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HIDDEN COMMUTATIVITY IN SEMI-FTVN SYSTEMS*

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Abstract. A semi-FTvN system (short for semi-Fan–Theobald–von Neumann system) is a triple $(\mathcal{V},\mathcal{W},\lambda)$, where $\mathcal{V}$ and $\mathcal{W}$ are real inner product spaces and the mapping $\lambda:\mathcal{V}\rightarrow\mathcal{W}$ satisfies the sharpened Cauchy–Schwarz inequality $\langle x,y\rangle\leq\langle\lambda(x),\lambda(y)\rangle\leq\|x\|\,\|y\|$. Such a system arises, for example, from a complete hyperbolic system and is a generalization of a Fan–Theobald–von Neumann system. In the setting of a semi-FTvN system, we introduce two commutativity concepts: Strong commutativity via the equality $\langle x,y\rangle=\langle\lambda(x),\lambda(y)\rangle$ and commutativity via the condition $\langle Dx,y\rangle=0$ for all D in the Lie algebra of the automorphism group of the system. We show that strong commutativity implies commutativity, and in the special case of a Euclidean Jordan algebra, commutativity and strong commutativity concepts reduce, respectively, to those of operator and strong operator commutativity. In the form of an application to optimization problems, we describe a commutation principle, where commutativity appears as an optimality condition.

Key words. Semi-FTvN system, FTvN system, Euclidean Jordan algebra, Hyperbolic system, (Strong) commutativity, Commutation principle.

AMS subject classifications. 15A27, 17C20, 46N10, 90C33.

1. Introduction.

A semi-Fan–Theobald–von Neumann (semi-FTvN) system is a triple $(\mathcal{V},\mathcal{W},\lambda)$, where $\mathcal{V}$ and $\mathcal{W}$ are real inner product spaces and $\lambda:\mathcal{V}\rightarrow\mathcal{W}$ is a mapping satisfying the "sharpened Cauchy–Schwarz inequality":

$$\langle x,y\rangle\leq\langle\lambda(x),\lambda(y)\rangle\leq\|x\|\,\|y\|\quad(\forall\ x,y\in\mathcal{V}).\tag{1.1}$$

Such a system is a generalization of an FTvN system and arises from a complete hyperbolic system. Our objective in this paper is to describe two commutativity concepts in a semi-FTvN system and show their relevance in certain commutation principles.

In a semi-FTvN system $(\mathcal{V},\mathcal{W},\lambda)$, we define *strong commutativity* of elements x and y in $\mathcal{V}$ by the condition

$$\langle x,y\rangle=\langle\lambda(x),\lambda(y)\rangle.\tag{1.2}$$

To define *commutativity* in this system, we consider the group $\mathrm{Aut}(\mathcal{V},\mathcal{W},\lambda)$ of all invertible bounded linear transformations $A:\mathcal{V}\rightarrow\mathcal{V}$ satisfying the condition $\lambda(Ax)=\lambda(x)$ for all $x\in\mathcal{V}$. Considering a closed subgroup $\mathcal{G}$ of $\mathrm{Aut}(\mathcal{V},\mathcal{W},\lambda)$, we define

$$\mathrm{Lie}(\mathcal{G}):=\{\mathrm{D}\in\mathrm{L}(\mathcal{V}):\exp(\mathrm{tD})\in\mathcal{G}\text{ for all }\mathrm{t}\in\mathbb{R}\},\tag{1.3}$$

where, for a given D, $\exp(\mathrm{tD})$ denotes the sum of the power series $\sum_{k=0}^{\infty}\frac{t^kD^k}{k!}$. (Here, $\mathrm{L}(\mathcal{V})$ is the space of all bounded linear transformations on $\mathcal{V}$, equipped with the operator norm. We observe that when $\mathcal{V}$ is finite-dimensional, $\mathrm{Lie}(\mathcal{G})$ is the Lie algebra of the matrix Lie group $\mathcal{G}$.) Given elements a and b in $\mathcal{V}$, we say that

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a commutes with b relative to $\mathcal{G}$ *if* $\langle Da, b\rangle = 0$ *for all* $\mathrm{D} \in \mathrm{Lie}(\mathcal{G})$.

We remark that these commutativity concepts are "hidden" as there is no explicit binary product $*$ on $\mathcal{V}$ to define commutativity in the form $x * y = y * x$.
A semi-FTvN system is a generalization of an FTvN system. We recall that an FTvN system (short for Fan–Theobald–von Neumann system) [6] is a triple $(\mathcal{V}, \mathcal{W}, \lambda)$, where $\mathcal{V}$ and $\mathcal{W}$ are real inner product spaces and $\lambda : \mathcal{V} \to \mathcal{W}$ is a mapping satisfying the condition

$$\max\Big\{\langle c, x\rangle : \, x \in [u]\Big\} = \langle \lambda(c), \lambda(u)\rangle \quad (\forall\, c, u \in \mathcal{V}), \tag{1.4}$$

with $[u] := \{x \in \mathcal{V} : \lambda(x) = \lambda(u)\}$ denoting the so-called λ-orbit of u. It is known (see [9], Section 3) that in such a system, the mapping λ satisfies (1.1). Thus, *every FTvN system is a semi-FTvN system.* Numerous examples and properties of FTvN systems are described in the articles [6, 9, 16, 17]. A simple example of an FTvN system is the triple $(\mathcal{V}, \mathbb{R}, \lambda)$, where $\mathcal{V}$ is a real inner product space and $\lambda(x) = ||x||$. Another example is the triple $(\mathbb{R}^n, \mathbb{R}^n, \lambda)$, where $\lambda(x) = x^{\downarrow}$ is the decreasing rearrangement of vector $x \in \mathbb{R}^n$. Nontrivial examples include Euclidean Jordan algebras, systems induced by complete and isometric hyperbolic polynomials, and normal decomposition systems (Eaton triples), see [6].

The concept of strong commutativity (1.2) was initially studied in the setting of FTvN systems (where it was just called commutativity). For example, it was shown in [6], Section 3.1, that in an FTvN system $(\mathcal{V}, \mathcal{W}, \lambda)$, for any $c \in \mathcal{V}$, $\phi : \mathcal{W} \to \mathbb{R}$, and any spectral set S (which is a set of the form $\lambda^{-1}(Q)$ for some $Q \subseteq \mathcal{W}$) in $\mathcal{V}$,

$$\sup_{x\in S}\Big\{\langle c, x\rangle + (\phi\circ\lambda)(x)\Big\} = \sup_{u\in\lambda(S)}\Big\{\langle\lambda(c), u\rangle + \phi(u)\Big\}, \tag{1.5}$$

with the attainment of one supremum implying the attainment of the other and additionally implying a relation of the form (1.2).

Semi-FTvN systems also arise from complete hyperbolic polynomials. We recall [5, 2] that given a finite-dimensional real vector space $\mathcal{V}$ and a nonzero element $e \in \mathcal{V}$, a real homogeneous polynomial p of degree n on $\mathcal{V}$ is said to be *hyperbolic relative to* e (or in direction e) if $p(e) \neq 0$ and for every $x \in \mathcal{V}$, the roots of the univariate polynomial $t \to p(te - x)$ are real. Given such a polynomial p, for any $x \in \mathcal{V}$, let $\lambda(x)$ denote the vector of roots of the polynomial $t \to p(te - x)$ with entries written in decreasing order. We say that p is complete if $\lambda(x) = 0 \Rightarrow x = 0$. Based on a construction due to Bauschke et al [2], we show below that *every complete hyperbolic polynomial induces a semi-FTvN system.* Euclidean Jordan algebras form a particular instance of hyperbolic systems. In the setting of a Euclidean Jordan algebra (carrying the trace inner product), the strong commutativity relation (1.2) reduces to the *strong operator commutativity* of x and y, which means that x and y have their spectral decompositions with respect to a common Jordan frame where their coefficients (eigenvalues) appear in decreasing order [1]. This commutativity is stronger than the *operator commutativity* where we merely require that elements have their spectral decompositions with respect to a common Jordan frame. We show, by rephrasing a recent result in [18], that *in the FTvN system induced by a Euclidean Jordan algebra, commutativity relative to the automorphism group is equivalent to operator commutativity.*

To highlight the importance of the above commutativity concepts, we present a commutation principle, which is a statement where commutativity appears as an optimality condition in certain optimization problems. More results of this type (over a transformation group) will be presented elsewhere [10].

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The organization of the paper is as follows. Section 2 and its various subsections deal with semi-FTvN systems, automorphisms, and commutativity concepts, and the result that strong commutativity implies commutativity. In this section, we also introduce the concepts of center and unit element and provide numerous examples. In Section 3, we consider the FTvN system induced by a Euclidean Jordan algebra. Here, we show that commutativity relative to the (algebra) automorphism group is equivalent to operator commutativity. In Section 4, we show that a complete hyperbolic polynomial induces a semi-FTvN system. In our final section, we describe a commutativity principle.

2. Semi-FTvN systems. In this section, we formally introduce semi-FTvN systems and describe some examples and results. In what follows, for simplicity, we use the same inner product and norm notation. We start with a slight reformulation of our earlier definition based on (1.1).

Definition 2.1. *We say that a triple $(\mathcal{V},\mathcal{W},\lambda)$ is a semi-FTvN system if $\mathcal{V}$ and $\mathcal{W}$ are real inner product spaces with $\lambda:\mathcal{V}\to\mathcal{W}$ satisfying the following conditions:*

(A1) λ is norm-preserving, that is, $||\lambda(x)||=||x||$ for all $x\in\mathcal{V}$;
(A2) λ is inner product expanding, that is, $\langle x,y\rangle\leq\big\langle\lambda(x),\lambda(y)\big\rangle$ for all $x,y\in\mathcal{V}$.

