

ON AN EQUALITY AND FOUR INEQUALITIES FOR GENERALIZED INVERSES OF HERMITIAN MATRICES*

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Abstract. A Hermitian matrix X is called a g -inverse of a Hermitian matrix A , denoted by A^- , if it satisfies $AXA = A$. In this paper, a group of explicit formulas are established for calculating the global maximum and minimum ranks and inertias of the difference $A^- - PN^-P^*$, where both A^- and N^- are Hermitian g -inverses of two Hermitian matrices A and N , respectively. As a consequence, necessary and sufficient conditions are derived for the matrix equality $A^- = PN^-P^*$ to hold, and the four matrix inequalities $A^- > (\geq, <, \leq) PN^-P^*$ in the Löwner partial ordering to hold, respectively. In addition, necessary and sufficient conditions are established for the Hermitian matrix equality $A^\dagger = PN^\dagger P^*$ to hold, and the four Hermitian matrix inequalities $A^\dagger > (\geq, <, \leq) PN^\dagger P^*$ to hold, respectively, where $(\cdot)^\dagger$ denotes the Moore-Penrose inverse of a matrix. As applications, identifying conditions are given for the additive decomposition of a Hermitian g -inverse $C^- = A^- + B^-$ (parallel sum of two Hermitian matrices) to hold, as well as the four matrix inequalities $C^- > (\geq, <, \leq) A^- + B^-$ in the Löwner partial ordering to hold, respectively.

Key words. Hermitian matrix, Generalized inverse of matrix, Moore-Penrose inverse, Schur complement, Rank, Inertia, Expansion formula, Equality, Inequality, Löwner partial ordering, Parallel sum of matrices, Shorted matrix.

AMS subject classifications. 15A03, 15A09, 15A24, 15B57, 65K10, 65K15.

1. Introduction. Throughout this paper, $\mathbb{C}^{m \times n}$ and \mathbb{C}_H^m denote the collections of all $m \times n$ complex matrices and all $m \times m$ complex Hermitian matrices, respectively. The symbols A^* , $r(A)$ and $\mathcal{R}(A)$ stand for the conjugate transpose, rank and range (column space) of a complex matrix A , respectively; I_m denotes the identity matrix of order m ; $[A, B]$ denotes a partitioned matrix consisting of A and B . Two Hermitian matrices A and B of the same size are said to be congruent if there is an invertible matrix S such that $SAS^* = B$. For an $A \in \mathbb{C}_H^m$, we write $A > 0$ ($A \geq 0$) if A is positive definite (nonnegative definite). Two $A, B \in \mathbb{C}_H^m$ are said to satisfy the inequality $A > B$ ($A \geq B$) in the Löwner partial ordering if $A - B$ is positive definite (nonnegative definite). The Moore-Penrose inverse of $A \in \mathbb{C}^{m \times n}$, denoted by A^\dagger , is defined to be the unique solution X of the following four matrix equations

$$(i) AXA = A, \quad (ii) XAX = X, \quad (iii) (AX)^* = AX, \quad (iv) (XA)^* = XA.$$

*Received by the editors on April 9, 2011. Accepted for publication on October 24, 2011. Handling Editor: Oskar Maria Baksalary.

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Further, define $E_A = I_m - AA^\dagger$ and $F_A = I_n - A^\dagger A$. A well-known property of the Moore-Penrose inverse is $(A^\dagger)^* = (A^*)^\dagger$. In addition, $AA^\dagger = A^\dagger A$ if $A = A^*$. We shall repeatedly use these facts in the latter part of this paper. A matrix X is called a Hermitian g -inverse of $A \in \mathbb{C}_H^m$, denoted by A^- , if it satisfies

$$AXA = A \quad \text{and} \quad X = X^*.$$

The collection of all possible Hermitian g -inverses of $A \in \mathbb{C}_H^m$ is denoted by $\{A^-\}$.

It is well known that the eigenvalues of a Hermitian matrix $A \in \mathbb{C}_H^m$ are all real, and the inertia of A is defined to be the triplet

$$\text{In}(A) = (i_+(A), i_-(A), i_0(A)),$$

where $i_+(A)$, $i_-(A)$ and $i_0(A)$ are the numbers of the positive, negative and zero eigenvalues of A counted with multiplicities, respectively. Both $i_+(A)$ and $i_-(A)$, usually called the partial inertia, can easily be computed by elementary congruence matrix operations. For a Hermitian matrix $A \in \mathbb{C}_H^m$, both $r(A) = i_+(A) + i_-(A)$ and $i_0(A) = m - r(A)$ hold.

The objective of this paper is to address the following problem:

PROBLEM 1.1. For three given matrices $A \in \mathbb{C}_H^m$, $N \in \mathbb{C}_H^n$ and $P \in \mathbb{C}^{m \times n}$, establish formulas for calculating the global maximum and minimum ranks and inertias of the difference $A^- - PN^-P^*$, where A^- and N^- are Hermitian g -inverses of A and N , and use the formulas to derive necessary and sufficient conditions for the following matrix equality and inequalities

$$(1.1) \quad \begin{cases} A^- = PN^-P^*, \\ A^- > PN^-P^*, \quad A^- \geq PN^-P^*, \\ A^- < PN^-P^*, \quad A^- \leq PN^-P^* \end{cases}$$

to hold, respectively.

Matrix equalities and inequalities with symmetric patterns that involve generalized inverses occur widely in matrix theory and applications. These equalities and inequalities can generally be formulated as

$$(1.2) \quad \begin{cases} p(A_1^-, \dots, A_s^-) = q(B_1^-, \dots, B_t^-), \\ p(A_1^-, \dots, A_s^-) > (\geq, <, \leq) q(B_1^-, \dots, B_t^-), \end{cases}$$

where A_1^-, \dots, A_s^- and B_1^-, \dots, B_t^- are (Hermitian) g -inverses of matrices. One of the fundamental and challenging research topics in the theory of generalized inverses is to establish necessary and sufficient conditions for these equalities and inequalities to hold. Obviously, (1.1) is a group of simplest forms of (1.2), while the equality $A^- =$

N^- and inequality $A^- > (\geq, <, \leq) N^-$ for g -inverses of two Hermitian matrices, the additive decomposition $C^- = A^- + B^-$ and the inequalities $C^- > (\geq, <, \leq) A^- + B^-$, and the reverse-order law $(PNP^*)^- = (P^\dagger)^* N^- P^\dagger$ and the inequalities $(PNP^*)^- > (\geq, <, \leq) (P^\dagger)^* N^- P^\dagger$, etc., are special cases of (1.1) as well.

In a recent paper [14], Liu and Tian considered the rank and inertia of the well-known Schur complement $D - B^* A^- B$, where both A and D are Hermitian and A^- is a Hermitian g -inverse of A , and obtained a group of explicit formulas for calculating the maximum and minimum ranks and inertias of $D - B^* A^- B$ with respect to the choice of A^- . In this paper, we shall use these rank and inertia formulas to solve Problem 1.1. As applications, we also derive necessary and sufficient conditions for the following matrix equality and inequalities

$$(1.3) \quad \begin{cases} C^- = A^- + B^-, \\ C^- > A^- + B^-, \quad C^- \geq A^- + B^-, \\ C^- < A^- + B^-, \quad C^- \leq A^- + B^- \end{cases}$$

to hold, respectively, and consider their extensions to the sum of k matrices.

The following are some simple or well-known facts and formulas for ranks and inertias of matrices and their consequences (see [32, 33] for their references), which will be used in the latter part of this paper.

LEMMA 1.2. *Let \mathcal{S} be a subset in $\mathbb{C}^{m \times n}$, and \mathcal{H} be a subset in \mathbb{C}_H^m . Then,*

- (a) *Under $m = n$, \mathcal{S} has a nonsingular matrix if and only if $\max_{X \in \mathcal{S}} r(X) = m$.*
- (b) *Under $m = n$, all $X \in \mathcal{S}$ are nonsingular if and only if $\min_{X \in \mathcal{S}} r(X) = m$.*
- (c) *$0 \in \mathcal{S}$ if and only if $\min_{X \in \mathcal{S}} r(X) = 0$.*
- (d) *$\mathcal{S} = \{0\}$ if and only if $\max_{X \in \mathcal{S}} r(X) = 0$.*
- (e) *\mathcal{H} has a matrix $X > 0$ ($X < 0$) if and only if*

$$\max_{X \in \mathcal{H}} i_+(X) = m \quad \left(\max_{X \in \mathcal{H}} i_-(X) = m \right).$$

- (f) *All $X \in \mathcal{H}$ satisfy $X > 0$ ($X < 0$) if and only if*

$$\min_{X \in \mathcal{H}} i_+(X) = m \quad \left(\min_{X \in \mathcal{H}} i_-(X) = m \right).$$

- (g) *\mathcal{H} has a matrix $X \geq 0$ ($X \leq 0$) if and only if*

$$\min_{X \in \mathcal{H}} i_-(X) = 0 \quad \left(\min_{X \in \mathcal{H}} i_+(X) = 0 \right).$$

- (h) *All $X \in \mathcal{H}$ satisfy $X \geq 0$ ($X \leq 0$) if and only if*

$$\max_{X \in \mathcal{H}} i_-(X) = 0 \quad \left(\max_{X \in \mathcal{H}} i_+(X) = 0 \right).$$

LEMMA 1.3. Let $A \in \mathbb{C}_H^m$, $B \in \mathbb{C}_H^n$, $Q \in \mathbb{C}^{m \times n}$, and assume that $P \in \mathbb{C}^{m \times m}$ is nonsingular. Then,

$$(1.4) \quad i_{\pm}(PAP^*) = i_{\pm}(A),$$

$$(1.5) \quad A(A^3)^{\dagger}A = A^{\dagger}, \quad r(A^3) = r(A),$$

$$(1.6) \quad i_{\pm}(A^3) = i_{\pm}(A), \quad i_{\pm}(A^{\dagger}) = i_{\pm}(A),$$

$$(1.7) \quad i_{\pm}(\lambda A) = \begin{cases} i_{\pm}(A) & \text{if } \lambda > 0 \\ i_{\mp}(A) & \text{if } \lambda < 0 \end{cases},$$

$$(1.8) \quad i_{\pm} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} = i_{\pm}(A) + i_{\pm}(B),$$

$$(1.9) \quad i_{+} \begin{bmatrix} 0 & Q \\ Q^* & 0 \end{bmatrix} = i_{-} \begin{bmatrix} 0 & Q \\ Q^* & 0 \end{bmatrix} = r(Q).$$

LEMMA 1.4. Let $A \in \mathbb{C}_H^m$, $B \in \mathbb{C}_H^n$, and $P, Q \in \mathbb{C}^{m \times n}$. Then,

$$(1.10) \quad i_{\pm}(P^*AP) \leq i_{\pm}(A).$$

In particular,

- (a) $r(P^*AP) = r(A)$ if and only if $i_{+}(P^*AP) = i_{+}(A)$ and $i_{-}(P^*AP) = i_{-}(A)$.
- (b) If $P^*AP = B$ and $QBQ^* = A$, then $i_{\pm}(A) = i_{\pm}(B)$ and $r(A) = r(B)$.

The two inequalities in (1.10) were first given in [25]; Theorem 1.4(a) and (b) were given in [32, Lemma 1.6].

