then there is $\alpha \in Inc_F(P)$ such that $\alpha\mathfrak{C}_P = \mathfrak{n}_P$ and $\det \alpha = 1$.*

Proof. Let $x_1, \ldots, x_r$ be a linear extension of P. For any $t \in \{0, 1, \ldots, r\}$, let C_t be the $P \times P$ matrix such that $C_t(x_i, \cdot) = \mathfrak{C}_P(x_i, \cdot)$ for $i \leq t$ and $C_t(x_i, \cdot) = \mathfrak{n}_P(x_i, \cdot)$ for $i > t$. Note that $C_r = \mathfrak{C}_P$ and $C_0 = \mathfrak{n}_P$. So, it suffices to show that there exists $\alpha_t \in Inc_F(P)$ satisfying $\alpha_t C_t = C_{t-1}$ and $\det \alpha_t = 1$ for any $t \in \{1, 2, \ldots, r\}$.

By assumption, there are $t_1, \ldots, t_n > t$ such that $x_{t_1}, \ldots, x_{t_n}$ are all elements of P which cover x_t. Since $\{y^{\uparrow_P} : y \in P, x <_P y\}$ is a hierarchy, we can further assume that there is m such that $x_{t_1}^{\uparrow}, \ldots, x_{t_m}^{\uparrow}$ are pairwise disjoint and for any $i > m$, there exists $j \leq m$ such that $x_{t_i}^{\uparrow} \subseteq x_{t_j}^{\uparrow}$. It now follows that $\alpha_t = T_{x_t x_{t_1}}(1) \cdots T_{x_t x_{t_m}}(1)$ is what we wanted, completing the proof. □

Proof of Theorem 5.1. Combining Example 5.5 with Theorem 5.6 yields the result. □

Let P be an interval order. From Theorem 5.1, we derive $\operatorname{rank}(\mathfrak{n}_P) = \operatorname{rank}(\mathfrak{C}_P)$ and hence $\mathfrak{n}_P$ and $\mathfrak{C}_P$ have the same number of Jordan blocks. We do not know of any example yet for which $\operatorname{rank}(\mathfrak{n}_P^2) \neq \operatorname{rank}(\mathfrak{C}_P^2)$. But the following example says that it is possible for $\operatorname{rank}(\mathfrak{n}_P^3) \neq \operatorname{rank}(\mathfrak{C}_P^3)$ to occur.

EXAMPLE 5.7. Consider the following family of intervals of the real line $\mathbb{R}$: $I_1 = [1, 2]$, $I_2 = [1, 4]$, $I_3 = [1, 4]$, $I_4 = [1, 6]$, $I_5 = [3, 8]$, $I_6 = [5, 10]$, $I_7 = [7, 12]$, $I_8 = [9, 14]$, $I_9 = [11, 14]$, $I_{10} = [11, 16]$, $I_{11} = [13, 16]$, $I_{12} = [15, 18]$, $I_{13} = [17, 20]$, $I_{14} = [17, 22]$, $I_{15} = [19, 24]$, $I_{16} = [21, 26]$, $I_{17} = [23, 28]$, $I_{18} = [25, 30]$, $I_{19} = [27, 30]$, $I_{20} = [27, 32]$, $I_{21} = [29, 34]$, $I_{22} = [31, 36]$, $I_{23} = [33, 36]$, $I_{24} = [35, 36]$. We depict these intervals in Fig. 5.1. Denote the corresponding interval poset by P, which can be shown to be a semiorder. A calculation by computer tells us that

$$\text{(5.1)} \qquad \begin{cases} (\operatorname{rank}_{\mathbb{Z}}(\mathfrak{n}_P), \operatorname{rank}_{\mathbb{Z}}(\mathfrak{n}_P^2), \ldots, \operatorname{rank}_{\mathbb{Z}}(\mathfrak{n}_P^7)) = (17, 13, 9, 6, 4, 2, 0), \\ (\operatorname{rank}_{\mathbb{Z}}(\mathfrak{C}_P), \operatorname{rank}_{\mathbb{Z}}(\mathfrak{C}_P^2), \ldots, \operatorname{rank}_{\mathbb{Z}}(\mathfrak{C}_P^7)) = (17, 13, 10, 7, 4, 2, 0). \end{cases}$$

Hence, the Jordan invariants of $\mathfrak{n}_P$ and $\mathfrak{C}_P$ are $(7, 7, 4, 3, 1, 1, 1)$ and $(7, 7, 5, 2, 1, 1, 1)$, respectively. Additionally, a computer enumeration shows that the cover matrix and the incidence matrix of any proper subposet of P have the same Jordan canonical form. Finally, let us point out that, a greedy search finds that $I_1 < I_5 < I_8 < I_{12} < I_{15} < I_{18} < I_{22}$ is a longest chain in P and hence Example 4.9 anticipates the fact

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that 7 is the size of the largest Jordan block in both $\mathfrak{n}_P$ and $\mathfrak{C}_P$.