We recall that (1.4) defines an FTvN system. In view of (1.4) implying (1.1), every FTvN system is a semi-FTvN system. The following statements are easy to verify:

- If $(\mathcal{V},\mathcal{W},\lambda)$ is a semi-FTvN system, then so is $(\mathcal{V},\mathcal{W},\widetilde{\lambda})$, where $\widetilde{\lambda}(x):=-\lambda(-x)$.
- If $(\mathcal{V},\mathcal{W},\lambda)$ is a semi-FTvN system and $\mathcal{U}$ is a (linear) subspace of $\mathcal{V}$, then $(\mathcal{U},\mathcal{W},\lambda)$, with λ restricted to $\mathcal{U}$, is also a semi-FTvN system. (A similar statement for FTvN systems is false, see Example 4.6 below.)
- If $(\mathcal{V},\mathcal{W},\lambda)$ and $(\mathcal{W},\mathcal{Z},\mu)$ are semi-FTvN systems, then so is their "composition" $(\mathcal{V},\mathcal{Z},\mu\circ\lambda)$. (A similar statement for FTvN systems is false, see Example 3.3 below.)
- If $(\mathcal{V}_1,\mathcal{W}_1,\lambda_1)$ and $(\mathcal{V}_2,\mathcal{W}_2,\lambda_2)$ are semi-FTvN systems, then so is their "product" $(\mathcal{V}_1\times\mathcal{V}_2,\mathcal{W}_1\times\mathcal{W}_2,\lambda_1\times\lambda_2)$.
- If $(\mathcal{V},\mathcal{W},\lambda)$ is a semi-FTvN system, then so is $(\overline{\mathcal{V}},\overline{\mathcal{W}},\overline{\lambda})$, where $\overline{\mathcal{V}}$ and $\overline{\mathcal{W}}$ are the Hilbert space completions of $\mathcal{V}$ and $\mathcal{W}$, respectively, and $\overline{\lambda}$ is the unique extension of λ. (We note that λ is Lipschitzian on $\mathcal{V}$, hence uniformly continuous.) It is not known if such a statement holds for FTvN systems, see Problem 2 on page 12 of [9].

The following properties are known in the setting of FTvN systems [6]. As we see below, they continue to hold in semi-FTvN systems.

Proposition 2.2. *Suppose $(\mathcal{V},\mathcal{W},\lambda)$ is a semi-FTvN system. Then, the following hold for all $x,y\in\mathcal{V}$ and $\alpha\in\mathbb{R}$:*

(*a*) $\lambda(\alpha\,x)=\alpha\,\lambda(x)$ *for all* $\alpha\geq 0$.
(*b*) $||\lambda(x)-\lambda(y)||\leq||x-y||$.
(c) $||\lambda(x+y)||\leq||\lambda(x)+\lambda(y)||$.

Note: Item (*b*) shows that λ is Lipschitz continuous.

Proof. (*a*) Let $\alpha\geq 0$. Using (*A*1) and (*A*2) (of Definition 2.1), we have

$$||\lambda(\alpha\,x)-\alpha\lambda(x)||^2=||\lambda(\alpha\,x)||^2+\alpha^2||\lambda(x)||^2-2\alpha\langle\lambda(\alpha x),\lambda(x)\rangle\leq||\alpha\,x||^2+\alpha^2||x||^2-2\alpha\langle\alpha x,x\rangle=0.$$

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This proves (a).
(b) Using $(A1)$ and $(A2)$, we have $||\lambda(x)-\lambda(y)||^2-||x-y||^2=2[\langle x,y\rangle-\langle\lambda(x),\lambda(y)\rangle]\leq 0$. The stated Lipschitzian property follows.
(c) From $(A1)$, $||\lambda(x+y)||^2=||x+y||^2=\langle x+y,x+y\rangle=\langle x+y,x\rangle+\langle x+y,y\rangle$.
From $(A2)$, $\langle x+y,x\rangle\leq\langle\lambda(x+y),\lambda(x)\rangle$ and $\langle x+y,y\rangle\leq\langle\lambda(x+y),\lambda(y)\rangle$; hence,

$$||\lambda(x+y)||^2\leq\langle\lambda(x+y),\lambda(x)+\lambda(y)\rangle\leq||\lambda(x+y)||\,||\lambda(x)+\lambda(y)||, \tag{2.6}$$

where the second inequality is due to the Cauchy–Schwarz inequality. This gives (c). □

2.1. Automorphisms and commutativity.

Definition 2.3. *In the setting of a semi-FTvN system $(\mathcal{V},\mathcal{W},\lambda)$, an invertible linear transformation $A:\mathcal{V}\to\mathcal{V}$ is said to be an automorphism if $\lambda(Ax)=\lambda(x)$ for all $x\in\mathcal{V}$.*

Consider a semi-FTvN system $(\mathcal{V},\mathcal{W},\lambda)$. Let $\mathrm{L}(\mathcal{V})$ denote the space of all bounded linear transformations on $\mathcal{V}$. Since λ is norm-preserving, each automorphism is a bounded linear transformation and an orthogonal transformation (i.e., inner product preserving). Letting $\mathrm{Aut}(\mathcal{V},\mathcal{W},\lambda)$ (or $\mathrm{Aut}(\mathcal{V})$, for short) denote the set of all automorphisms, we see that $\mathrm{Aut}(\mathcal{V})\subseteq\mathcal{O}(\mathcal{V})$, where $\mathcal{O}(\mathcal{V})$ (= the set of all invertible orthogonal linear transformations) denotes the orthogonal group of $\mathcal{V}$. As in (1.3), we let

$$\mathrm{Lie}(\mathrm{Aut}(\mathcal{V})):=\{D\in\mathrm{L}(\mathcal{V}):\exp(tD)\in\mathrm{Aut}(\mathcal{V})\text{ for all }t\in\mathbb{R}\}.$$

(So, in the above definition, we consider those $D\in\mathrm{L}(\mathcal{V})$ for which the power series $\sum_{k=0}^{\infty}\frac{t^kD^k}{k!}$ converges in $\mathrm{L}(\mathcal{V})$ for every $t\in\mathbb{R}$.) While many of our concepts and results below are based on the group $\mathrm{Aut}(\mathcal{V})$, in some settings, it may be more advantageous to work with a closed subgroup $\mathcal{G}$ of $\mathrm{Aut}(\mathcal{V})$ (such as the connected component of identity in $\mathrm{Aut}(\mathcal{V})$ when $\mathcal{V}$ is finite-dimensional or the one that appears in Example 2.18 below). In this case,

$$\mathrm{Lie}(\mathcal{G}):=\{D\in\mathrm{L}(\mathcal{V}):\exp(tD)\in\mathcal{G}\text{ for all }t\in\mathbb{R}\}\subseteq\mathrm{Lie}(\mathrm{Aut}(\mathcal{V})).$$

Suppose $D\in\mathrm{Lie}(\mathcal{G})$. Then, $\exp(tD)$ is orthogonal and so $\langle\exp(tD)x,\exp(tD)y\rangle=\langle x,y\rangle$ for all $x,y\in\mathcal{V}$ and $t\in\mathbb{R}$. This leads to

$$\langle Dx,y\rangle+\langle x,Dy\rangle=0. \tag{2.7}$$

We now define commutativity in a semi-FTvN system. Our motivation comes from Remark 3.7 in [7] and a recent formulation of operator commutativity in Euclidean Jordan algebras ([18], Proposition 2).

Definition 2.4. *Let $(\mathcal{V},\mathcal{W},\lambda)$ be a semi-FTvN system and $\mathcal{G}$ be a closed subgroup of* $\mathrm{Aut}(\mathcal{V})$. *Given $a,b\in\mathcal{V}$, we say that*

- (i) *a strongly commutes with b if $\big\langle a,b\big\rangle=\big\langle\lambda(a),\lambda(b)\big\rangle$.*
- (ii) *a commutes with b if $\langle Da,b\rangle=0$ for all $D\in\mathrm{Lie}(\mathrm{Aut}(\mathcal{V}))$.*
- (iii) *a commutes with b relative to $\mathcal{G}$ if $\langle Da,b\rangle=0$ for all $D\in\mathrm{Lie}(\mathcal{G})$.*

Remarks. (α) In our previous work on FTvN systems [6, 9, 17], we used the condition $\big\langle a,b\big\rangle=\big\langle\lambda(a),\lambda(b)\big\rangle$ to define commutativity. To use the terminologies compatible with those in Euclidean Jordan algebras, henceforth, we use the new term "strong commutativity."

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(β) *The above definitions are symmetric in a and b:* The first definition is clearly symmetric in a and b. From (2.7), $\langle Da, b\rangle = 0 \Rightarrow \langle Db, a\rangle = 0$ when $D \in \mathrm{Lie}(\mathcal{G})$; so, the other two definitions are also symmetric. Because of this symmetry, we use phrases such as "a and b (strongly) commute,"
We note that if a and b commute (relative to $\mathcal{G}$), then $\pm a$ and $\pm b$ commute (relative to $\mathcal{G}$). Also, *commutativity implies commutativity relative to any $\mathcal{G}$* as $\mathrm{Lie}(\mathcal{G}) \subseteq \mathrm{Lie}(\mathrm{Aut}(\mathcal{V}))$.
(γ) While strong commutativity of a and b is easily verifiable, checking commutativity requires the knowledge of the automorphism group $\mathrm{Aut}(\mathcal{V})$ and its Lie algebra.
(δ) The relation $\langle Da, b\rangle = 0$ for all $D \in \mathrm{Lie}(\mathcal{G})$ is a type of "orbital relation" defined in [26].
(ϵ) Suppose $\mathcal{V}$ is finite dimensional so that $\mathcal{G}$ is a matrix Lie group. It is well known that the connected component $\mathcal{G}_0$ of $\mathcal{G}$ is also a matrix Lie group (cf. [12], Prop. 1.8). Hence, if a and b commute relative to $\mathcal{G}$, then they commute relative to $\mathcal{G}_0$.