LEMMA 1.5 ([16]). Let $A \in \mathbb{C}^{m \times n}$, $B \in \mathbb{C}^{m \times n}$ and $C \in \mathbb{C}^{l \times n}$. Then, the following rank expansion formulas hold

$$(1.11) \quad r[A, B] = r(A) + r(E_A B) = r(B) + r(E_B A),$$

$$(1.12) \quad r \begin{bmatrix} A \\ C \end{bmatrix} = r(A) + r(CF_A) = r(C) + r(AF_C),$$

$$(1.13) \quad r \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = r(B) + r(C) + r(E_B AF_C).$$

LEMMA 1.6 ([32]). Let $M_1 = \begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix}$ and $M_2 = \begin{bmatrix} A & B \\ B^* & D \end{bmatrix}$, where $A \in \mathbb{C}_H^m$, $B \in \mathbb{C}^{m \times n}$ and $D \in \mathbb{C}_H^n$. Then, the following inertia expansion formulas

$$(1.14) \quad i_{+}(M_1) = r(B) + i_{+}(E_B A E_B), \quad i_{-}(M_1) = r(B) + i_{-}(E_B A E_B),$$

$$(1.15) \quad i_{\pm}(M_2) = i_{\pm}(A) + i_{\pm} \begin{bmatrix} 0 & E_A B \\ B^* E_A & D - B^* A^{\dagger} B \end{bmatrix}$$

hold. In particular,

(a) If $A \geq 0$, then

$$(1.16) \quad i_+(M_1) = r[A, B], \quad i_-(M_1) = r(B).$$

(b) If $A \leq 0$, then

$$(1.17) \quad i_+(M_1) = r(B), \quad i_-(M_1) = r[A, B].$$

(c) If $\mathcal{R}(B) \subseteq \mathcal{R}(A)$, then

$$(1.18) \quad i_{\pm}(M_2) = i_{\pm}(A) + i_{\pm}(D - B^*A^{\dagger}B).$$

(d) $r(M_2) = r(A) \Leftrightarrow r[A, B] = r(A)$ and $D = B^*A^{\dagger}B$.

Let $A \in \mathbb{C}_{\mathbb{H}}^m$. Then, the general expression for the Hermitian g -inverse of A can be written as

$$(1.19) \quad A^- = A^{\dagger} + F_A V + V^* F_A,$$

where $V \in \mathbb{C}^{m \times m}$ is arbitrary (see, e.g., [32]). For a partitioned Hermitian matrix

$$M = \begin{bmatrix} A & B \\ B^* & D \end{bmatrix},$$

where $A \in \mathbb{C}_{\mathbb{H}}^m$ and $D \in \mathbb{C}_{\mathbb{H}}^n$, the well-known Hermitian Schur complement of A in M is defined by

$$(1.20) \quad D - B^*A^-B.$$

This expression usually occurs in some decompositions of M that involve generalized inverses. The special case $D - B^*A^{\dagger}B$, as well as $D - B^*A^{-1}B$ when A is nonsingular, was extensively studied in the literature. In particular, some expansion formulas calculating for the rank and inertia of B^*A^-B and $D - B^*A^{-1}B$ can be found, e.g., in [5, 7, 8, 9, 10, 15, 26, 32, 38, 39]. Because the Hermitian g -inverse A^- is not necessarily unique, $D - B^*A^-B$ may vary with respect to the choice of A^- . Substituting the general expression of A^- in (1.19) into (1.20) yields

$$(1.21) \quad D - B^*A^-B = D - B^*A^{\dagger}B - B^*F_A V B - B^*V^*F_A B,$$

which means that (1.20) is in fact a Hermitian matrix function with a variable matrix V and its conjugate transpose. It is easily seen from (1.21) that

$$(1.22) \quad D - B^*A^-B \text{ is unique} \Leftrightarrow \mathcal{R}(B) \subseteq \mathcal{R}(A).$$

Concerning the global maximum and minimum ranks and inertias of (1.20), Liu and Tian [14] recently gave the following expansion formulas.

THEOREM 1.7 ([14]). *Let $A \in \mathbb{C}_H^m$, $B \in \mathbb{C}^{m \times n}$ and $D \in \mathbb{C}_H^n$, and let $M = \begin{bmatrix} A & B \\ B^* & D \end{bmatrix}$ and $N = \begin{bmatrix} A & 0 & B \\ 0 & B^* & D \end{bmatrix}$. Then,*

$$(1.23) \quad \max_{A^-} r(D - B^* A^- B) = \min \{r[B^*, D], r(M) - r(A)\},$$

$$(1.24) \quad \min_{A^-} r(D - B^* A^- B) = r(A) + 2r[B^*, D] + r(M) - 2r(N),$$

$$(1.25) \quad \max_{A^-} i_{\pm}(D - B^* A^- B) = i_{\pm}(M) - i_{\pm}(A),$$

$$(1.26) \quad \min_{A^-} i_{\pm}(D - B^* A^- B) = i_{\mp}(A) + r[B^*, D] + i_{\pm}(M) - r(N).$$

COROLLARY 1.8. *Let $A, D \in \mathbb{C}_H^m$, and let $M = \begin{bmatrix} A & I_m \\ I_m & D \end{bmatrix}$. Then,*

$$(1.27) \quad \max_{A^-} r(D - A^-) = \min \{m, r(M) - r(A)\},$$

$$(1.28) \quad \min_{A^-} r(D - A^-) = r(A) + r(M) - 2m,$$

$$(1.29) \quad \max_{A^-} i_{\pm}(D - A^-) = i_{\pm}(M) - i_{\pm}(A),$$

$$(1.30) \quad \min_{A^-} i_{\pm}(D - A^-) = i_{\mp}(A) + i_{\pm}(M) - m.$$

2. The rank of $A^- - PN^-P^*$ and the equality $A^- = PN^-P^*$. In a recent paper [35], Tian and Styan gave some closed-form formulas for calculating the maximum and minimum ranks of the difference $A^- - PN^-Q$, where $A \in \mathbb{C}^{m \times n}$, $N \in \mathbb{C}^{l \times k}$, $P \in \mathbb{C}^{n \times k}$ and $Q \in \mathbb{C}^{l \times m}$ are given, and used these formulas to derive necessary and sufficient conditions for the matrix equality $A^- = PN^-Q$ to hold in different settings. As a continuation, we consider in this section the following matrix equality

$$(2.1) \quad A^- = PN^-P^*$$

for the Hermitian g -inverses A^- and N^- , where $A \in \mathbb{C}_H^m$, $N \in \mathbb{C}_H^n$ and $P \in \mathbb{C}^{m \times n}$ are given. Note that the two Hermitian g -inverses A^- and N^- are not necessarily unique. Therefore, we can classify (2.1) as the following four reasonable cases:

$$\{A^-\} \cap \{PN^-P^*\} \neq \emptyset, \{A^-\} \subseteq \{PN^-P^*\}, \{A^-\} \supseteq \{PN^-P^*\}, \{A^-\} = \{PN^-P^*\}.$$

In what follows, we first derive a group of formulas for calculating the global maximum and minimum ranks of $A^- - PN^-P^*$ with respect to A^- and N^- . We then use these

formulas to derive necessary and sufficient conditions for the above four assertions to hold, respectively.

THEOREM 2.1. Let $A \in \mathbb{C}_H^m$, $N \in \mathbb{C}_H^n$ and $P \in \mathbb{C}^{m \times n}$. Then,

$$(2.2) \quad \max_{N^-} r(A - APN^-P^*A) = \min\{r(A), r(N - P^*AP) + r(A) - r(N)\},$$

$$(2.3) \quad \min_{N^-} r(A - APN^-P^*A) = r(N - P^*AP) + r(A) + r(N) - 2r[N, P^*AP].$$

As a consequence,

$$(a) \quad \{A^-\} \cap \{PN^-P^*\} \neq \emptyset \Leftrightarrow r(N - P^*AP) = 2r[N, P^*AP] - r(A) - r(N).$$

$$(b) \quad \{PN^-P^*\} \subseteq \{A^-\} \Leftrightarrow r(N - P^*AP) = r(N) - r(A).$$

Proof. It can be seen from the definition of Hermitian g -inverse of a matrix that

(i) There exists an N^- such that $PN^-P^* \in \{A^-\}$ if and only if $APN^-P^*A = A$, or equivalently by the rank of matrix, $\min_{N^-} r(A - APN^-P^*A) = 0$.

(ii) $\{PN^-P^*\} \subseteq \{A^-\}$ if and only if $APN^-P^*A = A$ for any N^- , or equivalently, by the rank of matrix $\max_{N^-} r(A - APN^-P^*A) = 0$.

Applying (1.23) and (1.24) to $A - APN^-P^*A$ and simplifying by elementary matrix operations, we obtain

$$(2.4) \quad \begin{aligned} & \max_{N^-} r(A - APN^-P^*A) \\ &= \min \left\{ r[AP, A], r \begin{bmatrix} N & P^*A \\ AP & A \end{bmatrix} - r(N) \right\} \\ &= \min\{r(A), r(N - P^*AP) + r(A) - r(N)\}, \end{aligned}$$

$$(2.5) \quad \begin{aligned} & \min_{N^-} r(A - APN^-P^*A) \\ &= r(N) + 2r[AP, A] + r \begin{bmatrix} N & P^*A \\ AP & A \end{bmatrix} - 2r \begin{bmatrix} N & 0 & P^*A \\ 0 & AP & A \end{bmatrix} \\ &= r(N - P^*AP) + r(A) + r(N) - 2r[N, P^*AP], \end{aligned}$$

establishing (2.2) and (2.3). Setting the right-hand sides of (2.2) and (2.3) to zero leads to (a) and (b), respectively. \square

Setting $N = P^*AP$ in Theorem 2.1 leads to the following consequence.

COROLLARY 2.2. Let $A \in \mathbb{C}_H^m$ and $P \in \mathbb{C}^{m \times n}$. Then,

(a) The rank of $A - AP(P^*AP)^-P^*A$ is invariant with respect to the choice of $(P^*AP)^-$.

(b) $\{P(P^*AP)^-P^*\} \subseteq \{A^-\}$ if and only if $r(P^*AP) = r(A)$.

Proof. Setting $N = P^*AP$ in (2.2) and (2.3) yields the following result,

$$\begin{aligned} \max_{(P^*AP)^-} r[A - AP(P^*AP)^-P^*A] &= \min_{(P^*AP)^-} r[A - AP(P^*AP)^-P^*A] \\ &= r(A) - r(P^*AP). \end{aligned}$$

Then, (a) and (b) follow, respectively. \square

The rank subtractivity equality for a pair of matrices A and B of the same order is defined by $r(B - A) = r(B) - r(A)$. This relation is usually called a minus partial ordering and is denoted by $A \leq^- B$. A well-known rank subtractivity equality associated with the difference $A - AP(P^*AP)^-P^*A$ is

$$(2.6) \quad r[A - AP(P^*AP)^-P^*A] = r(A) - r[AP(P^*AP)^-P^*A];$$

see [6, 26, 28, 29]. It can be derived from (1.23) and (1.24) that

$$(2.7) \quad \max_{(P^*AP)^-} r[AP(P^*AP)^-P^*A] = 2r(AP) - r(P^*AP),$$

$$(2.8) \quad \min_{(P^*AP)^-} r[AP(P^*AP)^-P^*A] = r(P^*AP).$$

Combining (2.6), (2.7) and (2.8) leads to the following result.