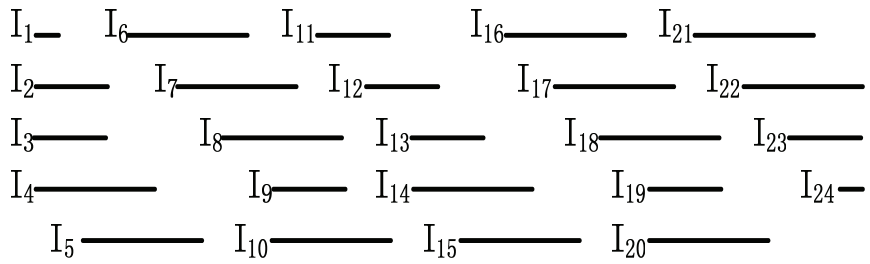

FIG. 5.1. *The relative distribution of the family of intervals in Example 5.7.*

We read from Eq. (5.1) that $\mathrm{rank}(\mathfrak{n}_P^i) \leq \mathrm{rank}(\mathfrak{C}_P^i)$ for all positive integers i. Let us give another example to illustrate that this is not true in general.

EXAMPLE 5.8. Consider the new poset P consisting of the following intervals in the real line: $I_1 = [1,2]$, $I_2 = [1,4]$, $I_3 = [3,6]$, $I_4 = [5,8]$, $I_5 = [7,10]$, $I_6 = [7,12]$, $I_7 = [9,10]$, $I_8 = [9,12]$, $I_9 = [11,14]$, $I_{10} = [13,16]$, $I_{11} = [15,18]$, $I_{12} = [17,18]$. See Fig. 5.2. It is easily verified that

$$\begin{cases} (\mathrm{rank}_{\mathbb{Z}}(\mathfrak{n}_P), \mathrm{rank}_{\mathbb{Z}}(\mathfrak{n}_P^2), \ldots, \mathrm{rank}_{\mathbb{Z}}(\mathfrak{n}_P^5)) = (8,5,3,1,0), \\ (\mathrm{rank}_{\mathbb{Z}}(\mathfrak{C}_P), \mathrm{rank}_{\mathbb{Z}}(\mathfrak{C}_P^2), \ldots, \mathrm{rank}_{\mathbb{Z}}(\mathfrak{C}_P^5)) = (8,5,2,1,0), \end{cases}$$

and the Jordan invariants of $\mathfrak{n}_P$ and $\mathfrak{C}_P$ are $(5,4,2,1)$ and $(5,3,3,1)$, respectively. By a computer enumeration we find that for any proper subposet of P its incidence matrix and cover matrix are similar to each other. It may be worth noting that P cannot be any semiorder, as can be certified by the intersection pattern of the intervals I_6, I_4, I_7, and I_9 – they correspond to a so-called **1**+**3** in the poset.

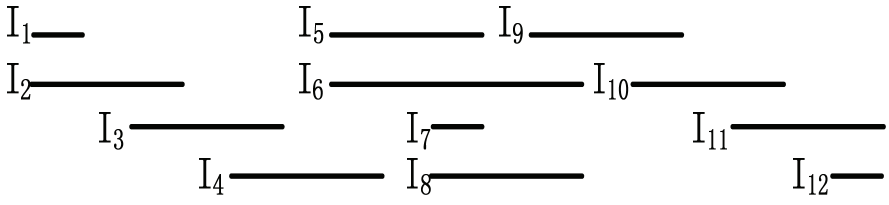

FIG. 5.2. *The relative distribution of the family of intervals in Example 5.8.*

EXAMPLE 5.9. Let P be any semiorder with at most 11 elements. A computer enumeration shows that $\mathrm{rank}_{\mathbb{Z}}(\mathfrak{n}_P^k) = \mathrm{rank}_{\mathbb{Z}}(\mathfrak{C}_P^k)$ for all positive integer k and hence $\mathfrak{n}_P$ and $\mathfrak{C}_P$ have the same Jordan canonical form over $\mathbb{C}$. It seems interesting to

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understand which kind of obstruction appeared in an interval order P can cause different Jordan invariants of $\mathfrak{C}_P$ and $\mathfrak{n}_P$.

Let P be a poset and let $L : x_1, x_2, \ldots, x_r$, be a linear extension of $\overline{P}$. We observe that any ordering of P such that the elements of x_i come earlier than those of x_j whenever $i < j$ is necessarily a linear extension of P. We say that the linear extension L of $\overline{P}$ is *blue* if for any $u, w \in P$ satisfying $|[u, w]_P| > 2$, $u \in x_i$ and $w \in x_j$, we can find a k such that

$$v <_P w \quad \text{for all} \quad v \in x_k \tag{5.2}$$

and

$$i = \max\{t : \ x_t <_{\overline{P}} x_k\}. \tag{5.3}$$

Note that in this case it surely holds

$$x_i \lessdot_{\overline{P}} x_k. \tag{5.4}$$

A *blue poset* is a poset whose quotient poset admits a blue linear extension.