In the theorem below, we show that strong commutativity implies commutativity. First, we describe strong commutativity in a semi-FTvN system; the following result and its proof are identical to those for FTvN systems ([6], Section 2).

Proposition 2.5. *Consider a semi-FTvN system $(\mathcal{V}, \mathcal{W}, \lambda)$. For $x, y \in \mathcal{V}$, the following are equivalent:*

(a) *x and y strongly commute, i.e., $\langle x, y\rangle = \langle \lambda(x), \lambda(y)\rangle$.*
(b) $||\lambda(x) - \lambda(y)|| = ||x - y||$.
(c) $||\lambda(x + y)|| = ||\lambda(x) + \lambda(y)||$.
(d) $\lambda(x + y) = \lambda(x) + \lambda(y)$.

Proof. Using ($A1$) (of Definition 2.1), we see that

$$||\lambda(x) - \lambda(y)||^2 - ||x - y||^2 = 2[\langle x, y\rangle - \langle \lambda(x), \lambda(y)\rangle] = ||\lambda(x + y)||^2 - ||\lambda(x) + \lambda(y)||^2.$$

The equivalence (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c) follows.
(c) $\Rightarrow$ (d): When (c) holds, we have equality in the Cauchy–Schwarz inequality (2.6). Hence, one of the vectors, $\lambda(x + y)$ or $\lambda(x) + \lambda(y)$, is a nonnegative multiple of the other. Since their norms are equal (as we are assuming (c)), these vectors are equal. Item (d) follows.
Finally, (d) $\Rightarrow$ (c) is obvious. □

Definition 2.6. *In a semi-FTvN system $(\mathcal{V}, \mathcal{W}, \lambda)$, we define the λ-orbit of an element $u \in \mathcal{V}$ by*

$$[u] := \{x \in \mathcal{V} : \lambda(x) = \lambda(u)\}.$$

One interesting consequence of Item (b) in the above proposition is: *If x, y in $[u]$ strongly commute, then $x = y$.*

We now state one of our main results.

Theorem 2.7. *In any semi-FTvN system* (*in particular, in any FTvN system*)*, strong commutativity implies commutativity* (*relative to any closed subgroup of* $\mathrm{Aut}(\mathcal{V})$).

Proof. Consider a semi-FTvN system $(\mathcal{V}, \mathcal{W}, \lambda)$ and fix a closed subgroup $\mathcal{G}$ of $\mathrm{Aut}(\mathcal{V})$. Suppose a and b strongly commute, that is, $\langle a, b\rangle = \langle \lambda(a), \lambda(b)\rangle$. Fix any $D \in \mathrm{Lie}(\mathcal{G})$. As the power series $\sum_{k=0}^{\infty} \frac{t^k D^k}{k!}$ converges in $\mathrm{L}(\mathcal{V})$ for every t in $\mathbb{R}$, the real-valued (analytic) function

$$\phi(t) := \big\langle \exp(tD)a, b\big\rangle = \sum_{k=0}^{\infty} \frac{t^k \langle D^k a, b\rangle}{k!}.$$

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is differentiable. Now,

$$\phi(t) = \big\langle \exp(tD)a, b\big\rangle \leq \big\langle\lambda(\exp(tD)a), \lambda(b)\big\rangle = \big\langle\lambda(a), \lambda(b)\big\rangle = \langle a, b\rangle = \phi(0),$$

where the first inequality is due to the definition of a semi-FTvN system, the second equality is due to the definition of an automorphism, and the third equality is due to the strong commutativity of a and b. So, ϕ attains its global maximum at $t = 0$. From $\phi'(0) = 0$, we have

$$\langle Da, b\rangle = \phi'(0) = 0.$$

Since D is arbitrary in $\mathrm{Lie}(\mathcal{G})$, a and b commute relative to $\mathcal{G}$. When $\mathcal{G} = \mathrm{Aut}(\mathcal{V})$, we get the commutativity of a and b. ∎

2.2. The center and unit element. The concepts of center and unit element have been defined and found to be useful in the setting of FTvN systems [9]. Now we extend these to semi-FTvN systems.

Definition 2.8. *In a semi-FTvN system $(\mathcal{V}, \mathcal{W}, \lambda)$, we define the center $\mathcal{C}$ and weak center $\mathcal{C}_w$ as follows:*

$$\mathcal{C} := \{x \in \mathcal{V} : x \textit{ strongly commutes with every } y \in \mathcal{V}\};$$

$$\mathcal{C}_w := \{x \in \mathcal{V} : x \ \textit{ commutes with every } y \in \mathcal{V}\} = \{x \in \mathcal{V} : Dx = 0 \textit{ for all } D \in \mathrm{Lie}(\mathrm{Aut}(\mathcal{V}))\}.$$

Note: The second equality in $\mathcal{C}_w$ is due to the fact that $\langle Dx, y\rangle = 0$ for all y implies that $Dx = 0$.

Proposition 2.9. *In a semi-FTvN system $(\mathcal{V}, \mathcal{W}, \lambda)$, the center $\mathcal{C}$ and the weak center $\mathcal{C}_w$ are closed (linear) subspaces of $\mathcal{V}$. Moreover, $\mathcal{C} \subseteq \mathcal{C}_w$ and λ is linear on $\mathcal{C}$.*

Proof. Clearly, $\mathcal{C}_w$ is a closed subspace of $\mathcal{V}$. That $\mathcal{C} \subseteq \mathcal{C}_w$ follows from Theorem 2.7. We now show that $\mathcal{C}$ is a closed subspace of $\mathcal{V}$ and that λ is linear on it. Suppose $a, b \in \mathcal{C}$. Then, by Proposition 2.2(a), any nonnegative multiple of a is also in $\mathcal{C}$. Moreover, as a strongly commutes with $-a$, by Proposition 2.5(d), $\lambda(-a) = -\lambda(a)$. It follows that any multiple of a is in $\mathcal{C}$. Now repeated use of Item (d) in Proposition 2.5 shows that $a + b \in \mathcal{C}$. That $\mathcal{C}$ is closed comes from the continuity of λ. Finally, Proposition 2.5(d) and positive homogeneity of λ give the linearity on $\mathcal{C}$. ∎

Definition 2.10. *In the setting of a semi-FTvN system $(\mathcal{V}, \mathcal{W}, \lambda)$, a nonzero element $e \in \mathcal{V}$ is said to be a unit if $\mathcal{C} = \mathbb{R}\, e$.*

Remarks. In an FTvN system, the center can be described in many ways, see Section 6 in [9]. In particular, it is shown in [9], Proposition 6.3, that $u \in \mathcal{C}$ if and only if $[u] = \{u\}$. We now prove a result of this type for semi-FTvN systems.

Proposition 2.11. *Let $(\mathcal{V}, \mathcal{W}, \lambda)$ be a semi-FTvN system. Consider the following statements:*

(i) $c \in \mathcal{C}$.
(ii) $[c] = \{c\}$.
(iii) $Ac = c$ *for all* $A \in \mathrm{Aut}(\mathcal{V})$.
(iv) $c \in \mathcal{C}_w$.

Then, (i) ⇒ (ii) ⇒ (iii) ⇒ (iv). In particular, if e is a unit element, then $Ae = e$ for all $A \in \mathrm{Aut}(\mathcal{V})$.

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Note: Example 2.17 below shows that the (reverse) implications $(ii) \Rightarrow (i)$ and $(iii) \Rightarrow (i)$ need not hold. Also, Example 2.16 shows that the (reverse) implication $(iv) \Rightarrow (iii)$ need not hold (even in an FTvN system). To see an example where (iii) does not imply (ii), we consider the Euclidean Jordan algebra $\mathcal{V} := \mathcal{S}^2 \times \mathbb{R}$, where $\mathcal{S}^2$ is the space of all 2×2 real symmetric matrices. Since $\mathcal{S}^2$ is nonisomorphic to $\mathbb{R}$, it is known, see [7], Corollary 5.6, that the automorphism group of $\mathcal{V}$ is the product of the automorphism groups of $\mathcal{S}^2$ and of $\mathbb{R}$. Then, the element $c := (\alpha I, \beta)$ with $\alpha \neq \beta$ satisfies condition (iii), but not (ii) as $[c]$ contains the element $u = (\mathrm{Diag}(\alpha, \beta), \alpha)$, where $\mathrm{Diag}(\alpha, \beta)$ denotes the diagonal matrix in $\mathcal{S}^2$ with diagonal entries α and β.