COROLLARY 2.3. *Let $A \in \mathbb{C}_H^m$ and $P \in \mathbb{C}^{m \times n}$. Then, there exists a $(P^*AP)^-$ such that (2.6) holds if and only if $r(P^*AP) = r(AP)$. As a consequence, (2.6) holds for any $(P^*AP)^-$.*

In order to derive necessary and sufficient conditions for the set inclusion $\{A^-\} \subseteq \{PN^-P^*\}$ to hold, we assume that P has full row rank because the maximum rank of A^- is equal to the size of A . It is obvious that $\{A^-\} \subseteq \{PN^-P^*\}$ is equivalent to

$$\max_{A^-} \min_{N^-} r(A^- - PN^-P^*) = 0.$$

Applying (1.23) and (1.24) to $A^- - PN^-P^*$ gives the following result.

THEOREM 2.4. *Let $A \in \mathbb{C}_H^m$, $N \in \mathbb{C}_H^n$ and $P \in \mathbb{C}^{m \times n}$ be given with $r(P) = m$. Then,*

$$(2.9) \quad \begin{aligned} \max_{A^-} \min_{N^-} r(A^- - PN^-P^*) &= \min\{m + r(N) - r[N, P^*], \\ &\quad 2m + r(N - P^*AP) + r(N) - r(A) - 2r[N, P^*]\}. \end{aligned}$$

As a consequence, the set inclusion $\{A^-\} \subseteq \{PN^-P^\}$ holds if and only if*

$$(2.10) \quad \mathcal{R}(N) \cap \mathcal{R}(P^*) = \{0\} \text{ or } r(N - P^*AP) = 2r[N, P^*] - r(N) + r(A) - 2m.$$

Proof. Under the given conditions, applying (1.24) to $A^- - PN^-P^*$ gives

$$\begin{aligned} \min_{N^-} r(A^- - PN^-P^*) &= r(N) + 2r[P, A^-] + r \begin{bmatrix} N & P^* \\ P & A^- \end{bmatrix} - 2r \begin{bmatrix} N & 0 & P^* \\ 0 & P & A^- \end{bmatrix} \\ &= r(N) - 2r[N, P^*] + r \begin{bmatrix} N & P^* \\ P & A^- \end{bmatrix}, \end{aligned}$$

so that

$$(2.11) \quad \max_{A^-} \min_{N^-} r(A^- - PN^-P^*) = r(N) - 2r[N, P^*] + \max_{A^-} r \begin{bmatrix} N & P^* \\ P & A^- \end{bmatrix}.$$

Further, applying (1.23) and simplifying by elementary matrix operations, we obtain

$$\begin{aligned} (2.12) \quad \max_{A^-} r \begin{bmatrix} N & P^* \\ P & A^- \end{bmatrix} &= \max_{A^-} r \left(\begin{bmatrix} N & P^* \\ P & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ I_m \end{bmatrix} A^- [0, I_m] \right) \\ &= \min \left\{ r \begin{bmatrix} N & P^* & 0 \\ P & 0 & I_m \end{bmatrix}, r \begin{bmatrix} -A & 0 & I_m \\ 0 & N & P^* \\ I_m & P & 0 \end{bmatrix} - r(A) \right\} \\ &= \min \{ m + r[N, P^*], r(N - P^*AP) - r(A) + 2m \}. \end{aligned}$$

Combining (2.11) and (2.12) yields (2.9). Setting the right-hand of (2.9) to zero yields (2.10). \square

Note that the first condition in (2.10) has no relation with A . Also note that $\{A^-\} = \mathbb{C}_H^m$ for $A = 0$. This implies $\{PN^-P^*\} = \mathbb{C}_H^m$ under first condition in (2.10). Conversely, if $\{PN^-P^*\} = \mathbb{C}_H^m$, then $\{PN^-P^*\} = \{0^-\}$. In this case, applying Theorem 2.4 leads to the first condition in (2.10). Thus, the first condition in (2.10) is a necessary and sufficient condition such that $\{PN^-P^*\} = \mathbb{C}_H^m$. In what follows, we assume that

$$(2.13) \quad \mathcal{R}(N) \cap \mathcal{R}(P^*) \neq \{0\}.$$

THEOREM 2.5. Let $A \in \mathbb{C}_H^m$, $N \in \mathbb{C}_H^n$ and $P \in \mathbb{C}^{m \times n}$ be given with $r(P) = m$. Then,

$$(2.14) \quad \{A^-\} = \{PN^-P^*\}$$

if and only if

$$(2.15) \quad r \begin{bmatrix} N & P^* \\ P & 0 \end{bmatrix} = r[N, P^*] + r(P) \quad \text{and} \quad A = -[0, I_m] \begin{bmatrix} N & P^* \\ P & 0 \end{bmatrix}^- \begin{bmatrix} 0 \\ I_m \end{bmatrix}.$$

In this case,

$$(2.16) \quad r(A) = \dim[\mathcal{R}(N) \cap \mathcal{R}(P^*)].$$

Proof. Suppose first that (2.14) holds. Then, this implies that

$$(2.17) \quad \min_{A^-} r(A^-) = \min_{N^-} r(PN^-P^*).$$

It can be derived from (1.24) that

$$\min_{A^-} r(A^-) = r(A) \quad \text{and} \quad \min_{N^-} r(PN^-P^*) = r(N) - 2r[N, P^*] + r \begin{bmatrix} N & P^* \\ P & 0 \end{bmatrix}.$$

Substituting these two equalities into (2.17) yields

$$(2.18) \quad r(A) = r(N) - 2r[N, P^*] + r \begin{bmatrix} N & P^* \\ P & 0 \end{bmatrix}.$$

On the other hand, (2.14) implies $\{A^-\} \subseteq \{PN^-P^*\}$. Thus, the second condition in (2.10) holds. Substituting (2.18) into the second condition in (2.10) gives

$$(2.19) \quad r(N - P^*AP) = r \begin{bmatrix} N & P^* \\ P & 0 \end{bmatrix} - 2m,$$

which is equivalent to

$$(2.20) \quad r \begin{bmatrix} N & P^* & 0 \\ P & 0 & I_m \\ 0 & I_m & -A \end{bmatrix} = r \begin{bmatrix} N & P^* \\ P & 0 \end{bmatrix}.$$

Applying Lemma 1.6(d) to (2.20) gives (2.15). Substituting the first rank equality in (2.15) into (2.18) results in (2.16).

Conversely, if (2.15) holds, we can see from Lemma 1.6(d) that (2.20), or equivalently, (2.19) holds. Combining the first rank equality in (2.15), (2.16) and (2.19), we see that $N - P^*AP$ satisfies $r(N - P^*AP) = r(N) - r(A)$ and the second rank equality in (2.10). This means by Theorems 2.1(b) and 2.4 that both $\{A^-\} \subseteq \{PN^-P^*\}$ and $\{A^-\} \supseteq \{PN^-P^*\}$ hold. Thus, the set equality in (2.14) holds. \square

It can be seen from (2.15) that the set equality in (2.14) is characterized by a rank additivity condition and a Hermitian g -inverse of the bordered matrix consisting of N and P . It is obvious that the rank additivity condition is easy to satisfy, for example,

- (i) If $\begin{bmatrix} N & P^* \\ P & 0 \end{bmatrix}$ is nonsingular, then the first rank equality in (2.15) holds.
- (ii) If N is nonnegative definite, then the first rank equality in (2.15) holds.

In particular, we have the following result.

COROLLARY 2.6. *Let $N \in \mathbb{C}_H^n$, and assume that $P \in \mathbb{C}^{n \times n}$ is nonsingular. Then, the set equality $\{(PNP^*)^-\} = \{(P^{-1})^*N^-P^{-1}\}$ holds.*

COROLLARY 2.7. *Let $A, B \in \mathbb{C}_H^m$. Then, $\{A^-\} = \{B^-\}$ if and only if $A = B$.*

3. The inertia of $A^- - PN^-P^*$ and the inequalities $A^- > (\geq, <, \leq) PN^-P^*$.

In this section, we establish a group of formulas for calculating the extremum inertias of the difference $A^- - PN^-P^*$, and then use the formulas to characterize the following four inequalities

$$(3.1) \quad A^- > PN^-P^*, \quad A^- \geq PN^-P^*, \quad A^- < PN^-P^*, \quad A^- \leq PN^-P^*,$$

where $A \in \mathbb{C}_H^m$, $N \in \mathbb{C}_H^n$ and $P \in \mathbb{C}^{m \times n}$ are given.

THEOREM 3.1. *Let $A \in \mathbb{C}_H^m$, $N \in \mathbb{C}_H^n$ and $P \in \mathbb{C}^{m \times n}$. Then,*

$$(3.2) \quad \max_{A^-, N^-} i_{\pm}(A^- - PN^-P^*) = m + i_{\mp}(P^*AP - N) - i_{\mp}(A) - i_{\pm}(N),$$

$$(3.3) \quad \min_{A^-, N^-} i_{\pm}(A^- - PN^-P^*) = i_{\mp}(P^*AP - N) + i_{\pm}(A) \\ + i_{\mp}(N) - r[P^*AP, N],$$

$$(3.4) \quad \max_{N^-} \min_{A^-} i_{\pm}(A^- - PN^-P^*) = i_{\mp}(P^*AP - N) + i_{\pm}(A) - i_{\pm}(N),$$

$$(3.5) \quad \min_{N^-} \max_{A^-} i_{\pm}(A^- - PN^-P^*) = m + i_{\mp}(P^*AP - N) + i_{\mp}(N) \\ - r[P^*AP, N] - i_{\mp}(A).$$

As a consequence,

(a) *There exist A^- and N^- such that $A^- > PN^-P^*$ ($A^- < PN^-P^*$) if and only if*

$$(3.6) \quad i_{-}(P^*AP - N) = i_{-}(A) + i_{+}(N) \quad (i_{+}(P^*AP - N) = i_{+}(A) + i_{-}(N)).$$

(b) *$A^- > PN^-P^*$ ($A^- < PN^-P^*$) for all A^- and N^- if and only if*

$$(3.7) \quad i_{-}(P^*AP - N) = r[P^*AP, N] - i_{+}(A) - i_{-}(N) + m$$

$$(3.8) \quad (i_{+}(P^*AP - N) = r[P^*AP, N] - i_{-}(A) - i_{+}(N) + m).$$

(c) *There exist A^- and N^- such that $A^- \geq PN^-P^*$ ($A^- \leq PN^-P^*$) if and only if*

$$(3.9) \quad i_{+}(P^*AP - N) = r[P^*AP, N] - i_{-}(A) - i_{+}(N)$$

$$(3.10) \quad (i_{-}(P^*AP - N) = r[P^*AP, N] - i_{+}(A) - i_{-}(N)).$$

(d) $A^- \geq PN^-P^*$ ($A^- \leq PN^-P^*$) for all A^- and N^- if and only if

$$(3.11) \quad i_+(P^*AP - N) = i_+(A) + i_-(N) - m$$

$$(3.12) \quad (i_-(P^*AP - N) = i_-(A) + i_+(N) - m).$$

(e) For any N^- , there exists an A^- such that $A^- > PN^-P^*$ ($A^- < PN^-P^*$) if and only if

$$(3.13) \quad i_-(P^*AP - N) = r[P^*AP, N] + i_-(A) - i_-(N)$$

$$(3.14) \quad (i_+(P^*AP - N) = r[P^*AP, N] + i_+(A) - i_+(N)).$$

(f) For any N^- , there exists an A^- such that $A^- \geq PN^-P^*$ ($A^- \leq PN^-P^*$) if and only if

$$(3.15) \quad i_-(N - P^*AP) = i_-(N) - i_-(A) \quad (i_+(N - P^*AP) = i_+(N) - i_+(A)).$$