The following lemma contains the main thrust of this note. Like Example 4.8, it recognizes certain condition under which a set of elements will fall into the same conjugacy class of $Inc_F(P)$. The proof of the lemma makes repeated use of the so-called *elementary combination similarity* [2, Fig. 7] which sends an element $A \in Inc_F(P)$ to TAT^{-1} for some transvection $T \in Inc_F(P)$.

LEMMA 5.10. *Let P be a blue poset. Let $\mathfrak{B}$ be the set of functions τ in the strict incidence algebra of P such that the following hold: (i) $\tau(u, v) = \tau(u', v)$ whenever $u \sim u'$ and $v \notin \langle u \rangle$; (ii) $\sum_{v' \in \langle v \rangle} \tau(u, v') \neq 0$ for every $u, v \in P$ satisfying $\langle u \rangle \lessdot_{\overline{P}} \langle v \rangle$. Take $f \in \mathfrak{B}$ and let g be the element in $Inc_F(P)$ that agrees with f on $\{(x, y) \in P \times P : \ |[x, y]_P| \leq 2\}$ and vanishes on $\{(x, y) \in P \times P : \ |[x, y]_P| \geq 3\}$. Then there is $\alpha \in Inc_F(P)$ such that $g = \alpha f \alpha^{-1}$.*

Proof. Let $\mathfrak{B}_f$ be the set of elements of $\mathfrak{B}$ which are conjugate to f in $Inc_F(P)$ and are equal to f when restricted on $\{(x, y) \in P \times P : \ |[x, y]| \leq 2\}$. Take a blue linear extension of $\overline{P}$, say $x_1, \ldots, x_r$.

Suppose the lemma was false. Then for any $\tau \in \mathfrak{B}_f$, we have two well-defined parameters,

$$\mathcal{C}_\tau = \min\{j : \ \exists w \in x_j, \exists u \in P, \tau(u, w) \neq 0, |[u, w]_P| > 2\}$$

and then

$$\mathcal{R}_\tau = \max\{i : \ \exists u \in x_i, \exists w \in x_{\mathcal{C}_\tau}, \tau(u, w) \neq 0, |[u, w]_P| > 2\}.$$

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We now choose $\eta \in \mathfrak{B}_f$ with $(\mathcal{R}_\eta, \mathcal{C}_\eta) = (i, j)$, where $j = \max_{\tau\in\mathfrak{B}_f} \mathcal{C}_\tau$ and $i = \min_{\tau\in\mathfrak{B}_f, \mathcal{C}_\tau = j} \mathcal{R}_\tau$. To derive a contradiction, it suffices to show that η is conjugate in $Inc_F(P)$ to an element $h \in \mathfrak{B}_f$ with either $\mathcal{C}_h > j$ or $\mathcal{C}_h = j$ but $\mathcal{R}_h < i$.

Fix any $u \in x_i$. Note that $|[x_i, x_j]_{\overline{P}}| > 2$ and L is blue. Consequently, for each $w \in x_j$ with $|[u, w]_P| > 2$, there is an integer $k = k(w)$ for which Eqs. (5.2), (5.3), and (5.4) are satisfied. In view of $\eta \in \mathfrak{B}$ and Eq. (5.4), condition (ii) for $\tau = \eta$ says that the number $\sum_{v\in x_{k(w)}} \eta(u, v)$ takes a nonzero value, which we denote by a_w. Let

$$\beta = \prod_{\substack{w\in x_j \\ |[u,w]_P|>2}} \prod_{v\in x_{k(w)}} T_{vw}(\frac{\eta(u, w)}{a_w}) \in Inc_F(P)$$

and let $h = \beta\eta\beta^{-1}$. It is not hard to see that $h \in \mathfrak{B}_f$. Moreover, in view of Eq. (5.3) for $k = k(w)$, where $w \in x_j$ and $|[u, w]_P| > 2$, we can check that either $\mathcal{C}_h > j$ or $\mathcal{C}_h = j$ but $\mathcal{R}_h < i$, arriving at the desired contradiction. □

REMARK 5.11. We follow the notation of Lemma 5.10. Let P' be a filter of the given blue poset P and assume that P' is a union of some $\sim$-equivalence classes of P. A moment's thought on the proof of Lemma 5.10 says that we can determine the required α to transform f to g step by step and $\alpha_{P'}$ can be constructed totally by $f_{P'}$ and $g_{P'}$.