Proof. $(i) \Rightarrow (ii)$: Assume (i), fix $c \in \mathcal{C}$; let $x \in [c]$ so that $\lambda(x) = \lambda(c)$. As c strongly commutes with x, from Proposition 2.5(b) , $||x - c|| = ||\lambda(x) - \lambda(c)|| = 0$. So, $x = c$. This gives (ii).
$(ii) \Rightarrow (iii)$: Assume (ii) and let $A \in \mathrm{Aut}(\mathcal{V})$. Then, $Ac \in [c] = \{c\}$, so $Ac = c$.
$(iii) \Rightarrow (iv)$: Assume $Ac = c$ for all $A \in \mathrm{Aut}(\mathcal{V})$. Then, for any $D \in \mathrm{Lie}(\mathrm{Aut}(\mathcal{V}))$ and $t \in \mathbb{R}$, we have $\exp(tD) \in \mathrm{Aut}(\mathcal{V})$, hence $\exp(tD)c = c$. Now, differentiation leads to $Dc = 0$. By Definition 2.8, $c \in \mathcal{C}_w$. ∎

Definition 2.12. *In a semi-FTvN system $(\mathcal{V}, \mathcal{W}, \lambda)$, a set $E \subseteq \mathcal{V}$ is said to be*

- *a spectral set if $x \in E \Rightarrow [x] \subseteq E$ or, equivalently, $E = \lambda^{-1}(Q)$ for some $Q \subseteq \mathcal{W}$;*
- *a weakly spectral set if $A(E) \subseteq E$ for all $A \in \mathrm{Aut}(\mathcal{V})$.*

Proposition 2.13. *In a semi-FTvN system, the center is a spectral set, and the weak center is a weakly spectral set.*

Proof. To simplify the notation, let $\mathcal{G} := \mathrm{Aut}(\mathcal{V})$.
If $x \in \mathcal{C}$, then by the above proposition, $[x] = \{x\} \subseteq \mathcal{C}$. Hence, $\mathcal{C}$ is a spectral set.
To show that $\mathcal{C}_w$ is weakly spectral, we need to show that $A(\mathcal{C}_w) \subseteq \mathcal{C}_w$ for all $A \in \mathcal{G}$. In view of Definition 2.8, we need to show:

$$\text{for any } c \in \mathcal{C}_w,\ A \in \mathcal{G}, \text{ and } D \in \mathrm{Lie}(\mathcal{G}), \text{ it holds that } D(Ac) = 0.$$

To see this, we fix $c \in \mathcal{C}_w$, $A \in \mathcal{G}$, and $D \in \mathrm{Lie}(\mathcal{G})$. Then, for any $t \in \mathbb{R}$, $exp(tD) \in \mathcal{G}$,

$$exp(tA^{-1}DA) = A^{-1}exp(tD)A \in \mathcal{G},$$

and so, $A^{-1}DA \in \mathrm{Lie}(\mathcal{G})$. As $c \in \mathcal{C}_w$, we have $A^{-1}DAc = 0$ and $DAc = 0$. As D is arbitrary in $\mathrm{Lie}(\mathcal{G})$, $Ac \in \mathcal{C}_w$. This completes the proof. ∎

We now describe some examples. Recall that in the setting of a semi-FTvN system $(\mathcal{V}, \mathcal{W}, \lambda)$, $\mathrm{Aut}(\mathcal{V})$ is an abbreviation for $\mathrm{Aut}(\mathcal{V}, \mathcal{W}, \lambda)$.

Example 2.14. *Consider the semi-FTvN system $(\mathbb{R}, \mathbb{R}, \lambda)$, where $\lambda(x) = |x|$. As $\mathrm{Aut}(\mathbb{R}) = \{I, -I\}$ and $\mathrm{Lie}(\mathrm{Aut}(\mathbb{R})) = \{0\}$, where I and 0 denote the identity and zero transformations, respectively, we see that any two elements $a, b \in \mathbb{R}$ commute. On the other hand, a and b strongly commute (that is, $ab = |a|\,|b|$) if and only if one of them (either a or b) is a nonnegative multiple of the other. Thus, in this system, $\mathcal{C} = \{0\}$ while $\mathcal{C}_w = \mathbb{R}$.*

Example 2.15. *(This is elaborated in Example 3.14, [10].) Consider the semi-FTvN system $(\mathcal{V}, \mathbb{R}, \lambda)$, where $\mathcal{V}$ is a real finite dimensional inner product space with $dim(\mathcal{V}) > 1$ and $\lambda(x) = ||x||$. By isomorphism considerations, we may assume that $\mathcal{V} = \mathbb{R}^n$ $(n > 1)$ with the usual inner product. It is shown in [10], Example 3.14, that a and b commute in the semi-FTvN system $(\mathbb{R}^n, \mathbb{R}, \lambda)$ if and only if b is a scalar multiple of a or a is a scalar multiple of b. Now suppose $a \in \mathcal{C}_w$. Let $\mathbf{1}$ be the vector of all 1s and d be a vector with*

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all entries distinct (recall that $n > 1$). Since a commutes with both $\mathbf{1}$ and d, we must have $a = 0$. Hence, $\mathcal{C}_w = \{0\}$. We note that a and b strongly commute in this system if and only if $\langle a, b\rangle = ||a||\,||b||$, in which case, a (or b) is a nonnegative multiple of b (respectively, a). Consequently, since $-a$ is not a nonnegative multiple of a unless $a = 0$, we see that (the center) $\mathcal{C} = \{0\}$. Thus, in this system, $\mathcal{C} = \mathcal{C}_w = \{0\}$. Note that this is different from the $n = 1$ case of Example 2.14.

Example 2.16. *Consider $\mathbb{R}^n$ with the usual inner product. For any $x \in \mathbb{R}^n$, let $x^\downarrow$ denote the vector obtained by rearranging the entries of x in decreasing order. As $\langle x, y\rangle \le \langle x^\downarrow, y^\downarrow\rangle$ (which is the Hardy–Littlewood–Polya rearrangement inequality, see [3], Cor. II.4.4) and $||x^\downarrow|| = ||x||$, we see that $(\mathbb{R}^n, \mathbb{R}^n, \lambda)$ is a semi-FTvN system, where $\lambda(x) = x^\downarrow$. (Actually, this is an FTvN system arising from the Euclidean Jordan algebra $(\mathbb{R}^n, \langle\cdot,\cdot\rangle, \circ)$, where, for $x, y \in \mathbb{R}^n$, $\langle x, y\rangle$ is the usual inner product and $x \circ y$ is the componentwise product.) As $\mathrm{Aut}(\mathbb{R}^n, \mathbb{R}^n, \lambda)$ is the group of all permutation matrices on $\mathbb{R}^n$, we see that the corresponding Lie algebra is $\{0\}$. Hence, any two elements in this semi-FTvN system commute. With $\mathbf{1}$ denoting the vector of ones in $\mathbb{R}^n$, we shall see (in Theorem 3.2 below) that $\mathcal{C} = \mathbb{R}\mathbf{1}$ while $\mathcal{C}_w = \mathbb{R}^n$. We note that when $n > 1$, the vectors $a = (1, 0, \ldots, 0)$ and $b = (0, 1, 0, \ldots, 0)$ commute, but not strongly: $\langle a, b\rangle = 0$, while $\langle \lambda(a), \lambda(b)\rangle = 1$.*

Example 2.17. *In $\mathbb{R}^2$, let $\mathcal{V} := \{r(1, 2) : r \in \mathbb{R}\}$. Consider $\lambda : \mathcal{V} \to \mathbb{R}^2$ with $\lambda(x) = x^\downarrow$. Then, $(\mathcal{V}, \mathbb{R}^2, \lambda)$ is a semi-FTvN system, where for any $x \in \mathcal{V}$, $[x] = \{y \in \mathcal{V} : \lambda(x) = \lambda(y)\} = \{x\}$. It follows that (1.4) fails and so, $(\mathcal{V}, \mathbb{R}^2, \lambda)$ is not an FTvN system. In this semi-FTvN system, $\mathrm{Aut}(\mathcal{V}) = \{I\}$, $\mathcal{C} = \{0\}$, and $\mathcal{C}_w = \mathcal{V}$.*

Note: It will be shown in the more general setting of a Euclidean Jordan algebra that commutativity is the same as operator commutativity. Consequently, in the Euclidean Jordan algebra $\mathcal{S}^n$ of all $n \times n$ real symmetric matrices, commutativity is the same as the usual commutativity of matrices.

Example 2.18. *(Normal decomposition system [20]) Let $\mathcal{V}$ be a real inner product space, $\mathcal{G}$ be a closed subgroup of the orthogonal group of $\mathcal{V}$, and $\gamma : \mathcal{V} \to \mathcal{V}$ be a map satisfying the following conditions:*

(*a*) *γ is $\mathcal{G}$-invariant, that is, $\gamma(Ax) = \gamma(x)$ for all $x \in \mathcal{V}$ and $A \in \mathcal{G}$.*
(*b*) *For each $x \in \mathcal{V}$, there exists $A \in \mathcal{G}$ such that $x = A\gamma(x)$.*
(*c*) *For all $x, y \in \mathcal{V}$, we have $\langle x, y\rangle \le \langle \gamma(x), \gamma(y)\rangle$.*

Then, $(\mathcal{V}, \mathcal{G}, \gamma)$ is called a normal decomposition system. In this setting, we have (cf. [20], Proposition 2.3 and Theorem 2.2): For any two elements x and y in $\mathcal{V}$,

$$\max_{A\in\mathcal{G}} \langle Ax, y\rangle = \langle \gamma(x), \gamma(y)\rangle,$$

and $\langle x, y\rangle = \langle \gamma(x), \gamma(y)\rangle$ if and only if there exists an $A \in \mathcal{G}$ such that $x = A\gamma(x)$ and $y = A\gamma(y)$.

As observed in [6], if $(\mathcal{V}, \mathcal{G}, \gamma)$ is a normal decomposition system, then $(\mathcal{V}, \mathcal{V}, \gamma)$ becomes an FTvN system. We refer to [9], Theorem 7.3, for a characterization of normal decomposition systems among FTvN systems. From the above result, we see a straightforward description of strong commutativity. For commutativity, we observe that $\mathcal{G}$ is a closed subgroup of the (possibly larger) group $\mathrm{Aut}(\mathcal{V}, \mathcal{V}, \gamma)$. Since the group $\mathcal{G}$ is readily available in this setting, it may be advantageous to work with "commutativity relative to $\mathcal{G}$." As in the proof of Theorem 2.7, we see that strong commutativity implies commutativity relative to $\mathcal{G}$.