Proof. Note from Lemma 1.2 that there exist A^- and PN^-P^* such that $A^- > PN^-P^*$ ($A^- < PN^-P^*$) if and only if

$$(3.16) \quad \max_{A^-, N^-} i_+(A^- - PN^-P^*) = m \quad \left(\max_{A^-, N^-} i_-(A^- - PN^-P^*) = m \right);$$

$A^- > PN^-P^*$ ($A^- < PN^-P^*$) for all A^- and N^- if and only if

$$(3.17) \quad \min_{A^-, N^-} i_+(A^- - PN^-P^*) = m \quad \left(\min_{A^-, N^-} i_-(A^- - PN^-P^*) = m \right);$$

there exist A^- and PN^-P^* such that $A^- \geq PN^-P^*$ ($A^- \leq PN^-P^*$) if and only if

$$(3.18) \quad \min_{A^-, N^-} i_-(A^- - PN^-P^*) = 0 \quad \left(\min_{A^-, N^-} i_+(A^- - PN^-P^*) = 0 \right);$$

$A^- \geq PN^-P^*$ ($A^- \leq PN^-P^*$) for all A^- and N^- if and only if

$$(3.19) \quad \max_{A^-, N^-} i_-(A^- - PN^-P^*) = 0 \quad \left(\max_{A^-, N^-} i_+(A^- - PN^-P^*) = 0 \right);$$

for any N^- , there exists an A^- such that $A^- > PN^-P^*$ ($A^- < PN^-P^*$) if and only if

$$(3.20) \quad \min_{N^-} \max_{A^-} i_+(A^- - PN^-P^*) = m \quad \left(\min_{N^-} \max_{A^-} i_-(A^- - PN^-P^*) = m \right);$$

for any N^- , there exists an A^- such that $A^- \geq PN^-P^*$ ($A^- \leq PN^-P^*$) if and only if

$$(3.21) \quad \max_{N^-} \min_{A^-} i_-(A^- - PN^-P^*) = 0 \quad \left(\max_{N^-} \min_{A^-} i_+(A^- - PN^-P^*) = 0 \right).$$

Applying (1.29) and (1.30) gives

$$(3.22) \quad \max_{A^-} i_{\pm}(A^- - PN^-P^*) = i_{\mp}(M) - i_{\mp}(A),$$

$$(3.23) \quad \min_{A^-} i_{\pm}(A^- - PN^-P^*) = i_{\mp}(M) + i_{\pm}(A) - m,$$

where

$$(3.24) \quad M = \begin{bmatrix} A & I_m \\ I_m & PN^-P^* \end{bmatrix} = \begin{bmatrix} A & I_m \\ I_m & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ P \end{bmatrix} N^- [0, P^*].$$

Applying (1.25) and (1.26) to the M and simplifying by elementary matrix operations and congruence matrix operations, and (1.15), we obtain

$$(3.25) \quad \begin{aligned} \max_{N^-} i_{\pm}(M) &= \max_{N^-} i_{\pm} \left(\begin{bmatrix} A & I_m \\ I_m & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ P \end{bmatrix} N^- [0, P^*] \right) \\ &= i_{\pm} \begin{bmatrix} -N & 0 & P^* \\ 0 & A & I_m \\ P & I_m & 0 \end{bmatrix} - i_{\mp}(N) \\ &= i_{\pm} \begin{bmatrix} P^*AP - N & 0 & 0 \\ 0 & 0 & I_m \\ 0 & I_m & 0 \end{bmatrix} - i_{\mp}(N) \\ &= m + i_{\pm}(P^*AP - N) - i_{\mp}(N), \end{aligned}$$

$$(3.26) \quad \begin{aligned} \min_{N^-} i_{\pm}(M) &= \min_{N^-} i_{\pm} \left(\begin{bmatrix} A & I_m \\ I_m & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ P \end{bmatrix} N^- [0, P^*] \right) \\ &= i_{\pm}(N) + i_{\pm} \begin{bmatrix} -N & 0 & P^* \\ 0 & A & I_m \\ P & I_m & 0 \end{bmatrix} + r \begin{bmatrix} 0 & A & I_m \\ P & I_m & 0 \end{bmatrix} \\ &\quad - r \begin{bmatrix} -N & 0 & 0 & P^* \\ 0 & 0 & A & I_m \\ 0 & P & I_m & 0 \end{bmatrix} \\ &= m + i_{\pm}(P^*AP - N) + i_{\pm}(N) - r[P^*AP, N]. \end{aligned}$$

Substituting (3.25) and (3.26) into (3.22) and (3.23) gives (3.2)–(3.5). Setting the right-hand sides of (3.2)–(3.5) to zero leads to the results in (a)–(f). \square

COROLLARY 3.2. *Let $0 \leq A \in \mathbb{C}_H^m$, $0 \leq N \in \mathbb{C}_H^n$ and $P \in \mathbb{C}^{m \times n}$. Then,*

(a) *There exist A^- and N^- such that $A^- > PN^-P^*$ ($A^- < PN^-P^*$) if and only if*

$$(3.27) \quad i_-(P^*AP - N) = r(N) \quad (i_+(P^*AP - N) = r(A)).$$

(b) *$A^- > PN^-P^*$ ($A^- < PN^-P^*$) for all A^- and N^- if and only if*

$$(3.28) \quad i_-(P^*AP - N) = r[P^*A, N] - r(A) + m$$

$$(3.29) \quad (i_+(P^*AP - N) = r[P^*A, N] - r(N) + m).$$

(c) *There exist A^- and N^- such that $A^- \geq PN^-P^*$ ($A^- \leq PN^-P^*$) if and only if*

$$(3.30) \quad i_+(P^*AP - N) = r[P^*A, N] - r(N)$$

$$(3.31) \quad (i_-(P^*AP - N) = r[P^*A, N] - r(A)).$$

(d) *$A^- \geq PN^-P^*$ ($A^- \leq PN^-P^*$) for all A^- and N^- if and only if*

$$(3.32) \quad i_+(P^*AP - N) = r(A) - m \quad (i_-(P^*AP - N) = r(N) - m).$$

(e) *For any N^- , there exists an A^- such that $A^- > PN^-P^*$ ($A^- < PN^-P^*$) if and only if*

$$(3.33) \quad N \geq P^*AP \text{ and } r(P^*AP - N) = r[P^*AP, N]$$

$$(3.34) \quad (i_+(P^*AP - N) = r[P^*AP, N] + r(A) - r(N)).$$

(f) *For any N^- , there exists an A^- such that $A^- \geq PN^-P^*$ ($A^- \leq PN^-P^*$) if and only if*

$$(3.35) \quad N \geq P^*AP \quad (i_+(N - P^*AP) = r(N) - r(A)).$$

Setting $N = P^*AP$ in Theorem 3.1 leads to the following consequence.

COROLLARY 3.3. *Let $A \in \mathbb{C}_H^m$ and $P \in \mathbb{C}^{m \times n}$. Then,*

(a) *The following statements are equivalent:*

(i) *There exist A^- and $(P^*AP)^-$ such that*

$$A^- > P(P^*AP)^-P^* \quad (A^- < P(P^*AP)^-P^*).$$

(ii) For any $(P^*AP)^-$, there exists an A^- such that

$$A^- > P(P^*AP)^-P^* \quad (A^- < P(P^*AP)^-P^*).$$

(iii) $A \geq 0$ and $AP = 0$ ($A \leq 0$ and $AP = 0$).

(b) The following statements are equivalent:

(i) There exist A^- and $(P^*AP)^-$ such that

$$A^- \geq P(P^*AP)^-P^* \quad (A^- \leq P(P^*AP)^-P^*).$$

(ii) For any $(P^*AP)^-$, there exists an A^- such that

$$A^- \geq P(P^*AP)^-P^* \quad (A^- \leq P(P^*AP)^-P^*).$$

(iii) $i_-(P^*AP) = i_-(A)$ ($i_+(P^*AP) = i_+(A)$).

(c) $A^- > P(P^*AP)^-P^*$ ($A^- < P(P^*AP)^-P^*$) for all A^- and $(P^*AP)^-$ if and only if $A > 0$ and $P = 0$ ($A < 0$ and $P = 0$).

(d) $A^- \geq P(P^*AP)^-P^*$ ($A^- \leq P(P^*AP)^-P^*$) for all A^- and $(P^*AP)^-$ if and only if $i_+(A) + i_-(P^*AP) = m$ ($i_-(A) + i_+(P^*AP) = m$).

The partial inertia of difference of two Hermitian generalized inverses of Hermitian matrices of the same size, as well as the Löwner partial ordering of Hermitian generalized inverses of two Hermitian matrices were studied by some authors; see, e.g., [2, 3, 4, 11, 12, 13, 18, 24, 36, 37]. Setting $P = I_m$ in Theorem 3.1, we obtain the following results on the partial inertia of $A^- - B^-$ and their consequences.

COROLLARY 3.4. Let $A, B \in \mathbb{C}_H^m$. Then,

$$(3.36) \quad \max_{A^-, B^-} i_{\pm}(A^- - B^-) = m + i_{\mp}(A - B) - i_{\mp}(A) - i_{\pm}(B),$$

$$(3.37) \quad \min_{A^-, B^-} i_{\pm}(A^- - B^-) = i_{\mp}(A - B) + i_{\pm}(A) + i_{\mp}(B) - r[A, B],$$

$$(3.38) \quad \max_{B^-} \min_{A^-} i_{\pm}(A^- - B^-) = i_{\mp}(A - B) + i_{\pm}(A) - i_{\pm}(B),$$

$$(3.39) \quad \min_{B^-} \max_{A^-} i_{\pm}(A^- - B^-) = m + i_{\mp}(A - B) + i_{\mp}(B) - r[A, B] - i_{\mp}(A).$$

As a consequence,

(a) There exist A^- and B^- such that $A^- > B^-$ ($A^- < B^-$) if and only if $i_-(A - B) = i_-(A) + i_+(B)$ ($i_+(A - B) = i_+(A) + i_-(B)$).

(b) $A^- > B^-$ ($A^- < B^-$) for all A^- and B^- if and only if $i_-(A - B) = r[A, B] - i_+(A) - i_-(B) + m$ ($i_+(A - B) = r[A, B] - i_-(A) - i_+(B) + m$).