Like Theorem 5.6, the next lemma furnishes further example of the combinatorial regularity implied by the nested property.

LEMMA 5.12. *The nested interval order is a blue poset.*

Proof. Let P be a nested interval poset for a linear order Q. Let $r = |\overline{P}|$ and take an ordering

$$L : \ x_1 = \langle y_1 \rangle, \ \ldots, \ x_r = \langle y_r \rangle$$

of $\overline{P}$, where $y_t = [a_t, b_t]$, $t = 1, \ldots, r$, such that

$$y_\ell^{\uparrow_P} \subsetneq y_m^{\uparrow_P} \text{ implies } \ \ell > m. \tag{5.5}$$

It is clear that L gives rise to a linear extension of $\overline{P}$. Our task is to prove that L is blue. Suppose $u \in x_i$ and $w \in x_j$ satisfy $|[u, w]_P| > 2$. Accordingly, we take

$$k = \min\{t : \ \exists v \in x_t, u <_P v <_P w\} \tag{5.6}$$

and want to verify Eqs. (5.2) and (5.3). Eq. (5.2) is trivially true. To finish the proof, we want to derive a contradiction under the assumption that Eq. (5.3) fails, namely

$$i < h = \max\{t : \ x_t <_{\overline{P}} x_k\}. \tag{5.7}$$

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On account of Eq. (5.7), we can assume without loss of generality that

$$y_h <_P y_k, \tag{5.8}$$

from which it follows that

$$a_h \leq_Q b_h <_Q a_k. \tag{5.9}$$

By Eqs. (5.5) and (5.7) we can find a number n and with no loss of generality assume that

$$b_i <_Q a_n \leq_Q b_h. \tag{5.10}$$

We obtain from Eq. (5.6) that

$$y_i \lessdot_P y_k. \tag{5.11}$$

Henceforth, from Eq. (5.8) we see that it is impossible to happen $y_i <_P y_h$ and so, considering that $[a_i, b_i]$ and $[a_h, b_h]$ are nested, Eq. (5.10) implies

$$a_h \leq_Q a_i. \tag{5.12}$$

In addition, Eqs. (5.9), (5.10) and (5.12) tell us that $a_h \leq_Q a_i \leq_Q b_i <_Q a_n \leq_Q b_h <_Q a_k$. As $[a_n, b_n]$ and $[a_h, b_h]$ are nested, we get $b_i <_Q a_n \leq_Q b_n \leq_Q b_h <_Q a_k$ and hence $y_i <_P y_n <_P y_k$. This leads to a contradiction with Eq. (5.11), as desired. □

Proof of Theorem 5.2. This follows from Lemmas 5.10 and 5.12. □

Note that Examples 5.7 and 5.8 assert that the nested condition in Theorem 5.2 cannot be dropped. However, we do not find any example yet for which the characteristic 0 condition cannot be relaxed. Another direction to pursue if to see if the work reported by Theorem 5.2 can be generalized to nested interval posets of UPP posets. It would be desirable to combine Example 4.11 with Remark 5.11 to shed some light on the next question.

QUESTION 5.13. Let P be a nested interval poset for a UPP poset and F a characteristic 0 field. Are $\mathbb{n}_P$ and $\mathfrak{C}_P$ always in the same conjugacy class of $Inc_F(P)$? Do they at least have the same Jordan invariants when viewed as integer matrices?

6. Tensor product constructions. For four matrices A_1, A_2, B_1, and B_2, if A_1 and A_2 are similar and B_1 and B_2 are similar, then we surely know that the tensor product $A_1 \otimes B_1$ is similar to $A_2 \otimes B_2$. We try to address more simple cases of Stanley's problem in this section and the underlying idea can be said to be this fairly easy observation on tensor product.

Let P be a poset. We write $T(P)$ and $B(P)$ for the set of sources and the set of sinks in $\Gamma(P)$, respectively. We call the elements of $T(P)$ *top elements* of P and those

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of $B(P)$ the *bottom elements* of P. Let $M(P) = P \setminus (T(P) \cup B(P))$. For any positive integer n, the set $\{1, 2, \ldots, n\}$ is denoted by $[n]$. For any three positive integers n_1, n_2 and n_3, the poset P^{n_1,n_2,n_3} has $(T(P) \times [n_1]) \cup (M(P) \times [n_2]) \cup (B(P) \times [n_3])$ as its ground set and $(u,i) <_{P^{n_1,n_2,n_3}} (v,j)$ if and only if $u <_P v$. Notice that any element from $B(P) \cap T(P)$ has $n_1 + n_3$ copies in P^{n_1,n_2,n_3}. Given any $f \in Inc_F(P)$, the function $f^{n_1,n_2,n_3} \in Inc_F(P^{n_1,n_2,n_3})$ is specified by $f^{n_1,n_2,n_3}((x,i),(y,j)) = f(x,y)$.