Example 2.19. *Let $M_{m,n}(\mathbb{C})$ denote the space of all $m \times n$ complex matrices under the inner product $\langle X, Y\rangle = \mathrm{Re}\,\mathrm{tr}(XY^*)$, where Y^* denotes the conjugate-transpose of X. With $\lambda(X) := \mathrm{Diag}(s(X))$ denoting the (rectangular) diagonal matrix of the singular values of X written in decreasing order, we recall von*

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Neumann's result, see [14], Theorem 7.4.1.1 and the discussion in [25], pp. 432: For $X, Y \in M_{m,n}(\mathbb{C})$,

$$\langle X, Y\rangle \leq \langle \lambda(X), \lambda(Y)\rangle,$$

with equality if and only if there exist unitary matrices U *and* V *such that* $X = U\,\mathrm{Diag}(s(X))\,V^*$ *and* $Y = U\,\mathrm{Diag}(s(Y))\,V^*$. *Using the above inequality, for any given* $A, B \in M_{m,n}(\mathbb{C})$ *with* $A = U\,\mathrm{Diag}(s(A))\,V^*$, *we can define* $X := U\,\mathrm{Diag}(s(B))V^*$ *to verify that*

$$\max\Big\{\langle A, X\rangle :\, X \in [B]\Big\} = \langle \lambda(A), \lambda(B)\rangle, \tag{2.8}$$

where $[B] := \{Z \in M_{m,n} : \lambda(Z) = \lambda(B)\}$. *Thus,* $\big(M_{m,n}(\mathbb{C}), M_{m,n}(\mathbb{C}), \lambda\big)$ *is an FTvN system. In this system, von Neumann's result (mentioned above) characterizes strong commutativity. In particular, we have the following:*

If $X, Y \in M_{m,n}(\mathbb{C})$ *and* $\langle X, Y\rangle = \langle \lambda(X), \lambda(Y)\rangle$, *then* XY^* *and* X^*Y *are Hermitian.*

By replacing complex numbers with real numbers, we can consider the space $M_{m,n}(\mathbb{R})$ and obtain analogous results. In particular, we see that $\big(M_{m,n}(\mathbb{R}), M_{m,n}(\mathbb{R}), \lambda\big)$ is an FTvN system. (Note: The unitary matrices become orthogonal matrices, see [14], Theorem 7.3.2(c) or [15], Theorem 3.1.1, and [21], Theorem 4.6.) Moreover,

If $X, Y \in M_{m,n}(\mathbb{R})$ *and* $\langle X, Y\rangle = \langle \lambda(X), \lambda(Y)\rangle$, *then* XY^T *and* X^TY *are symmetric.*

We draw attention to the following characterization result due to Lewis and Sendov, see [21], Lemma 4.3. We state the result (only) for $M_n := M_{n,n}(\mathbb{R})$. Recall that a signed permutation is a square matrix where each row and column has exactly one nonzero entry, which is ± 1.

THEOREM 2.20. *Let* $X, Y \in M_n$. *Then,* XY^T *and* X^TY *are symmetric if and only if there is a signed permutation* Q *and orthogonal matrices* U *and* V *such that* $X = U\,\mathrm{Diag}(Qs(X))V^T$ *and* $Y = U\,\mathrm{Diag}(s(Y))V^T$.

We end this section by describing the center $\mathcal{C}$ and weak center $\mathcal{C}_w$ of the FTvN system (M_n, M_n, λ). For this, we need the following result from [10].

PROPOSITION 2.21. *On* (M_n, M_n, λ), *consider the group* $\mathcal{G}$ *of linear transformations of the form* $X \mapsto UXV$, *where* U *and* V *are orthogonal. Then,*

(*i*) $\mathrm{Lie}(\mathcal{G})$ *consists of transformations of the form* $X \mapsto AX + XB$, *where* A *and* B *are skew-symmetric.*
(*ii*) $X, Y \in M_n$ *commute relative to* $\mathcal{G}$ *if and only if* XY^T *and* X^TY *are symmetric.*

Suppose $C \in \mathcal{C}$. Then, $\langle C, X\rangle = \langle \lambda(C), \lambda(X)\rangle$ for all $X \in M_n(\mathbb{R})$. In particular, we have that $\langle C, -C\rangle = \langle \lambda(C), \lambda(-C)\rangle$. As $\lambda(C) = \lambda(-C)$, we get $\langle C, -C\rangle \geq 0$ and, consequently, $C = 0$. Thus, $\mathcal{C} = \{0\}$.
To describe $\mathcal{C}_w$, we consider two cases: Let $n = 1$. Then, $M_1 = \mathbb{R}$ and $\lambda(x) = |x|$. From Example 2.14, $\mathcal{C}_w = \mathbb{R}$. On the other hand, suppose $n > 1$ and $X \in \mathcal{C}_w$. As $\mathcal{G} \subseteq \mathrm{Aut}(M_n, M_n, \lambda)$, from the above proposition and Definition 2.8, $AX + XB = 0$, where A and B are arbitrary skew-symmetric matrices. We see that $AX = 0$ for every skew-symmetric matrix A, consequently, $X = 0$. This means that when $n > 1$, $\mathcal{C}_w = \{0\}$. To summarize: *In the FTvN system* (M_n, M_n, λ), *the center is* $\{0\}$; *the weak center is* $\mathbb{R}$ *when* $n = 1$ *and* $\{0\}$ *when* $n > 1$.

3. The FTvN system induced by a Euclidean Jordan algebra. Suppose $\mathcal{A} = (\mathcal{V}, \langle\cdot,\cdot\rangle, \circ)$ is a Euclidean Jordan algebra of rank n with unit element e [4] . Here, for any two elements x and y, $\langle x, y\rangle$

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denotes the inner product and $x \circ y$ denotes the Jordan product. We recall some basic concepts. A nonzero element c is said to be an idempotent if $c^2 = c$; it is a primitive idempotent if it is not the sum of two other idempotents. A Jordan frame is a finite set of mutually orthogonal primitive idempotents with sum e. For any $x \in \mathcal{V}$, by the spectral decomposition theorem ([4], Theorem III.1.2), we can write

$$x = x_1 e_1 + x_2 e_2 + \cdots + x_n e_n, \tag{3.9}$$

where $\{e_1, e_2, \ldots, e_n\}$ is a Jordan frame and $x_1, x_2, \ldots, x_n$ are (unique) real numbers (called the eigenvalues of x). We define the trace of x as $\mathrm{tr}(x) := x_1 + x_2 + \cdots + x_n$ and trace inner product over $\mathcal{V}$ by $\langle x, y\rangle_{\mathrm{tr}} := \mathrm{tr}(x \circ y)$. This inner product is compatible with the Jordan product. Moreover, Jordan frames, eigenvalues, and spectral decompositions remain the same when we replace the given inner product with the trace inner product. *Henceforth, in the setting of a Euclidean Jordan algebra, we assume that $\mathcal{V}$ carries this trace inner product.*

Let $\mathrm{Aut}(\mathcal{A})$ denote the group of all invertible linear transformations $A : \mathcal{V} \to \mathcal{V}$ satisfying the condition

$$A(x \circ y) = Ax \circ Ay \text{ for all } x, y \in \mathcal{V}.$$

Elements of $\mathrm{Aut}(\mathcal{A})$ are called algebra automorphisms of $\mathcal{A}$.

Given the spectral decomposition of x (as above), let $\lambda(x)$ in $\mathbb{R}^n$ denote the vector of eigenvalues of x written in decreasing order. Then, it is shown in [9], Section 4, Example 4.6, that $(\mathcal{V}, \mathbb{R}^n, \lambda)$ *is an FTvN system.* In this setting, it is known, see [9], Section 7, Example 4.6, that

$$A \in \mathrm{Aut}(\mathcal{V}, \mathbb{R}^n, \lambda) \textit{ if and only if } A \in \mathrm{Aut}(\mathcal{A}).$$

So, in the setting of a Euclidean Jordan algebra, the automorphisms of the FTvN system $(\mathcal{V}, \mathbb{R}^n, \lambda)$ are the same as the algebra automorphisms.

We recall that in a Euclidean Jordan algebra, two elements a and b *operator commute* if $L_a L_b = L_b L_a$ (where L_a is the linear operator defined by $L_a(x) := a \circ x$ for all $x \in \mathcal{V}$), or equivalently, a and b have their spectral representations with respect to the same Jordan frame ([4], Lemma X.2.2). We say that a and b *strongly operator commute* if there exists a Jordan frame $\{e_1, e_2, \ldots, e_n\}$ such that a and b have spectral decompositions

$$a = a_1 e_1 + a_2 e_2 + \cdots + a_n e_n \quad \text{and} \quad b = b_1 e_1 + b_2 e_2 + \cdots + b_n e_n,$$

with $a_1 \geq a_2 \geq \cdots \geq a_n$ and $b_1 \geq b_2 \geq \cdots \geq b_n$. It is shown in [6], Proposition 4.4, that *in the induced FTvN system $(\mathcal{V}, \mathbb{R}^n, \lambda)$ a and b strongly commute if and only if they strongly operator commute in the Euclidean Jordan algebra $\mathcal{V}$.* We now prove a similar result for commutativity.