(c) *There exist A^- and B^- such that $A^- \geq B^-$ ($A^- \leq B^-$) if and only if $i_+(A - B) = r[A, B] - i_-(A) - i_+(B)$ ($i_-(A - B) = r[A, B] - i_+(A) - i_-(B)$).*

(d) *$A^- \geq B^-$ ($A^- \leq B^-$) for all A^- and B^- if and only if $i_+(A - B) = i_+(A) + i_-(B) - m$ ($i_-(A - B) = i_-(A) + i_+(B) - m$).*

(e) *For any B^- , there exists an A^- such that $A^- > B^-$ ($A^- < B^-$) if and only if $i_-(A - B) = r[A, B] + i_-(A) - i_-(B)$ ($i_+(A - B) = r[A, B] + i_+(A) - i_+(B)$).*

(f) *For any B^- , there exists an A^- such that $A^- \geq B^-$ ($A^- \leq B^-$) if and only if $i_-(B - A) = i_-(B) - i_-(A)$ ($i_+(B - A) = i_+(B) - i_+(A)$).*

Under the conditions that $A \geq 0$ and $B \geq 0$,

(g) *There exist A^- and B^- such that $A^- > B^-$ ($A^- < B^-$) if and only if $i_-(A - B) = r(B)$ ($i_+(A - B) = r(A)$).*

(h) *$A^- > B^-$ ($A^- < B^-$) for all A^- and B^- if and only if*

$$i_-(A - B) = r[A, B] - r(A) + m \quad (i_+(A - B) = r[A, B] - r(B) + m).$$

(i) *There exist A^- and B^- such that $A^- \geq B^-$ ($A^- \leq B^-$) if and only if $i_+(A - B) = r[A, B] - r(B)$ ($i_-(A - B) = r[A, B] - r(A)$).*

(j) *$A^- \geq B^-$ ($A^- \leq B^-$) for all A^- and B^- if and only if $A \leq B$ ($A \geq B$).*

(k) *For any B^- , there exists an A^- such that $A^- > B^-$ ($A^- < B^-$) if and only if $i_-(A - B) = r[A, B]$ ($i_+(A - B) = r[A, B] + r(A) - r(B)$).*

(l) *For any B^- , there exists an A^- such that $A^- \geq B^-$ ($A^- \leq B^-$) if and only if $B \geq A$ ($i_+(B - A) = r(B) - r(A)$).*

4. The rank and inertia of $A^\dagger - PN^\dagger P^*$, the equality $A^\dagger = PN^\dagger P^*$ and the inequalities $A^\dagger > (\geq, < \leq) PN^\dagger P^*$. For the special cases of (2.1) and (3.1) corresponding to the Moore-Penrose inverses, we have the following several results.

THEOREM 4.1. *Let $A \in \mathbb{C}_H^m$, $P \in \mathbb{C}^{m \times n}$ and $N \in \mathbb{C}_H^n$, and let*

$$(4.1) \quad M = \begin{bmatrix} A^3 & 0 & A \\ 0 & -N^3 & NP^* \\ A & PN & 0 \end{bmatrix}.$$

Then, the following expansion formulas

$$(4.2) \quad i_\pm(A^\dagger - PN^\dagger P^*) = i_\mp(M) - i_\mp(A) - i_\pm(N),$$

$$(4.3) \quad r(A^\dagger - PN^\dagger P^*) = r(M) - r(A) - r(N)$$

hold. As a consequence,

- (a) $A^\dagger > PN^\dagger P^*$ if and only if $i_-(M) = i_-(A) + i_+(N) + m$.
- (b) $A^\dagger \geq PN^\dagger P^*$ if and only if $i_+(M) = i_+(A) + i_-(N)$.
- (c) $A^\dagger < PN^\dagger P^*$ if and only if $i_+(M) = i_+(A) + i_-(N) + m$.
- (d) $A^\dagger \leq PN^\dagger P^*$ if and only if $i_-(M) = i_-(A) + i_+(N)$.
- (e) $A^\dagger = PN^\dagger P^*$ if and only if $r(M) = r(A) + r(N)$.

Proof. Note that $\mathcal{R}(A) = \mathcal{R}(A^3)$ and $\mathcal{R}(N) = \mathcal{R}(N^3)$. Then, applying (1.16) and (1.18) to (4.1) and simplifying by (1.4)–(1.8), we obtain

$$\begin{aligned} i_\pm(M) &= i_\pm \begin{bmatrix} A^3 & 0 & A \\ 0 & -N^3 & NP^* \\ A & PN & 0 \end{bmatrix} = i_\pm \begin{bmatrix} A^3 & 0 & 0 \\ 0 & -N^3 & 0 \\ 0 & 0 & -A(A^3)^\dagger A + PN(N^3)^\dagger NP^* \end{bmatrix} \\ &= i_\pm(A) + i_\mp(N) + i_\mp(A^\dagger - PN^\dagger P^*), \end{aligned}$$

establishing (4.2) and (4.3). Setting the right-hand sides of (4.2) and (4.3) equal to m or zero, respectively, leads to (a)–(e). \square

Setting $N = P^*AP$ in (4.1) and applying congruence operations and (1.9), we obtain

$$\begin{aligned} i_\pm(M) &= i_\pm \begin{bmatrix} A^3 & 0 & A \\ 0 & -P^*APP^*APP^*AP & P^*APP^* \\ A & PP^*AP & 0 \end{bmatrix} \\ &= i_\pm \begin{bmatrix} 0 & 0 & A \\ 0 & 0 & P^*APP^* \\ A & PP^*AP & 0 \end{bmatrix} = r[A, PP^*AP]. \end{aligned}$$

Thus, Theorem 4.1 reduces to the following result.

COROLLARY 4.2 ([32]). *Let $A \in \mathbb{C}_H^m$ and $P \in \mathbb{C}^{m \times n}$. Then,*

(a) *The following expansion formulas*

$$(4.4) \quad i_\pm[A^\dagger - P(P^*AP)^\dagger P^*] = r[A, PP^*AP] - i_\mp(A) - i_\pm(P^*AP),$$

$$(4.5) \quad r[A^\dagger - P(P^*AP)^\dagger P^*] = 2r[A, PP^*AP] - r(A) - r(P^*AP)$$

hold. As a consequence,

- (i) $A^\dagger > P(P^*AP)^\dagger P^*$ if and only if $A > 0$ and $P = 0$.
- (ii) $A^\dagger \geq P(P^*AP)^\dagger P^*$ if and only if $r[A, PP^*AP] = i_+(A) + i_-(P^*AP)$.
- (iii) $A^\dagger < P(P^*AP)^\dagger P^*$ if and only if $A < 0$ and $P = 0$.
- (iv) $A^\dagger \leq P(P^*AP)^\dagger P^*$ if and only if $r[A, PP^*AP] = i_-(A) + i_+(P^*AP)$.
- (v) $A^\dagger = P(P^*AP)^\dagger P^*$ if and only if $\mathcal{R}(A) = \mathcal{R}(PP^*AP)$.

(b) *Under the condition $A \geq 0$, the following expansion formulas*

$$(4.6) \quad i_+[A^\dagger - P(P^*AP)^\dagger P^*] = r[A, PP^*A] - r(AP),$$

$$(4.7) \quad i_-[A^\dagger - P(P^*AP)^\dagger P^*] = r[A, PP^*A] - r(A),$$

$$(4.8) \quad r[A^\dagger - P(P^*AP)^\dagger P^*] = 2r[A, PP^*A] - r(A) - r(AP)$$

hold. As a consequence,

$$(i) \quad A^\dagger \geq P(P^*AP)^\dagger P^* \Leftrightarrow \mathcal{R}(PP^*A) \subseteq \mathcal{R}(A).$$

$$(ii) \quad A^\dagger \leq P(P^*AP)^\dagger P^* \Leftrightarrow A^\dagger = P(P^*AP)^\dagger P^* \Leftrightarrow \mathcal{R}(PP^*A) = \mathcal{R}(A).$$

THEOREM 4.3. Let $A, B \in \mathbb{C}_H^m$. Then,

(a) The following equalities hold

$$(4.9) \quad i_\pm(A^\dagger - B^\dagger) = i_\pm(A) - i_\pm(B) + i_\pm \begin{bmatrix} 0 & E_A B \\ BE_A & B^3 - BAB \end{bmatrix},$$

$$(4.10) \quad i_\pm(A^\dagger - B^\dagger) = r(A^\dagger - B^\dagger) - i_\mp(A) + i_\mp(B) - i_\mp \begin{bmatrix} 0 & E_A B \\ BE_A & B^3 - BAB \end{bmatrix},$$

$$(4.11) \quad r(A^\dagger - B^\dagger) = r(A) - r(B) + r \begin{bmatrix} 0 & E_A B \\ BE_A & B^3 - BAB \end{bmatrix}.$$

(b) The following inequalities hold

$$(4.12) \quad \max\{s_1, s_2\} \leq i_\pm(A^\dagger - B^\dagger) \leq \min\{s_3, s_4\},$$

$$(4.13) \quad \max\{t_1, t_2\} \leq r(A^\dagger - B^\dagger) \leq \min\{t_3, t_4\},$$

where

$$s_1 = r[A, B] - i_\mp(A) - i_\pm(B),$$

$$s_2 = i_\pm(B^3 - BAB) + i_\pm(A) - i_\pm(B),$$

$$s_3 = r(A^\dagger - B^\dagger) + i_\mp(A) + i_\mp(B) - r[A, B],$$

$$s_4 = i_\pm(B^3 - BAB) + r[A, B] - i_\mp(A) - i_\pm(B),$$

$$t_1 = 2r[A, B] - r(A) - r(B),$$

$$t_2 = r(B^3 - BAB) + r(A) - r(B),$$

$$t_3 = 2r(A^\dagger - B^\dagger) + r(A) + r(B) - 2r[A, B],$$

$$t_4 = r(B^3 - BAB) + 2r[A, B] - r(A) - r(B).$$

In particular,

(c) If $\mathcal{R}(B) \subseteq \mathcal{R}(A)$, then

$$(4.14) \quad i_\pm(A^\dagger - B^\dagger) = i_\mp(BAB - B^3) + i_\pm(A) - i_\pm(B),$$

$$(4.15) \quad r(A^\dagger - B^\dagger) = r(BAB - B^3) + r(A) - r(B).$$

(d) If $BAB = B^3$, then

$$\begin{aligned} i_{\pm}(A^{\dagger} - B^{\dagger}) &= r[A, B] - i_{\mp}(A) - i_{\pm}(B), \\ r(A^{\dagger} - B^{\dagger}) &= 2r[A, B] - r(A) - r(B). \end{aligned}$$

(e) [24] If $\mathcal{R}(A) = \mathcal{R}(B)$, then

$$(4.16) \quad i_{\pm}(A^{\dagger} - B^{\dagger}) = i_{\mp}(A - B) + i_{\pm}(A) - i_{\pm}(B),$$

$$(4.17) \quad r(A^{\dagger} - B^{\dagger}) = r(A - B) + r(A) - r(B).$$

Therefore, $A^{\dagger} \geq B^{\dagger}$ ($A^{\dagger} \leq B^{\dagger}$) if and only if

$$i_{-}(B - A) = i_{-}(B) - i_{-}(A) \quad (i_{+}(B - A) = i_{+}(B) - i_{+}(A)).$$

(f) $i_{+}(A^{\dagger} - B^{\dagger}) = i_{+}(A) - i_{+}(B)$ if and only if $\mathcal{R}(B) \subseteq \mathcal{R}(A)$ and $B^3 - BAB \leq 0$.

(g) $i_{-}(A^{\dagger} - B^{\dagger}) = i_{-}(A) - i_{-}(B)$ if and only if $\mathcal{R}(B) \subseteq \mathcal{R}(A)$ and $B^3 - BAB \geq 0$.

(h) Both $i_{+}(A^{\dagger} - B^{\dagger}) = i_{+}(A) - i_{+}(B)$ and $i_{-}(A^{\dagger} - B^{\dagger}) = i_{-}(A) - i_{-}(B) \Leftrightarrow r(A^{\dagger} - B^{\dagger}) = r(A) - r(B) \Leftrightarrow \mathcal{R}(B) \subseteq \mathcal{R}(A)$ and $BAB = B^3$.