LEMMA 6.1. *Let f and g be two elements in the strict incidence algebra of a poset P over a field F. If there is an $\alpha \in Inc_F(P)$ such that*

$$\alpha(M(P), T(P)) = \mathbf{0} \text{ and } \alpha^{-1} f \alpha = g,$$

then f^{n_1,n_2,n_3} and g^{n_1,n_2,n_3} are conjugate in $Inc_F(P^{n_1,n_2,n_3})$.

Proof. For ease of notation, put $Q = P^{n_1,n_2,n_3}$, $f^Q = f^{n_1,n_2,n_3}$, $g^Q = g^{n_1,n_2,n_3}$, $\alpha^{-1} = \widehat{\alpha}$, $T = T(P)$, $B = B(P)$, $M = M(P)$, $T_i = T \times i$, $B_i = B \times i$, $M_i = M \times i$. Fix any mappings $\psi \in [n_2]^{[n_1]}$ and define $\beta, \widehat{\beta} \in Inc_F(Q)$ by setting

$$\begin{cases} \widehat{\beta}(B_i, B_j) = & \delta(i,j)\widehat{\alpha}_B; \\ \widehat{\beta}(B_i, M_j) = & \delta(j, \psi(i))\widehat{\alpha}(B, M); \\ \widehat{\beta}(B_i, T_j) = & \widehat{\alpha}(B, T); \\ \widehat{\beta}(M_i, M_j) = & \delta(i,j)\widehat{\alpha}_M; \\ \widehat{\beta}(M_i, T_j) = & \mathbf{0}; \\ \widehat{\beta}(T_i, T_j) = & \delta(i,j)\widehat{\alpha}_T; \end{cases} \qquad \begin{cases} \beta(B_i, B_j) = & \delta(i,j)\alpha_B; \\ \beta(B_i, M_j) = & \delta(j, \psi(i))\alpha(B, M); \\ \beta(B_i, T_j) = & \alpha(B, T); \\ \beta(M_i, M_j) = & \delta(i,j)\alpha_M; \\ \beta(M_i, T_j) = & \mathbf{0}; \\ \beta(T_i, T_j) = & \delta(i,j)\alpha_T. \end{cases}$$

Here are some easily-checked properties of the two functions β and $\widehat{\beta}$:

$$\begin{aligned} * \ \widehat{\beta}\beta(B_i, B_j) &= \delta(i,j)\widehat{\beta}_{B_i}\beta_{B_j} \\ &= \delta(i,j)\widehat{\alpha}_B\alpha_B \\ &= \delta(i,j)\delta_{B_i}; \end{aligned}$$

$$\begin{aligned} * \ \widehat{\beta}\beta(B_i, M_j) &= \delta(j, \psi(i))(\widehat{\alpha}_B\alpha(B, M) + \widehat{\alpha}(B, M)\alpha_M) \\ &= \delta(j, \psi(i))\widehat{\alpha}\alpha(B, M) \\ &= \mathbf{0}; \end{aligned}$$

$$\begin{aligned} * \ \widehat{\beta}\beta(B_i, T_j) &= \widehat{\alpha}_B\alpha(B, T) + \widehat{\alpha}(B, T)\alpha_T \\ &= \widehat{\alpha}\alpha(B, T) \qquad (\text{by } \alpha(M, T) = \mathbf{0}) \\ &= \mathbf{0}; \end{aligned}$$

$$\begin{aligned} * \ \widehat{\beta}\beta(M_i, M_j) &= \delta(i,j)\widehat{\alpha}_M\alpha_M \\ &= \delta(i,j)\widehat{\alpha}\alpha(M, M) \\ &= \delta(i,j)\delta_{M_i}; \end{aligned}$$

$$\begin{aligned} * \ \widehat{\beta}\beta(M_i, T_j) &= \widehat{\beta}_{M_i}\beta(M_i, T_j) + \widehat{\beta}(M_i, T_j)\beta_{T_j} \\ &= \mathbf{0}; \end{aligned}$$

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$$\begin{aligned} \divideontimes\ \widehat{\beta}\beta(T_i,T_j) &= \delta(i,j)\widehat{\alpha}_T\alpha_T \\ &= \delta(i,j)\delta_{T_i}. \end{aligned}$$

Combining the above six items enables us to get $\widehat{\beta}\beta = \delta_Q$. Thus, to infer that f^Q and g^Q are conjugate in $Inc_F(Q)$, it remains to verify $\widehat{\beta}f^Q\beta = g^Q$. Taking into consideration that f lies in the strict incidence algebra of P, we have

$$f_B = \mathbf{0},\ f_T = \mathbf{0},\ f^Q_{B\times[n_3]} = \mathbf{0},\ f^Q_{T\times[n_1]} = \mathbf{0}. \tag{6.1}$$