Theorem 3.1. *Consider a Euclidean Jordan algebra $\mathcal{A} = (\mathcal{V}, \langle\cdot,\cdot\rangle, \circ)$ of rank n with unit element e. Let $a, b \in \mathcal{V}$. Then, a and b commute in the FTvN system $(\mathcal{V}, \mathbb{R}^n, \lambda)$ if and only if they operator commute in the Euclidean Jordan algebra $\mathcal{A}$.*

Proof. We have noted previously that $(\mathcal{V}, \mathbb{R}^n, \lambda)$ is an FTvN system. Furthermore, as noted above, $\mathrm{Aut}(\mathcal{V}, \mathbb{R}^n, \lambda) = \mathrm{Aut}(\mathcal{A})$. It is known ([19], Lemma 7 and Theorem 8) that a transformation $D \in \mathrm{Lie}(\mathrm{Aut}(\mathcal{A}))$ (such a D is called a derivation) if and only if it is a sum of transformations of the form $L_u L_v - L_v L_u$. So, a and b commute if and only if $\langle (L_u L_v - L_v L_u)a, b\rangle = 0$ for all $u, v \in \mathcal{V}$. Since the latter statement simplifies to $\langle (L_a L_b - L_b L_a)u, v\rangle = 0$, and u, v are arbitrary, commutativity of a and b reduces to the equality $L_a L_b - L_b L_a = 0$, which is the operator commutativity of a and b.

The following result provides a description of the center and weak center in the FTvN system corresponding to a Euclidean Jordan algebra. Recall that the unit element e in the algebra $\mathcal{V}$ satisfies the condition $x \circ e = x$ for all x.

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Theorem 3.2. *Suppose $\mathcal{A}$ is a Euclidean Jordan algebra of rank n with unit element e. In the corresponding FTvN system $(\mathcal{V}, \mathbb{R}^n, \lambda)$, let $\mathcal{C}$ and $\mathcal{C}_w$ denote, respectively, the center and weak center. Then, the following statements hold:*

(*i*) $\mathcal{C} = \mathbb{R}\, e$;
(*ii*) *If $\mathcal{A}$ is simple, then* $\mathcal{C}_w = \mathbb{R}\, e$;
(*iii*) *If $\mathcal{A} = \Pi_{i=1}^k \mathcal{A}_i$, where $\mathcal{A}_i$ is a simple Euclidean Jordan algebra with unit e_i, then $\mathcal{C}_w = \Pi_{i=1}^k \mathbb{R}\, e_i$. In particular, when $\mathcal{V} = \mathbb{R}^n$ with the usual inner product/componentwise product, we have $\mathcal{C}_w = \mathbb{R}^n$.*

Proof. Item (i) is known, see [9], Section 6, Example 4.6.
(ii) Clearly, $\mathbb{R}\, e \subseteq \mathcal{C}_w$. To see the reverse inclusion, suppose that $\mathcal{V}$ is simple and $a \in \mathcal{C}_w$. By the previous theorem, a operator commutes with every $x \in \mathcal{V}$. In particular, a operator commutes with every primitive idempotent c in $\mathcal{V}$. Writing the spectral decompositions of a and (the primitive idempotent) c with respect to a common Jordan frame, we see that (every) c must appear in some spectral decomposition of a. Suppose, to get a contradiction, a is not a multiple of e, that is, a has two distinct eigenvalues. Let $a_1, a_2, \ldots, a_k$ $(k > 1)$, be the distinct eigenvalues of a. Then, the closed sets

$$E_i := \{x \in \mathcal{V} : \langle a, x \rangle = a_i\} \ (1 \leq i \leq k),$$

are disjoint. Since every primitive idempotent appears in some spectral decomposition of a and $\mathcal{V}$ carries the trace inner product, we see that every primitive idempotent is in some E_i. This means that the set $\mathcal{T}(\mathcal{V})$ of all primitive idempotents in $\mathcal{V}$ is contained in the disjoint union of closed sets E_i. However, $\mathcal{T}(\mathcal{V})$ is connected in (any) simple algebra $\mathcal{V}$, see [4], Page 71 or [8], Theorem 3.1. We reach a contradiction. Thus, a must be a multiple of e, proving our assertion.
(iii) Suppose $\mathcal{A} = \Pi_{i=1}^k \mathcal{A}_i$, where each $\mathcal{A}_i$ is simple. For $a, b \in \mathcal{V}$, we write $a = (a_1, a_2, \ldots, a_k)$ and $b = (b_1, b_2, \ldots, b_k)$ and note that $\langle a, b \rangle = \sum_{i=1}^k \langle a_i, b_i \rangle$ and $a \circ b = (a_1 \circ b_1, a_2 \circ b_2, \ldots, a_k \circ b_k)$. Then, a and b operator commute if and only if a_i and b_i operator commute in $\mathcal{A}_i$ for each i. It follows that a is in the weak center of $\mathcal{A}$ (that is, the FTvN system corresponding to $\mathcal{A}$) if and only if a_i is in the weak center of $\mathcal{A}_i$ for each i. As $\mathcal{A}_i$ is simple, from (ii), the weak center of $\mathcal{A}_i$ is $\mathbb{R}\, e_i$, where e_i is the unit element of $\mathcal{A}_i$. We see that $\mathcal{C}_w$ is the product of $\mathbb{R}\, e_i$, $i = 1, 2, \ldots, k$. In particular, when $\mathcal{V} = \mathbb{R}^n$, we have $\mathcal{V}_i = \mathbb{R}$ with $e_i = 1$. Hence, in this setting, the weak center is $\mathbb{R}^n$.

Remarks. As pointed out by Michael Orlitzky (in a private communication), Item (ii) in the above theorem can also be seen by imitating the proof of Lemma VIII.5.1 in [4].

We have observed before that the composition of two semi-FTvN systems is again a semi-FTvN system. The following example shows that a similar statement fails for FTvN systems, that is, *composition of FTvN systems need not be an FTvN system.*

Example 3.3. *Let $\widetilde{\mathbb{R}^2}$ denote the inner product space consisting of the vector space $\mathbb{R}^2$ with the inner product $\langle x, y \rangle_* := 2\langle x, y \rangle$. (Recall that $\mathbb{R}^2$ carries the usual inner product.) Consider two FTvN systems $(\widetilde{\mathbb{R}^2}, \widetilde{\mathbb{R}^2}, \lambda)$ and $(\widetilde{\mathbb{R}^2}, \mathbb{R}^2, \mu)$, where for any $x = (x_1, x_2)$, we define $\lambda(x) := x^\downarrow$ and $\mu(x) := (x_1 + |x_2|, x_1 - |x_2|)$. In fact, $(\widetilde{\mathbb{R}^2}, \widetilde{\mathbb{R}^2}, \lambda)$ is the scaled version (on $\mathbb{R}^2$) of the FTvN system of Example 2.16 and $(\widetilde{\mathbb{R}^2}, \mathbb{R}^2, \mu)$ is the scaled version of the Jordan spin algebra on $\mathbb{R}^2$, see Example 4.14 in [9]. Let $\nu := \mu \circ \lambda$. Clearly, $(\widetilde{\mathbb{R}^2}, \mathbb{R}^2, \nu)$ is a semi-FTvN system. We claim that $(\widetilde{\mathbb{R}^2}, \mathbb{R}^2, \nu)$ is not an FTvN system. Consider the vectors $c = (1,1)$ and $u = (1, -2)$ so that $\nu(c) = (2, 0)$ and $\nu(u) = (3, -1)$. Suppose $(\widetilde{\mathbb{R}^2}, \mathbb{R}^2, \nu)$ is an FTvN system so that condition (1.4) holds. Then, there is an $x = (x_1, x_2)$ such that $\nu(x) = \nu(u)$ and $\langle c, x \rangle_* = \langle \nu(c), \nu(u) \rangle = 6$. Writing $\lambda(x) = x^\downarrow = (x_1^\downarrow, x_2^\downarrow)$, these translate to $(x_1^\downarrow + |x_2^\downarrow|, x_1^\downarrow - |x_2^\downarrow|) = (3, -1)$ and $x_1 + x_2 = 3$. As $x_1^\downarrow = 1$,*

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$|x_2^{\downarrow}| = 2$, *and* $x_1^{\downarrow} \geq x_2^{\downarrow}$, *we must have* $x^{\downarrow} = (1, -2)$. *But then, the condition* $x_1 + x_2 = 3$ *cannot hold. Thus, (1.4) fails to hold for* ν.

4. The semi-FTvN system induced by a complete hyperbolic polynomial. Going beyond Euclidean Jordan algebras, in this section, we consider hyperbolic systems. Let $\mathcal{V}$ be a nonzero finite-dimensional real vector space, $0 \neq e \in \mathcal{V}$, and p be a real homogeneous polynomial of degree n on $\mathcal{V}$. We say that p *is hyperbolic relative to* e if $p(e) \neq 0$ and for every $x \in \mathcal{V}$, the roots of the univariate polynomial $t \to p(te - x)$ are all real. Given such a polynomial p, for any $x \in \mathcal{V}$, let $\lambda(x)$ denote the vector of roots of the polynomial $t \to p(te - x)$ with entries written in decreasing order (so, $\lambda(x) \in \mathbb{R}^n$). We shall call $\lambda : \mathcal{V} \to \mathbb{R}^n$ the *eigenvalue map* of p (relative to e).

Given the above setting, without imposing additional conditions on p, we list some known properties:

(P1) Proposition 2.2 continues to hold for the eigenvalue map of p (though the proofs are not elementary), see [2].