(i) If $A \geq 0$ and $B \geq 0$, then

$$(4.18) \quad i_{-}(A^{\dagger} - B^{\dagger}) = i_{-} \begin{bmatrix} 0 & E_A B \\ BE_A & B^3 - BAB \end{bmatrix}.$$

Therefore, $A^{\dagger} \geq B^{\dagger}$ if and only if $\mathcal{R}(B) \subseteq \mathcal{R}(A)$ and $B^3 - BAB \geq 0$.

(j) If $A \geq B \geq 0$, then

$$(4.19) \quad i_{+}(A^{\dagger} - B^{\dagger}) = r(A) - r(B),$$

$$(4.20) \quad i_{-}(A^{\dagger} - B^{\dagger}) = r(B^3 - BAB),$$

$$(4.21) \quad r(A^{\dagger} - B^{\dagger}) = r(B^3 - BAB) + r(A) - r(B).$$

Therefore, $A^{\dagger} \geq B^{\dagger}$ if and only if $B^3 = BAB$.

(k) [27] If A and B are nonsingular, then

$$(4.22) \quad i_{\pm}(A^{-1} - B^{-1}) = i_{\mp}(A - B) + i_{\pm}(A) - i_{\pm}(B).$$

Therefore, $A^{-1} \geq B^{-1}$ ($A^{-1} \leq B^{-1}$) if and only if

$$i_{-}(B - A) = i_{-}(B) - i_{-}(A) \quad (i_{+}(B - A) = i_{+}(B) - i_{+}(A)).$$

(l) $i_{+}(A^{-1} - B^{-1}) = i_{+}(A) - i_{+}(B)$ if and only if $B - A \leq 0$ ($i_{-}(A^{-1} - B^{-1}) = i_{-}(A) - i_{-}(B)$ if and only if $B - A \geq 0$).

(m) [18, 36] Under the condition $A \geq 0$ and $B \geq 0$, any two of the following conditions:

- (i) $A \geq B \geq 0$,
- (ii) $B^\dagger \geq A^\dagger \geq 0$,
- (iii) $r(A) = r(B)$

imply the third condition.

Proof. Applying (1.15) to the matrix $\begin{bmatrix} A^\dagger & B^\dagger \\ B^\dagger & B^\dagger \end{bmatrix}$ and simplifying by (1.6) and (1.8), we obtain the following equalities

$$(4.23) \quad i_\pm \begin{bmatrix} A^\dagger & B^\dagger \\ B^\dagger & B^\dagger \end{bmatrix} = i_\pm(A) + i_\pm \begin{bmatrix} 0 & E_A B^\dagger \\ B^\dagger E_A & B^\dagger - B^\dagger A B^\dagger \end{bmatrix},$$

$$(4.24) \quad i_\pm \begin{bmatrix} A^\dagger & B^\dagger \\ B^\dagger & B^\dagger \end{bmatrix} = i_\pm \begin{bmatrix} A^\dagger - B^\dagger & 0 \\ 0 & B^\dagger \end{bmatrix} = i_\pm(A^\dagger - B^\dagger) + i_\pm(B).$$

It is easy to see from Lemma 1.4(b) that

$$(4.25) \quad i_\pm \begin{bmatrix} 0 & E_A B^\dagger \\ B^\dagger E_A & B^\dagger - B^\dagger A B^\dagger \end{bmatrix} = i_\pm \begin{bmatrix} 0 & E_A B \\ B E_A & B^3 - B A B \end{bmatrix},$$

$$(4.26) \quad r \begin{bmatrix} 0 & E_A B^\dagger \\ B^\dagger E_A & B^\dagger - B^\dagger A B^\dagger \end{bmatrix} = r \begin{bmatrix} 0 & E_A B \\ B E_A & B^3 - B A B \end{bmatrix}.$$

Substituting (4.25) and (4.26) into (4.23) and (4.24) yields (4.9)–(4.11). Results (c)–(m) follow from (4.9)–(4.11) and Lemma 1.2, and the details are omitted. \square

Assume that $A, B \in \mathbb{C}_H^m$ are given such that $A + B$ is nonsingular. Then it can be derived from (4.14) and (4.15) that

$$\begin{aligned} i_\pm[(A + B)^{-1} - A^\dagger] &= i_\pm(A + B) - i_\pm(A) + i_\mp(ABA), \\ r[(A + B)^{-1} - A^\dagger] &= m - r(A) + r(ABA). \end{aligned}$$

5. The rank of $C^- - A^- - B^-$ and the additive decomposition $C^- = A^- + B^-$. It is well known that the parallel sum of a pair of Hermitian nonnegative definite matrices A and B of the same size is defined to be

$$A : B \stackrel{\text{def}}{=} A(A + B)^\dagger B;$$

see [1]. Later, the parallel sum was also extended to any pair of matrices A and B of the same size as

$$p(A, B) \stackrel{\text{def}}{=} A(A + B)^- B$$

whenever this product is invariant with respect to the choice of $(A + B)^-$. Parallel sums of matrices and various related topics were widely investigated; see, e.g., [17, 19,

20, 21, 22, 23, 30, 35]. One of the well-known properties of the parallel sum $p(A, B)$ is

$$\{p^-(A, B)\} = \{A^- + B^-\};$$

see [23]. In a recent paper [35], the matrix equality $C^- = A^- + B^-$ was studied and a variety of results on parallel sum of matrices were derived through some expansion formulas for ranks of matrices. As a continuation, we consider in this section the matrix equality

$$(5.1) \quad C^- = A^- + B^-$$

for Hermitian generalized inverses of Hermitian matrices A , B and C .

LEMMA 5.1. *The set equality*

$$(5.2) \quad \{A^- + B^-\} = \{PN^-P^*\}$$

holds for any $A, B \in \mathbb{C}_H^m$, where $P = [I_m, I_m]$ and $N = \text{diag}\{A, B\}$.

Proof. From (1.19), the general expression of $A^- + B^-$ can be written as

$$(5.3) \quad A^- + B^- = A^\dagger + B^\dagger + F_A V + V^* F_A + F_B W + W^* F_B,$$

where V and W are arbitrary, while the general expression of N^- is

$$\begin{aligned} N^- &= N^\dagger + F_N S + S^* F_N \\ &= \begin{bmatrix} A^\dagger & 0 \\ 0 & B^\dagger \end{bmatrix} + \begin{bmatrix} F_A & 0 \\ 0 & F_B \end{bmatrix} \begin{bmatrix} S_1 & S_2 \\ S_3 & S_4 \end{bmatrix} + \begin{bmatrix} S_1^* & S_3^* \\ S_2^* & S_4^* \end{bmatrix} \begin{bmatrix} F_A & 0 \\ 0 & F_B \end{bmatrix} \\ &= \begin{bmatrix} A^\dagger + F_A S_1 + S_1^* F_A & F_A S_2 + S_3^* F_B \\ S_2^* F_A + F_B S_3 & B^\dagger + F_B S_4 + S_4^* F_B \end{bmatrix}, \end{aligned}$$

where S_1, \dots, S_4 are arbitrary. Hence, we obtain

$$PN^-P^* = A^\dagger + B^\dagger + F_A(S_1 + S_2) + (S_1^* + S_2^*)F_A + F_B(S_3 + S_4) + (S_3^* + S_4^*)F_B,$$

which is the same as (5.3). Thus, (5.2) holds. \square

Applying the results in Section 3 and Lemma 5.1 to $\{C^-\}$ and $\{PN^-P^*\}$ in (5.2) gives the following results. The proofs are omitted.

THEOREM 5.2. *Let $A, B, C \in \mathbb{C}_H^m$. Then,*

$$(5.4) \quad \max_{A^-, B^-} r[C - C(A^- + B^-)C] = \min \left\{ r(C), r \begin{bmatrix} C & A \\ B & A + B \end{bmatrix} + r(C) - r(A) - r(B) \right\},$$

$$(5.5) \quad \min_{A^-, B^-} r[C - C(A^- + B^-)C] = \min_{C^-, A^-, B^-} r(C^- - A^- - B^-) \\ = r(A) + r(B) + r(C) + r \begin{bmatrix} C & A \\ B & A + B \end{bmatrix} - 2r \begin{bmatrix} A & 0 & C \\ 0 & B & C \end{bmatrix}.$$

As a consequence,

(a) There exist A^- and B^- such that $A^- + B^- \in \{C^-\}$ if and only if

$$(5.6) \quad r \begin{bmatrix} C & A \\ B & A + B \end{bmatrix} = 2r \begin{bmatrix} A & 0 & C \\ 0 & B & C \end{bmatrix} - r(A) - r(B) - r(C).$$

(b) The following statements are equivalent:

(i) $\{A^- + B^-\} \subseteq \{C^-\}$.

(ii) $r \begin{bmatrix} C & A \\ B & A + B \end{bmatrix} = r(A) + r(B) - r(C)$.

(iii) $r \begin{bmatrix} A & 0 & C \\ 0 & B & C \\ C & C & C \end{bmatrix} = r \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$.

(iv) $\mathcal{R}(C) \subseteq \mathcal{R}(A)$, $\mathcal{R}(C) \subseteq \mathcal{R}(B)$ and $C = CA^-C + CB^-C$.

THEOREM 5.3. Let $A, B, C \in \mathbb{C}_H^m$. Then, $\{C^-\} \subseteq \{A^- + B^-\}$ if and only if

$$(5.7) \quad \mathcal{R}(A) \cap \mathcal{R}(B) = \{0\} \text{ or } r \begin{bmatrix} C & A \\ B & A + B \end{bmatrix} = 2r[A, B] + r(C) - r(A) - r(B).$$

A special case of Theorem 5.3 is

$$(5.8) \quad \mathcal{R}(A) \cap \mathcal{R}(B) = \{0\} \Leftrightarrow \{A^- + B^-\} = \{0^-\} = \mathbb{C}_H^m.$$

In what follows, we assume that

$$(5.9) \quad \mathcal{R}(A) \cap \mathcal{R}(B) \neq \{0\}.$$

In this case, combining Theorems 5.2 and 5.3 yields the following result.

THEOREM 5.4. Let $A, B, C \in \mathbb{C}_H^m$. Then,

$$(5.10) \quad \{A^- + B^-\} = \{C^-\}$$

if and only if

$$(5.11) \quad \mathcal{R}(A) \subseteq \mathcal{R}(A + B), \quad \mathcal{R}(B) \subseteq \mathcal{R}(A + B) \text{ and } C = A(A + B)^-B.$$

In this case,

$$(5.12) \quad r(C) = r(A) + r(B) - r(A + B) \quad \text{and} \quad \mathcal{R}(C) = \mathcal{R}(A) \cap \mathcal{R}(B).$$

Theorems 5.2, 5.3 and 5.4 can be used to derive various additive decompositions of Hermitian g -inverses of matrices. For example, if $A + B$ is nonsingular and $C = A(A + B)^{-1}B$, then (5.10) holds. Setting $C = A + B$ in Theorems 5.2, 5.3 and 5.4 gives the following consequences.