Now, the final proof is accomplished as follows:

- $$\begin{aligned} \widehat{\beta}f^Q\beta(B_i,B_j) &= \widehat{\beta}_{B_i}f^Q(B_i,B_j)\beta_{B_j} \\ &= \widehat{\alpha}_B f_B\alpha_B \\ &= g_B \\ &= g^Q(B_i,B_j); \end{aligned}$$

- $$\begin{aligned} \widehat{\beta}f^Q\beta(B_i,M_j) &= \widehat{\beta}_{B_i}f^Q(B_i,M_j)\beta_{M_j} + \\ &\quad \widehat{\beta}(B_i,M_{\psi(i)})f^Q(M_{\psi(i)},M_j)\beta_{M_j} \quad \text{(by Eq. (6.1))} \\ &= \widehat{\alpha}_B f(B,M)\alpha_M + \widehat{\alpha}(B,M)f_M\alpha_M \\ &= \widehat{\alpha}f\alpha(B,M) \quad \text{(by Eq. (6.1))} \\ &= g(B,M) \\ &= g^Q(B_i,M_j); \end{aligned}$$

- $$\begin{aligned} \widehat{\beta}f^Q\beta(B_i,T_j) &= \widehat{\beta}_{B_i}f^Q(B_i,T_j)\beta_{T_j} + \\ &\quad \widehat{\beta}(B_i,M_{\psi(i)})f^Q(M_{\psi(i)},T_j)\beta_{T_j} \quad \text{(by Eq. (6.1))} \\ &= \widehat{\alpha}_B f(B,T)\alpha_T + \widehat{\alpha}(B,M)f(M,T)\alpha_T \\ &= \widehat{\alpha}f\alpha(B,T) \quad \text{(by Eq. (6.1) and } \alpha(M,T)=\mathbf{0}) \\ &= g(B,T) \\ &= g^Q(B_i,T_j); \end{aligned}$$

- $$\begin{aligned} \widehat{\beta}f^Q\beta(M_i,M_j) &= \widehat{\beta}_{M_i}f^Q(M_i,M_j)\beta_{M_j} \\ &= \widehat{\alpha}_M f_M\alpha_M \\ &= g_M \\ &= g^Q(M_i,M_j); \end{aligned}$$

- $$\begin{aligned} \widehat{\beta}f^Q\beta(M_i,T_j) &= \widehat{\beta}_{M_i}f^Q(M_i,T_j)\beta_{T_j} \quad \text{(by Eq. (6.1))} \\ &= \widehat{\alpha}_M f(M,T)\alpha_T \\ &= \widehat{\alpha}f\alpha(M,T) \quad \text{(by Eq. (6.1) and } \alpha(M,T)=\mathbf{0}) \\ &= g(M,T) \\ &= g^Q(M_i,T_j); \end{aligned}$$

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- $\begin{aligned} \widehat{\beta} f^Q \beta(T_i, T_j) &= \widehat{\beta}_{T_i} f^Q(T_i, T_j) \beta_{T_j} \\ &= \widehat{\alpha}_T f_T \alpha_T \\ &= g_T \\ &= g^Q(T_i, T_j). \quad \square \end{aligned}$

We are now ready to show that the characteristic 0 requirement for Example 5.3 is indeed unnecessary.

COROLLARY 6.2. *Let Q be the Bruhat order of the dihedral group of order $2m$ for $m \geq 3$. Then $\mathtt{n}_Q$ and $\mathfrak{C}_Q$ are conjugate to each other in $Inc_F(Q)$ over any field F.*

Proof. Let P be the total order of $m+1$ elements. In view of the nested interval representation of Q given in Example 5.3, it is clear that Q is just $P^{1,2,1}$. Consequently, by taking $f = \mathtt{n}_P$ and $g = \mathfrak{C}_P$, the result follows from Example 3.2 and Lemma 6.1. $\square$

Corollary 6.2 is concerned with the Bruhat orders of dihedral groups. Let us pursue the ideas of its proof and establish a similar result for weak orders instead of Bruhat orders.

For any three positive integers n_1, n_2, n_3, and any $f \in Inc_F(P)$, let $Q = (T(P) \times [n_1]) \cup (M(P) \times [n_2]) \cup (B(T) \times [n_3])$ and let $f(n_1, n_2, n_3)$ be the function on $Q \times Q$ that takes value 0 on $\{((x,i),(y,j)) : \ x, y \in M(P), i \neq j\}$ and coincides with f^{n_1,n_2,n_3} elsewhere. The poset $P(n_1, n_2, n_3)$ is the one satisfying $\mathtt{n}_{P(n_1,n_2,n_3)} = \mathtt{n}_P(n_1, n_2, n_3)$. The next lemma is a slight modification of Lemma 6.1.