(P2) For any two elements $x, y \in \mathcal{V}$, there exist real symmetric $n \times n$ matrices A and B such that

$$\lambda(tx + sy) = \lambda(tA + sB) \quad (t, s \in \mathbb{R}), \tag{4.10}$$

where the right-hand side denotes the eigenvalue vector of a symmetric matrix. This is an observation due to Gurvits [11] based on the validity of the Lax conjecture (see [13, 22]).
One important consequence of (4.10) is the validity of the Lidskii inequality (from a similar inequality for real symmetric matrices):

$$\lambda(x) - \lambda(y) \prec \lambda(x - y) \text{ for all } x,\, y \in \mathcal{V}. \tag{4.11}$$

(P3) The set $K := \{x \in \mathcal{V} : \lambda(x) \geq 0\}$ is a closed convex cone with nonempty interior, called the *hyperbolicity cone* corresponding to p and e. For any $d \in K^{\circ}$ (the interior of K), p is hyperbolic relative to d and the corresponding hyperbolicity cone is (still) K. In fact, K° is the connected component of the set $\{x \in \mathcal{V} : p(x) \neq 0\}$ that contains e. These results are shown in [5]. Moreover, from [2], Fact. 2.9, we have

$$K \cap -K = \{x \in \mathcal{V} : \lambda(x) = 0\}. \tag{4.12}$$

Definition 4.1. *([2], Definition 2.8) Let* p *be hyperbolic on* $\mathcal{V}$ *relative to* e. *Let* λ *be the corresponding eigenvalue map. We say that* p *is complete (relative to* e*) if the implication* $\lambda(x) = 0 \Rightarrow x = 0$ *holds.*

Remarks. Let p (hyperbolic on $\mathcal{V}$ relative to e) be complete. In view of (4.12), this completeness property of p is equivalent to (the corresponding hyperbolicity cone) K being pointed (meaning $K \cap -K = \{0\}$). Since K is the hyperbolicity cone of p relative to any $d \in K^{\circ}$, we see that the p is complete relative to any d in K°.

Theorem 4.2. *Suppose* p *is a complete hyperbolic polynomial on* $\mathcal{V}$ *relative to* e. *Let* λ *be the corresponding eigenvalue map. Define*

$$\langle x, y \rangle := \frac{1}{4}\Big\{||\lambda(x+y)||^2 - ||\lambda(x-y)||^2\Big\} \quad (x, y \in \mathcal{V}), \tag{4.13}$$

(where the right-hand side is computed in $\mathbb{R}^n$ *equipped with the usual norm). Then, relative to the above,* $\mathcal{V}$ *becomes a (real, finite-dimensional) inner product space and the triple* $(\mathcal{V}, \mathbb{R}^n, \lambda)$ *becomes a semi-FTvN system. Moreover, the hyperbolicity cone* $\{x \in \mathcal{V} : \lambda(x) \geq 0\}$ *is a proper cone.*

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Proof. That $\langle x, y\rangle$ defines an inner product on $\mathcal{V}$, $\lambda : \mathcal{V} \to \mathbb{R}^n$ is norm-preserving, and $\langle x, y\rangle \leq \langle \lambda(x), \lambda(y)\rangle$ for all $x, y \in \mathcal{V}$, are shown in [2], Theorem 4.2 and Proposition 4.4. Additionally, the pointedness of the hyperbolicity cone comes from (4.12). As K is a pointed closed convex cone with nonempty interior, by definition, it is a proper cone. ∎

Remarks. Suppose p is a complete hyperbolic polynomial on $\mathcal{V}$ and consider the induced semi-FTvN system $(\mathcal{V}, \mathbb{R}^n, \lambda)$ defined above. Then, following [2], Definition 5.1, we say that p is *isometric* if for all $y, z \in \mathcal{V}$, there exists $x \in \mathcal{V}$ such that

$$\lambda(x) = \lambda(z) \quad \text{and} \quad \lambda(x + y) = \lambda(x) + \lambda(y).$$

Then, Proposition 5.3 in [2] shows that when p is complete and isometric, $(\mathcal{V}, \mathbb{R}^n, \lambda)$ satisfies (1.4), hence an FTvN system. In summary,
Every complete hyperbolic polynomial induces a semi-FTvN system and every complete isometric hyperbolic polynomial induces an FTvN system.

We list below a few examples:

Example 4.3. *On $\mathbb{R}^n$ with $x = (x_1, x_2, \ldots, x_n)$, $p(x) := x_1 x_2 \cdots x_n$ is hyperbolic with respect to $e = (1, 1, \ldots, 1)$. Here, $\lambda(x) = x^{\downarrow}$. This p is complete. The semi-FTvN system reduces to the one given in Example 2.16.*

Example 4.4. *On $\mathbb{R}^n$, $n > 1$, $q(x) := x_1^2 - (x_2^2 + x_3^2 + \cdots + x_n^2)$ is hyperbolic with respect to $e = (1, 0, \ldots, 0)$. This q is complete. The induced semi-FTvN system corresponds to the Jordan spin algebra $\mathcal{L}^n$ that appears in the study of Euclidean Jordan algebras.*

Example 4.5. *Let $\mathcal{A}$ be a Euclidean Jordan algebra of rank n and unit element e. Then, $p(x) := \det(x)$ is hyperbolic with respect to e. In this case, the inner product defined in (4.13) reduces to the trace inner product.*

For the rest of this section, we assume that p *is a complete hyperbolic polynomial on $\mathcal{V}$ relative to e with eigenvalue map λ, K is the corresponding hyperbolicity cone, and $(\mathcal{V}, \mathbb{R}^n, \lambda)$ is the induced semi-FTvN system.*

We now record some properties in the semi-FTvN system $(\mathcal{V}, \mathbb{R}^n, \lambda)$ induced by p and e.

- $\lambda(e) = \mathbf{1}$ (the vector of 1s in $\mathbb{R}^n$) and $\lambda(x + re) = \lambda(x) + r\mathbf{1}$ for all $r \in \mathbb{R}$.
- $\lambda(x) = \mathbf{1} \Rightarrow x = e$ (by the completeness of p).
- $\langle x, e\rangle = \langle \lambda(x), \lambda(e)\rangle$ for all $x \in \mathcal{V}$. So, e strongly commutes with every x in $\mathcal{V}$.
- $\mathcal{C} = \mathbb{R}\, e$. As in [9], Example 4.8, this can be seen as follows: From above, $\mathbb{R}\, e \subseteq \mathcal{C}$. If $x \in \mathcal{C}$, then x strongly commutes with $-x$; hence, by Proposition 2.5, $\lambda(x) + \lambda(-x) = \lambda(x + (-x)) = \lambda(0) = 0$. Thus, $\lambda(x) = -\lambda(-x)$. Since the entries of $\lambda(x)$ are decreasing and those of $-\lambda(-x)$ are increasing, $\lambda(x)$ must be a multiple of $\mathbf{1}$. By completeness of p, x is a multiple of e. Hence, $\mathcal{C} = \mathbb{R}\, e$.
- $De = 0$ for every $D \in \mathrm{Lie}(\mathrm{Aut}(\mathcal{V}))$. (We recall that e strongly commutes, hence, commutes with every $x \in \mathcal{V}$.)

One way of interpreting condition (1.4) is: For every $c, u \in \mathcal{V}$, there is an x in the λ-orbit of u that strongly commutes with c. Based on this (and Item (d) in Proposition 2.5), it was shown in [6], Corollary 2.8, that when $(\mathcal{V}, \mathcal{W}, \lambda)$ is an FTvN system, $\lambda(\mathcal{V})$ is a convex cone in $\mathcal{W}$ (additionally closed when $\mathcal{V}$ is finite-dimensional). The following example shows that such a property may not hold when $(\mathcal{V}, \mathcal{W}, \lambda)$ is merely a semi-FTvN system.

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EXAMPLE 4.6. *This example is taken from [2]. Consider the FTvN system* $(\mathbb{R}^3,\mathbb{R}^3,\lambda)$ *with* $\lambda(x)=x^{\downarrow}$. *This corresponds to the complete hyperbolic polynomial* $p(x)=x_1x_2x_3$, *where* $x=(x_1,x_2,x_3)$. *Let* $\mathcal{U}$ *be the span of vectors* $(1,1,1)$ *and* $(3,1,0)$. *For any* $x=\alpha(1,1,1)+\beta(3,1,0)$ *in* $\mathcal{U}$ *with* $\alpha,\beta\in\mathbb{R}$, *we see that* $\lambda(x)=x$ *if* $\beta\geq 0$ *and* $\lambda(x)=\alpha(1,1,1)+\beta(0,1,3)$ *if* $\beta<0$. *As* $(\mathbb{R}^3,\mathbb{R}^3,\lambda)$ *is an FTvN system,* $(\mathcal{U},\mathbb{R}^3,\lambda)$ *is a semi-FTvN system. It has been observed in [2], Example 5.2, that the range of* λ *is not convex, hence* p *(restricted to* $\mathcal{U}$*) is not isometric. Additionally, condition (1.4) fails to hold, e.g., with* $c=(3,1,0)$, $u=-(3,1,0)$, *and* $\lambda(u)=(0,-1,-3)$. *In conclusion,* $(\mathcal{U},\mathbb{R}^3,\lambda)$ *is a semi-FTvN system in which* $\lambda(\mathcal{U})$ *is not convex. This example also shows that a subspace of an FTvN system need not be an FTvN system.*

We end this section with a characterization of strong commutativity in a semi-FTvN system induced by a complete hyperbolic polynomial.