COROLLARY 5.5. Let $A, B \in \mathbb{C}_H^m$. Then,

(a) There exist A^- and B^- such that $A^- + B^- \in \{(A + B)^-\}$ if and only if

$$r \begin{bmatrix} A + B & A \\ B & A + B \end{bmatrix} = 2r \begin{bmatrix} A & 0 & B \\ 0 & B & A \end{bmatrix} - r(A + B) - r(A) - r(B).$$

(b) Under $A + B \neq 0$, the following statements are equivalent:

(i) $\{(A + B)^-\} \supseteq \{A^- + B^-\}$.

(ii) $r \begin{bmatrix} A + B & A \\ B & A + B \end{bmatrix} = r(A) + r(B) - r(A + B)$.

(iii) $\mathcal{R}(A) = \mathcal{R}(B)$ and $A + B = -\frac{1}{2}(AB^-A + BA^-B)$.

(c) $\{(A + B)^-\} \subseteq \{A^- + B^-\}$ if and only if $\mathcal{R}(A) \cap \mathcal{R}(B) = \{0\}$ or

$$r \begin{bmatrix} A + B & A \\ B & A + B \end{bmatrix} = 2r[A, B] + r(A + B) - r(A) - r(B).$$

Theorem 5.4 can be used to derive some additive decompositions for Hermitian g -inverses of matrices. For example, applying Theorem 5.4 to $A - A^2$ and $I_m - A^2$ gives the following consequences.

COROLLARY 5.6. Let $A \in \mathbb{C}_H^m$. Then,

(a) The set equality $\{(A - A^2)^-\} = \{A^- + (I_m - A)^-\}$ always holds.

(b) The set equality $\{2(I_m - A^2)^-\} = \{(I_m + A)^- + (I_m - A)^-\}$ always holds.

6. The matrix inequalities $C^- > (\geq, <, \leq) A^- + B^-$. Applying Theorem 3.1 to C^- and PN^-P^* in (5.2) yields the following result.

THEOREM 6.1. Let $A, B, C \in \mathbb{C}_H^m$. Then,

(a) *There exist A^- , B^- and C^- such that $C^- > A^- + B^-$ ($C^- < A^- + B^-$) if and only if*

$$\begin{aligned} i_- \begin{bmatrix} C-A & C \\ C & C-B \end{bmatrix} &= i_+(A) + i_+(B) + i_-(C) \\ \left(i_+ \begin{bmatrix} C-A & C \\ C & C-B \end{bmatrix} \right) &= i_-(A) + i_-(B) + i_+(C). \end{aligned}$$

(b) *$C^- > A^- + B^-$ ($C^- < A^- + B^-$) for all A^- , B^- and C^- if and only if*

$$\begin{aligned} i_- \begin{bmatrix} C-A & C \\ C & C-B \end{bmatrix} &= r \begin{bmatrix} A & 0 & C \\ 0 & B & C \end{bmatrix} - i_-(A) - i_-(B) - i_+(C) + m \\ \left(i_+ \begin{bmatrix} C-A & C \\ C & C-B \end{bmatrix} \right) &= r \begin{bmatrix} A & 0 & C \\ 0 & B & C \end{bmatrix} - i_+(A) - i_+(B) - i_-(C) + m. \end{aligned}$$

(c) *There exist A^- , B^- and C^- such that $C^- \geq A^- + B^-$ ($C^- \leq A^- + B^-$) if and only if*

$$\begin{aligned} i_+ \begin{bmatrix} C-A & C \\ C & C-B \end{bmatrix} &= r \begin{bmatrix} A & 0 & C \\ 0 & B & C \end{bmatrix} - i_+(A) - i_+(B) - i_-(C) \\ \left(i_- \begin{bmatrix} C-A & C \\ C & C-B \end{bmatrix} \right) &= r \begin{bmatrix} A & 0 & C \\ 0 & B & C \end{bmatrix} - i_-(A) - i_-(B) - i_+(C). \end{aligned}$$

(d) *$C^- \geq A^- + B^-$ ($C^- \leq A^- + B^-$) for all A^- , B^- and C^- if and only if*

$$\begin{aligned} i_+ \begin{bmatrix} C-A & C \\ C & C-B \end{bmatrix} &= i_-(A) + i_-(B) + i_+(C) - m \\ \left(i_- \begin{bmatrix} C-A & C \\ C & C-B \end{bmatrix} \right) &= i_+(A) + i_+(B) + i_-(C) - m. \end{aligned}$$

(e) *For any A^- and B^- , there exists a C^- such that*

$$C^- > A^- + B^- \quad (C^- < A^- + B^-)$$

if and only if

$$\begin{aligned} i_- \begin{bmatrix} C-A & C \\ C & C-B \end{bmatrix} &= r \begin{bmatrix} A & 0 & C \\ 0 & B & C \end{bmatrix} + i_-(C) - i_-(A) - i_-(B) \\ \left(i_+ \begin{bmatrix} C-A & C \\ C & C-B \end{bmatrix} \right) &= r \begin{bmatrix} A & 0 & C \\ 0 & B & C \end{bmatrix} + i_+(C) - i_+(A) - i_+(B). \end{aligned}$$

(f) For any A^- and B^- , there exists a C^- such that

$$C^- \geq A^- + B^- \quad (C^- \leq A^- + B^-)$$

if and only if

$$\begin{aligned} i_+ \begin{bmatrix} C - A & C \\ C & C - B \end{bmatrix} &= i_-(A) + i_-(B) - i_-(C) \\ \left(i_- \begin{bmatrix} C - A & C \\ C & C - B \end{bmatrix} \right) &= i_+(A) + i_+(B) - i_+(C). \end{aligned}$$

COROLLARY 6.2. Let $A, B \in \mathbb{C}_H^m$. Then,

(a) There exist A^- , B^- and $(A + B)^-$ such that

$$(A + B)^- > A^- + B^- \quad ((A + B)^- < A^- + B^-)$$

if and only if

$$\begin{aligned} i_- \begin{bmatrix} A & A + B \\ A + B & B \end{bmatrix} &= i_+(A) + i_+(B) + i_-(A + B) \\ \left(i_+ \begin{bmatrix} A & A + B \\ A + B & B \end{bmatrix} \right) &= i_-(A) + i_-(B) + i_+(A + B). \end{aligned}$$

(b) $(A + B)^- > A^- + B^-$ ($(A + B)^- < A^- + B^-$) for all A^- , B^- and $(A + B)^-$ if and only if

$$\begin{aligned} i_- \begin{bmatrix} A & A + B \\ A + B & B \end{bmatrix} &= r \begin{bmatrix} A & 0 & B \\ 0 & B & A \end{bmatrix} - i_-(A) - i_-(B) - i_+(A + B) + m \\ \left(i_+ \begin{bmatrix} A & A + B \\ A + B & B \end{bmatrix} \right) &= r \begin{bmatrix} A & 0 & B \\ 0 & B & A \end{bmatrix} - i_+(A) - i_+(B) - i_-(A + B) + m. \end{aligned}$$

(c) There exist A^- , B^- and $(A + B)^-$ such that

$$(A + B)^- \geq A^- + B^- \quad ((A + B)^- \leq A^- + B^-)$$

if and only if

$$\begin{aligned} i_+ \begin{bmatrix} A & A + B \\ A + B & B \end{bmatrix} &= r \begin{bmatrix} A & 0 & B \\ 0 & B & A \end{bmatrix} - i_+(A) - i_+(B) - i_-(A + B) \\ \left(i_- \begin{bmatrix} A & A + B \\ A + B & B \end{bmatrix} \right) &= r \begin{bmatrix} A & 0 & B \\ 0 & B & A \end{bmatrix} - i_-(A) - i_-(B) - i_+(A + B). \end{aligned}$$

(d) $(A+B)^- \geq A^- + B^-$ ($(A+B)^- \leq A^- + B^-$) for all A^- , B^- and $(A+B)^-$ if and only if

$$\begin{aligned} i_+ \begin{bmatrix} A & A+B \\ A+B & B \end{bmatrix} &= i_-(A) + i_-(B) + i_+(A+B) - m \\ \left(i_- \begin{bmatrix} A & A+B \\ A+B & B \end{bmatrix} \right) &= i_+(A) + i_+(B) + i_-(A+B) - m. \end{aligned}$$

(e) For any A^- and B^- , there exists a $(A+B)^-$ such that

$$(A+B)^- > A^- + B^- \quad ((A+B)^- < A^- + B^-)$$

if and only if

$$\begin{aligned} i_- \begin{bmatrix} A & A+B \\ A+B & B \end{bmatrix} &= r \begin{bmatrix} A & 0 & B \\ 0 & B & A \end{bmatrix} + i_-(A+B) - i_-(A) - i_-(B) \\ \left(i_+ \begin{bmatrix} A & A+B \\ A+B & B \end{bmatrix} \right) &= r \begin{bmatrix} A & 0 & B \\ 0 & B & A \end{bmatrix} + i_+(A+B) - i_+(A) - i_+(B). \end{aligned}$$

(f) For any A^- and B^- , there exists a $(A+B)^-$ such that

$$(A+B)^- \geq A^- + B^- \quad ((A+B)^- \leq A^- + B^-)$$

if and only if

$$\begin{aligned} i_+ \begin{bmatrix} A & A+B \\ A+B & B \end{bmatrix} &= i_-(A) + i_-(B) - i_-(A+B) \\ \left(i_- \begin{bmatrix} A & A+B \\ A+B & B \end{bmatrix} \right) &= i_+(A) + i_+(B) - i_+(A+B). \end{aligned}$$

7. The additive matrix decomposition $A^- = A_1^- + \cdots + A_k^-$. The results in Sections 5 and 6 can easily be extended to the sum of k Hermitian g -inverses of Hermitian matrices. In fact, it is easy to verify that

$$(7.1) \quad \{A_1^- + \cdots + A_k^-\} = \{PN^-P^*\},$$

where $N = \text{diag}(A_1, \dots, A_k)$ and $P = [I_m, \dots, I_m]$. A useful formula for the dimension of intersection of ranges of k matrices is given below.

LEMMA 7.1 ([31]). Let $[A_1, \dots, A_k] \in \mathbb{C}^{m \times t}$, and N and P be as given in (7.1). Then, the dimension of intersection of the ranges of A_1, \dots, A_k is

$$(7.2) \quad \dim[\mathcal{R}(A_1) \cap \cdots \cap \mathcal{R}(A_k)] = r(N) + r(P^*) - r[N, P^*].$$

In particular,

$$(7.3) \quad \mathcal{R}(A_1) \cap \cdots \cap \mathcal{R}(A_k) = \{0\} \Leftrightarrow \mathcal{R}(N) \cap \mathcal{R}(P^*) = \{0\}.$$

Applying the results in Section 2 to $\{A^-\}$ and $\{PN^-P^*\}$ in (7.1) gives the following results. The proofs are omitted.

THEOREM 7.2. *Let $A_1, \dots, A_k, A \in \mathbb{C}_H^m$, and N and P be as given in (7.1). Then,*

$$(7.4) \quad \max_{A_1^-, \dots, A_k^-} r[A - A(A_1^- + \cdots + A_k^-)A] \\ = \min \{ r(A), \quad r(N - P^*AP) + r(A) - r(N) \},$$

$$(7.5) \quad \min_{A_1^-, \dots, A_k^-} r[A - A(A_1^- + \cdots + A_k^-)A] \\ = r(A) + r(N) + r(N - P^*AP) - 2r[N, P^*A],$$

$$(7.6) \quad \max_{A^-} \min_{A_1^-, \dots, A_k^-} r(A^- - A_1^- - \cdots - A_k^-) = \min \{ r(N) + m - r[N, P^*], \\ 2m + r(N - P^*AP) + r(N) - r(A) - 2r[N, P^*] \}.$$

As a consequence,

(a) $\{A_1^- + \cdots + A_k^-\} \cap \{A^-\} \neq \emptyset$ if and only if $r(N - P^*AP) = 2r[N, P^*A] - r(N) - r(A)$.