LEMMA 6.3. *Let f and g be two elements in the strict incidence algebra of a poset P over a field F. Suppose*

$$f(B(P), T(P)) = g(B(P), T(P)) = \mathbf{0},$$

and there exists an $\alpha' \in Inc_F(P)$ such that

$$\alpha'^{-1}(B(P), M(P)) f(M(P), T(P)) = \mathbf{0}, \ \alpha'(M(P), T(P)) = \mathbf{0} \text{ and } \alpha'^{-1} f \alpha' = g.$$

Then $f(n_1, n_2, n_3)$ and $g(n_1, n_2, n_3)$ are conjugate in $Inc_F(P(n_1, n_2, n_3))$.

Proof. Before embarking on the proof, we explain some short-hand notations to be used for simplicity's sake: $Q = P(n_1, n_2, n_3)$, $f^Q = f(n_1, n_2, n_3)$, $g^Q = g(n_1, n_2, n_3)$, $\alpha'^{-1} = \widehat{\alpha'}$, $T = T(P)$, $B = B(P)$, $M = M(P)$, $T_i = T \times i$, $B_i = B \times i$, $M_i = M \times i$. In addition, we should note that Eq. (6.1) is still valid in the current situation.

Now, let us start the proof by introducing two functions $\beta, \widehat{\beta} \in Inc_F(Q)$ given

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by:

$$\begin{cases} \widehat{\beta}(B_i,B_j)= & \delta(i,j)\widehat{\alpha'}_B;\\ \widehat{\beta}(B_i,M_j)= & \widehat{\alpha'}(B,M);\\ \widehat{\beta}(B_i,T_j)= & \widehat{\alpha'}(B,T);\\ \widehat{\beta}(M_i,M_j)= & \delta(i,j)\widehat{\alpha'}_M;\\ \widehat{\beta}(M_i,T_j)= & \mathbf{0};\\ \widehat{\beta}(T_i,T_j)= & \delta(i,j)\widehat{\alpha'}_T; \end{cases} \quad \begin{cases} \beta(B_i,B_j)= & \delta(i,j)\alpha'_B;\\ \beta(B_i,M_j)= & \alpha'(B,M);\\ \beta(B_i,T_j)= & \alpha'(B,T);\\ \beta(M_i,M_j)= & \delta(i,j)\alpha'_M;\\ \beta(M_i,T_j)= & \mathbf{0};\\ \beta(T_i,T_j)= & \delta(i,j)\alpha'_T. \end{cases}$$

We can check that $\widehat{\beta}\beta=\delta_Q$:

$$\begin{aligned} \divideontimes\ \widehat{\beta}\beta(B_i,B_j) &= \delta(i,j)\widehat{\alpha'}(B_i,B_i)\alpha'(B_j,B_j)\\ &= \delta(i,j)\widehat{\alpha'}_B\alpha'_B\\ &= \delta(i,j)\delta_{B_i};\end{aligned}$$

$$\begin{aligned} \divideontimes\ \widehat{\beta}\beta(B_i,M_j) &= \widehat{\alpha'}_B\alpha'(B,M)+\widehat{\alpha'}(B,M)\alpha'_M\\ &= \widehat{\alpha'}\alpha'(B,M)\\ &= \mathbf{0};\end{aligned}$$

$$\begin{aligned} \divideontimes\ \widehat{\beta}\beta(B_i,T_j) &= \widehat{\beta}_{B_i}\beta(B_i,T_j)+\widehat{\beta}(B_i,T_j)\beta_{T_j}\\ &= \widehat{\alpha'}_B\alpha'(B,T)+\widehat{\alpha'}(B,T)\alpha'_T\\ &= \widehat{\alpha'}\alpha'(B,T) \qquad (\text{by } \alpha'(M,T)=\mathbf{0})\\ &= \mathbf{0};\end{aligned}$$

$$\begin{aligned} \divideontimes\ \widehat{\beta}\beta(M_i,M_j) &= \delta(i,j)\widehat{\beta}_{M_i}\beta_{M_j}\\ &= \delta(i,j)\widehat{\alpha'}_M\alpha'_M\\ &= \delta(i,j)\widehat{\alpha'}\alpha'(M,M)\\ &= \delta(i,j)\delta_{M_i};\end{aligned}$$

$$\begin{aligned} \divideontimes\ \widehat{\beta}\beta(M_i,T_j) &= \widehat{\beta}_{M_i}\beta(M_i,T_j)+\widehat{\beta}(M_i,T_j)\beta_{T_j}\\ &= \mathbf{0};\end{aligned}$$

$$\begin{aligned} \divideontimes\ \widehat{\beta}\beta(T_i,T_j) &= \delta(i,j)\widehat{\beta}_{T_i}\beta_{T_j}\\ &= \delta(i,j)\widehat{\alpha'}_T\alpha'_T\\ &= \delta(i,j)\delta_{T_i}.\end{aligned}$$