THEOREM 4.7. *Consider the semi-FTvN system induced by a complete hyperbolic polynomial. Then,* a *and* b *strongly commute in this system if and only if*

$$[\lambda(a)-\lambda(b)]^{\downarrow}=\lambda(a-b).$$

Proof. Suppose $[\lambda(a)-\lambda(b)]^{\downarrow}=\lambda(a-b)$. Then, $||\lambda(a)-\lambda(b)||=||\lambda(a-b)||=||a-b||$. By Proposition 2.5, a and b strongly commute.
Now suppose a and b strongly commute so that (by Proposition 2.5) $||\lambda(a)-\lambda(b)||=||a-b||$. By (4.11) and the Hardy–Littlewood–Polya theorem ([3], Theorem II.1.10), there exists a doubly stochastic matrix A such that

$$\lambda(a)-\lambda(b)=A\lambda(a-b).$$

Moreover, by Birkhoff's theorem ([3], Theorem II.2.3), A could be written as a convex combination of permutation matrices. Thus,

$$\lambda(a)-\lambda(b)=\left(\sum_{k=1}^{N}t_kP_k\right)\lambda(a-b),$$

where t_ks are positive with sum 1 and P_ks are permutation matrices. Now,

$$||a-b||=||\lambda(a)-\lambda(b)||=\left\|\left(\sum_{k=1}^{N}t_kP_k\right)\lambda(a-b)\right\|\leq$$

$$\sum_{k=1}^{N}t_k||P_k\big(\lambda(a-b)\big)||=\sum_{k=1}^{N}t_k||\lambda(a-b))||=\sum_{k=1}^{N}t_k||a-b||=||a-b||.$$

By the strict convexity of the (usual) norm, $P_k\lambda(a-b)=P_1\lambda(a-b)$ for all k. Then, $\lambda(a)-\lambda(b)=P_1\big(\lambda(a-b)\big)$, hence, $[\lambda(a)-\lambda(b)]^{\downarrow}=\lambda(a-b)$. This completes the proof. ∎

Remarks. Consider a Euclidean Jordan algebra that corresponds to a complete hyperbolic polynomial. In this setting, the above result says that $[\lambda(a)-\lambda(b)]^{\downarrow}=\lambda(a-b)$ if and only if a and b strongly operator commute. When this is specialized to the algebra of complex Hermitian matrices, we recover a result of Massey et al., see [23], Theorem 2.3.

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5. A commutation principle. In this section, we describe a commutation principle where the focus is on commutativity instead of strong commutativity. Our goal is to formulate a generalization of the following commutation principle of Ramírez et al., from Euclidean Jordan algebras to semi-FTvN settings.

THEOREM 5.1. *(Ramírez, Seeger, and Sossa [24]) Let $\mathcal{V}$ be a Euclidean Jordan algebra, E be a spectral set in $\mathcal{V}$, and $F : \mathcal{V} \to \mathbb{R}$ be a spectral function. Let $\Theta : \mathcal{V} \to \mathbb{R}$ be Fréchet differentiable. If a is a local minimizer/maximizer of the map*

$$x \in E \to \Theta(x) + F(x),$$

then a and $\Theta'(a)$ operator commute.

In [7], Theorem 1.2, Gowda and Jeong extended the above result by assuming that E and F are weakly spectral. Now, we extend this further by working with the automorphism group of a semi-FTvN system. Many such results and their proofs—formulated in the setting of transformation groups—can be found in [10]. We mention the following result for completeness.

We recall that for a map $h : \mathcal{V} \to \mathcal{V}$ and a set $E \subseteq \mathcal{V}$, the *variational inequality problem*, $\mathrm{VI}(h, E)$, is to

$$\text{find } x^* \in E \text{ such that } \langle h(x^*), x - x^* \rangle \geq 0 \text{ for all } x \in E.$$

THEOREM 5.2. *Let $(\mathcal{V}, \mathcal{W}, \lambda)$ be a semi-FTvN system, $\mathcal{G}$ be a closed subgroup of $\mathrm{Aut}(\mathcal{V}, \mathcal{W}, \lambda)$. Let*

- *E be a set in $\mathcal{V}$ that is $\mathcal{G}$-invariant, i.e., $A(E) \subseteq E$ for all $A \in \mathcal{G}$,*
- *$F : \mathcal{V} \to \mathbb{R}$ be $\mathcal{G}$-invariant, i.e., $F(Ax) = F(x)$ for all $A \in \mathcal{G}$ and $x \in \mathcal{V}$,*
- *$\Theta : \mathcal{V} \to \mathbb{R}$ be Fréchet differentiable and $h : \mathcal{V} \to \mathcal{V}$,*
- *$a \in E$ and $c \in \mathcal{V}$.*

Then, the following statements hold:

(i) If a is a local minimizer/maximizer of the map

$$x \in E \to \Theta(x) + F(x),$$

then a commutes with $\Theta'(a)$ relative to $\mathcal{G}$.

(ii) If a is an optimizer of

$$\max_{x\in E}/\min \quad \langle c, x \rangle,$$

then a commutes with c relative to $\mathcal{G}$;

(iii) If a is a solution of the variational inequality problem $\mathrm{VI}(h, E)$, then a commutes with $h(a)$ relative to $\mathcal{G}$.

In the proof below, for any $D \in \mathrm{Lie}(\mathcal{G})$, we require the continuity and differentiability of the exponential function $t \mapsto \exp(\mathrm{tD})$ at $t = 0$. When $\mathcal{V}$ is a Hilbert space (in which case, $\mathrm{L}(\mathcal{V})$ is a Banach algebra), this is a consequence of the well-known result that, on a Banach algebra, the exponential function is Fréchet differentiable. The following lemma shows that these two properties hold in any inner product space $\mathcal{V}$.

LEMMA 5.3. *Suppose $D \in \mathrm{L}(\mathcal{V})$ such that $\exp(\mathrm{tD}) := \sum_{k=0}^{\infty} \frac{t^k D^k}{k!}$ converges in $\mathrm{L}(\mathcal{V})$ for every $t \in \mathbb{R}$. Then the mapping $t \mapsto \exp(\mathrm{tD})$ is Fréchet differentiable (hence continuous) at $t = 0$.*

Proof. Let $\overline{\mathcal{V}}$ denote the completion of (the inner product space) $\mathcal{V}$. Then, for any $T \in \mathrm{L}(\mathcal{V})$, there is a unique extension $\overline{T} \in \mathrm{L}(\overline{\mathcal{V}})$; moreover, the (linear) mapping $T \mapsto \overline{T}$ is norm-preserving, i.e., $||T||_{\mathcal{V}} = ||\overline{T}||_{\overline{\mathcal{V}}}$. Hence,

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$$\left\|\sum_{k=0}^{N}\frac{t^kD^k}{k!}-\exp(\mathrm{tD})\right\|_{\mathcal{V}}=\left\|\sum_{k=0}^{N}\frac{t^k\overline{D}^k}{k!}-\overline{\exp(\mathrm{tD})}\right\|_{\overline{\mathcal{V}}}.$$

Since $\sum_{k=0}^{\infty}\frac{t^kD^k}{k!}$ converges to $\exp(\mathrm{tD})\in \mathrm{L}(\mathcal{V})$ for every t, letting $N\to\infty$ in the above, we see that $\overline{\exp(\mathrm{tD})}=\exp(\mathrm{t}\overline{\mathrm{D}})\in \mathrm{L}(\overline{\mathcal{V}})$. Now in the Banach algebra $\mathrm{L}(\overline{\mathcal{V}})$, $t\mapsto \exp(t\overline{D})$ is differentiable at $t=0$ with derivative $\overline{D}$. Hence,

$$0=\lim_{t\to 0}\left|\left|\frac{\exp(\mathrm{t}\overline{\mathrm{D}})-I}{t}-\overline{D}\right|\right|_{\overline{\mathcal{V}}}=\lim_{t\to 0}\left|\left|\frac{\exp(\mathrm{tD})-I}{t}-D\right|\right|_{\mathcal{V}}.$$

This shows that $\exp(\mathrm{tD})$ is differentiable at $t=0$ with derivative D. $\square$

We now come to the proof of the theorem.

Proof. This is a shortened version of proof given in [10]; it is provided here for completeness.
(i) We (only) consider the case when a is a local minimizer of $\Theta+F$ over E; for the other case, we consider $-(\Theta+F)$. Then, we have

$$\Theta(a)+F(a)\le\Theta(x)+F(x)\text{ for all }x\in B_a\cap E,$$

where B_a denotes some open ball around a. Let $D\in\mathrm{Lie}(\mathcal{G})$ so that for all $t\in\mathbb{R}$, $\exp(tD)\in\mathcal{G}$. As E and F are $\mathcal{G}$-invariant, we have $\exp(tD)a\in E$ and $F(\exp(tD)a)=F(a)$ for all $t\in\mathbb{R}$. Moreover, by the continuity of $\exp(tD)$ in t, $\exp(tD)a\in B_a\cap E$ for all t close to zero. For such t,

$$\Theta(a)+F(a)\le\Theta\big(\exp(tD)a\big)+F\big(\exp(tD)a\big)=\Theta\big(\exp(tD)a\big)+F(a).$$

Then, $\Theta(a)\le\Theta(\exp(tD)a)$ for all t close to 0; consequently, $t=0$ is a local minimizer of the map

$$t\in\mathbb{R}\mapsto\phi(t):=\Theta(\exp(tD)a).$$

As Θ is Fréchet differentiable,

$$0=\phi'(0)=\langle Da,\Theta'(a)\rangle.$$

As this holds for all $D\in\mathrm{Lie}(\mathcal{G})$, we see that a commutes with $\Theta'(a)$ relative to $\mathcal{G}$.
(ii) This follows from (i) by putting $F=0$ and $\Theta(x):=\langle c,x\rangle$.
(iii) Suppose a solves $\mathrm{VI}(h,E)$. Then, a minimizes $\langle h(a),x\rangle$ over E. Now the stated assertion follows from (ii).

Concluding remarks. In this paper, we introduced two commutativity concepts in the setting of semi-FTvN systems and demonstrated their appearance as optimality conditions in certain optimization problems. Additionally, we have shown that these concepts reduce to the well-known operator commutativity concepts in the setting of Euclidean Jordan algebras.

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