(b) The following statements are equivalent:

- (i) $\{A_1^- + \cdots + A_k^-\} \subseteq \{A^-\}$.
- (ii) $r(N - P^*AP) = r(N) - r(P^*AP)$.
- (iii) $r \begin{bmatrix} N & P^*A \\ AP & A \end{bmatrix} = r(N)$.
- (iv) $\mathcal{R}(A) \subseteq \mathcal{R}(A_i)$, $\mathcal{R}(A^*) \subseteq \mathcal{R}(A_i^*)$, $i = 1, \dots, k$, and $A = APN^-P^*A$.

(c) $\{A_1^- + \cdots + A_k^-\} \supseteq \{A^-\}$ if and only if

$$\mathcal{R}(A_1) \cap \cdots \cap \mathcal{R}(A_k) = \{0\} \quad \text{or} \quad r(N - P^*AP) = 2r[N, P^*] - r(N) + r(A) - 2m.$$

Theorem 7.2(c) implies the following special case.

COROLLARY 7.3. *Let $A_1, \dots, A_k \in \mathbb{C}_H^m$ be given. Then, $\{A_1^- + \cdots + A_k^-\} = \mathbb{C}_H^m$ if and only if $\mathcal{R}(A_1) \cap \cdots \cap \mathcal{R}(A_k) = \{0\}$.*

In what follows, we assume that $\bigcap_{i=1}^k \mathcal{R}(A_i) \neq \{0\}$.

THEOREM 7.4. *Let $A_1, \dots, A_k, A \in \mathbb{C}_H^m$ be given, and N and P be as given in*

(7.1). Then,

$$(7.7) \quad \{A_1^- + \cdots + A_k^-\} = \{A^-\}$$

if and only if

$$(7.8) \quad r \begin{bmatrix} N & P^* \\ P & 0 \end{bmatrix} = r[N, P^*] + r(P) \quad \text{and} \quad A = -[0, I_m] \begin{bmatrix} N & P^* \\ P & 0 \end{bmatrix}^- \begin{bmatrix} 0 \\ I_m \end{bmatrix}.$$

In this case,

$$(7.9) \quad r(A) = \dim \left[\bigcap_{i=1}^k \mathcal{R}(A_i) \right] \quad \text{and} \quad \mathcal{R}(A) = \mathcal{R}(A_1) \cap \cdots \cap \mathcal{R}(A_k).$$

By Theorem 7.4, we now are able to define the parallel sums of k Hermitian matrices of the same size.

DEFINITION 7.5. Matrices $A_1, \dots, A_k \in \mathbb{C}_H^m$ are said to be parallel summable if the matrix product

$$(7.10) \quad -[0, I_m] \begin{bmatrix} N & P^* \\ P & 0 \end{bmatrix}^- \begin{bmatrix} 0 \\ I_m \end{bmatrix}$$

is invariant with respect to the choice of the Hermitian g -inverse, where N and P are as given in (7.1). In this case, (7.10) is called the parallel sum of A_1, \dots, A_k and is denoted by $p(A_1, \dots, A_k)$.

Applying (1.22) to (7.10) leads to the following result.

THEOREM 7.6. Non-null matrices $A_1, \dots, A_k \in \mathbb{C}_H^m$ are parallel summable if and only if

$$(7.11) \quad \mathcal{R} \begin{bmatrix} 0 \\ I_m \end{bmatrix} \subseteq \mathcal{R} \begin{bmatrix} N & P^* \\ P & 0 \end{bmatrix},$$

or equivalently,

$$(7.12) \quad r \begin{bmatrix} N & P^* \\ P & 0 \end{bmatrix} = r[N, P^*] + r(P),$$

where N and P are as given in (7.1). In particular, k nonnegative definite matrices of the same size are always parallel summable.

We next examine the relations between the parallel sum of k Hermitian matrices and shorted matrices. Let

$$(7.13) \quad V = \{X \in \mathbb{C}_H^m \mid \mathcal{R}(X) \subseteq \mathcal{R}(B)\}.$$

The shorted matrix of $A \in \mathbb{C}_H^m$ relative to $\mathcal{R}(B)$, denoted by $S[A|\mathcal{R}(B)]$, is defined to be a matrix

$$(7.14) \quad X_0 = \operatorname{argmin}_{X \in V} r(A - X).$$

Note that $X \in V$ can be written as $X = BZB^*$ for some $Z \in \mathbb{C}_H^n$. Hence, $S[A|\mathcal{R}(B)]$ can be written as

$$(7.15) \quad S[A|\mathcal{R}(B)] = BZB^*, \quad Z = \operatorname{argmin}_{Z \in \mathbb{C}_H^n} r(A - BZB^*).$$

The general expressions of the matrices Z and BZB^* satisfying (7.15) were given in [34]. It was shown in [34] that the matrix BZB^* , i.e., the shorted matrix $S[A|\mathcal{R}(B)]$, is unique if and only if A and B satisfy the rank additivity condition

$$(7.16) \quad r \begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix} = r[A, B] + r(B).$$

In this case, the unique shorted matrix can be written as

$$(7.17) \quad S[A|\mathcal{R}(B)] = -[0, B] \begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix}^- \begin{bmatrix} 0 \\ B^* \end{bmatrix} = A - [A, 0] \begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix}^- \begin{bmatrix} A \\ 0 \end{bmatrix}.$$

Applying these results to Theorem 7.6, we immediately obtain the following result.

THEOREM 7.7. *Assume that $A_1, \dots, A_k \in \mathbb{C}_H^m$ are all non-null, and let N and P be as given in (7.1). Then, A_1, \dots, A_k are parallel summable if and only if the shorted matrix of N relative to $\mathcal{R}(P^*)$ is unique. In this case, the parallel sum $p(A_1, \dots, A_k)$ and the shorted matrix $S[N|\mathcal{R}(P^*)]$ satisfy the equalities*

$$(7.18) \quad S[N|\mathcal{R}(P^*)] = P^* p(A_1, \dots, A_k) P,$$

$$(7.19) \quad p(A_1, \dots, A_k) = \frac{1}{k^2} P S[N|\mathcal{R}(P^*)] P^*,$$

$$(7.20) \quad p(A_1, \dots, A_k) = \frac{1}{k^2} \left(A - [A_1, \dots, A_k, 0] \begin{bmatrix} N & P^* \\ P & 0 \end{bmatrix}^- \begin{bmatrix} A_1 \\ \vdots \\ A_k \\ 0 \end{bmatrix} \right),$$

where $A = A_1 + \dots + A_k$.

Proof. Eq. (7.18) follows from contrasting (7.10) with $S[A|\mathcal{R}(P^*)]$ in the first equality in (7.17). Pre- and post-multiplying P and P^* on the both sides of (7.18) and noticing that $PP^* = kI_m$, we obtain (7.19) from (7.18). Finally, substituting the $S[A|\mathcal{R}(P^*)]$ in the second equality in (7.17) into (7.19) yields (7.20). \square

THEOREM 7.8. *Assume that $A_1, \dots, A_k \in \mathbb{C}_H^m$ are all non-null, and parallel summable. Then,*

$$(a) \{ [p(A_1, \dots, A_k)]^- \} = \{ A_1^- + \dots + A_k^- \}.$$

(b) $p(A_1, \dots, A_k) = p(A_{i_1}, \dots, A_{i_k})$, where i_1, \dots, i_k are any permutation of $1, \dots, k$.

Proof. Result (a) follows from Theorems 7.4 and 7.6. Result (b) follows from Corollary 2.7 and $\{ A_1^- + \dots + A_k^- \} = \{ A_{i_1}^- + \dots + A_{i_k}^- \}$. \square

THEOREM 7.9. Assume that $A_1, \dots, A_k \in \mathbb{C}_H^m$ are all non-null, and let $B \in \mathbb{C}^{m \times m}$ be a nonsingular matrix. Then, A_1, \dots, A_k are parallel summable if and only if BA_1B^*, \dots, BA_kB^* are parallel summable. In this case,

$$(7.21) \quad p(BA_1B^*, \dots, BA_kB^*) = Bp(A_1, \dots, A_k)B^*.$$

Proof. Denote $\widehat{B} = \text{diag}(B, \dots, B)$. Since B is nonsingular, \widehat{B} is nonsingular, too. Thus, it is easy to verify that

$$\begin{aligned} r \begin{bmatrix} \widehat{B}N\widehat{B}^* & P^* \\ P & 0 \end{bmatrix} &= r \begin{bmatrix} N & \widehat{B}^{-1}P^* \\ P(\widehat{B}^*)^{-1} & 0 \end{bmatrix} \\ &= r \begin{bmatrix} N & \widehat{B}^{-1}P^*B \\ B^*P(\widehat{B}^*)^{-1} & 0 \end{bmatrix} = r \begin{bmatrix} N & P^* \\ P & 0 \end{bmatrix}, \\ r[\widehat{B}N\widehat{B}^*, P^*] &= r[N, \widehat{B}^{-1}P^*] = r[N, \widehat{B}^{-1}P^*B] = r[N, P^*]. \end{aligned}$$

Combining these two rank equalities with (7.12) shows that A_1, \dots, A_k are parallel summable if and only if BA_1B^*, \dots, BA_kB^* are parallel summable. From the nonsingularity of B , we also see that

$$\begin{bmatrix} \widehat{B}N\widehat{B}^* & P^* \\ P & 0 \end{bmatrix}^- = \begin{bmatrix} (\widehat{B}^*)^{-1} & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} N & P^* \\ P & 0 \end{bmatrix}^- \begin{bmatrix} \widehat{B}^{-1} & 0 \\ 0 & B^* \end{bmatrix}.$$

Thus, it follows from (7.10) that

$$\begin{aligned} p(BA_1B^*, \dots, BA_kB^*) &= -[0, I_m] \begin{bmatrix} \widehat{B}N\widehat{B}^* & P^* \\ P & 0 \end{bmatrix}^- \begin{bmatrix} 0 \\ I_m \end{bmatrix} \\ &= -[0, I_m] \begin{bmatrix} (\widehat{B}^*)^{-1} & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} N & P^* \\ P & 0 \end{bmatrix}^- \begin{bmatrix} \widehat{B}^{-1} & 0 \\ 0 & B^* \end{bmatrix} \begin{bmatrix} 0 \\ I_m \end{bmatrix} \\ &= -B[0, I_m] \begin{bmatrix} N & P^* \\ P & 0 \end{bmatrix}^- \begin{bmatrix} 0 \\ I_m \end{bmatrix} B^* = Bp(A_1, \dots, A_k)B^*, \end{aligned}$$

establishing (7.21). \square

It is also easy to establish formulas for calculating the extremum inertias of $A^- - A_1^- - \cdots - A_k^-$, and to extend the results in Section 6 to the matrix inequalities $A^- > (\geq, <, \leq) A_1^- + \cdots + A_k^-$ in the Löwner partial ordering. The details are left for the reader.

Acknowledgment. The author is grateful to anonymous referees and the handling editor for their careful reading and helpful comments and suggestions that improved the quality of the paper.

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