To complete the proof, it then suffices to verify $\widehat{\beta}f^Q\beta=g^Q$ in the following six steps:

$$\begin{aligned} \bullet\ \widehat{\beta}f^Q\beta(B_i,B_j) &= \widehat{\beta}_{B_i}f^Q(B_i,B_j)\beta_{B_j}\\ &= \widehat{\alpha'}_Bf_B\alpha'_B\\ &= g_B\\ &= g^Q(B_i,B_j);\end{aligned}$$

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- $\widehat{\beta} f^Q \beta(B_i, M_j) = \widehat{\beta}_{B_i} f^Q(B_i, M_j)\beta_{M_j} + \widehat{\beta}(B_i, M_j) f^Q(M_j, M_j)\beta_{M_j}$ (by Eq. (6.1))
  $= \widehat{\alpha'}_B f(B, M)\alpha'_M + \widehat{\alpha'}(B, M) f_M \alpha'_M$
  $= g(B, M)$ (by Eq. (6.1))
  $= g^Q(B_i, M_j)$;

- $\widehat{\beta} f^Q \beta(B_i, T_j) = \widehat{\beta}_{B_i} f^Q(B_i, T_j)\beta_{T_j} + \sum_k \widehat{\beta}(B_i, M_k) f^Q(M_k, T_j)\beta_{T_j}$ (by Eq. (6.1))
  $= \sum_k \widehat{\alpha'}(B, M) f(M, T)\alpha'_T$ (by $f(B, T) = \mathbf{0}$)
  $= \mathbf{0}$ (by $\widehat{\alpha'}(B, M) f(M, T) = \mathbf{0}$)
  $= g(B, T)$
  $= g^Q(B_i, T_j)$;

- $\widehat{\beta} f^Q \beta(M_i, M_j) = \widehat{\beta}_{M_i} f^Q(M_i, M_j)\beta_{M_j}$
  $= \delta(i, j)\widehat{\alpha'}_M f_M \alpha'_M$
  $= \delta(i, j) g_M$
  $= g^Q(M_i, M_j)$;

- $\widehat{\beta} f^Q \beta(M_i, T_j) = \widehat{\beta}_{M_i} f^Q(M_i, T_j)\beta_{T_j}$ (by Eq. (6.1))
  $= \widehat{\alpha'}_M f(M, T)\alpha'_T$
  $= g(M, T)$ (by Eq. (6.1) and $\alpha'(M, T) = \mathbf{0}$)
  $= g^Q(M_i, T_j)$;

- $\widehat{\beta} f^Q \beta(T_i, T_j) = \widehat{\beta}_{T_i} f^Q(T_i, T_j)\beta_{T_j}$
  $= \widehat{\alpha'}_T f_T \alpha'_T$
  $= g_T$
  $= g^Q(T_i, T_j)$. □

COROLLARY 6.4. *Let Q be the weak order of the dihedral group of order $2m$, $m \geq 3$. Then $\mathbb{n}_Q$ and $\mathfrak{C}_Q$ are conjugate to each other in $Inc_F(Q)$ over any field F.*

Proof. Let P be the total order of $m+1$ elements, say $1 < 2 < \cdots < m+1$. It is not difficult to find that Q is just $P(1, 2, 1)$. Let x be the unique bottom element of Q and y be one of the two elements which are covered by the unique top element of Q. Let $n = m+1$ and put $\alpha' = T_{1,n-1}(-1)T(n-1)\cdots T(2)$, where $T(j)$, $j = 2, \ldots, n-1$, are defined in Eq. (3.1). A closer look on Example 3.2 says that we can apply Lemma 6.3 for $f = T_{1,n-1}(-1)\mathbb{n}_P T_{1,n-1}(1)$, $g = \mathfrak{C}_P$, and the above-mentioned α', to conclude that $g(1, 2, 1) = \mathfrak{C}_Q$ is conjugate to $f(1, 2, 1) = T_{xy}(-1)\mathbb{n}_Q T_{xy}(1)$, and therefore the claim follows. □

Corollaries 6.2 and 6.4 are about the Bruhat orders and the weak orders, respectively, of types $I_2(m), m \geq 3$. These, together with the sporadic observations in Example 4.12, suggest that it may serve to further exploit other Bruhat orders and

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weak orders of Coxeter groups and see if their incidence matrices and cover matrices are always conjugate to each other in the corresponding incidence algebras. To this end, a good knowledge of the structure of these posets may be crucial. We close this paper by noting that a detailed description of the underlying graph of the Hasse diagrams of the weak orders of the Coxeter groups of types B_n and $\tilde{A}_2$ can be found in [15].

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