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LAPLACIAN SPECTRAL RADIUS AND INTEGRALITY OF TOKEN GRAPHS*

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Abstract. Let G be a simple graph on n vertices. A graph is a cograph if and only if it contains no induced path on four vertices. The k-token graph $F_k(G)$ of G is the graph whose vertices are the k-subsets of $V(G)$, and two of them are adjacent whenever their symmetric difference is a pair of adjacent vertices in G. It is known that the algebraic connectivity (the second smallest Laplacian eigenvalue) of G is equal to that of $F_k(G)$. Motivated by this, Barik and Verma [Linear Algebra Appl., 687:181–206, (2024)] posed the following question: for which graphs does the Laplacian spectral radius $\rho_L(G)$ (the largest Laplacian eigenvalue) of G equal the Laplacian spectral radius $\rho_L(F_k(G))$ of $F_k(G)$? In this article, we answer this question by proving that $\rho_L(F_k(G)) = \rho_L(G)$ if and only if G is a star graph. We also ask the following question: if G is Laplacian integral (all its Laplacian eigenvalues are integers) on n vertices, then is $F_k(G)$ also Laplacian integral for any integer k such that $2 \leq k \leq \frac{n}{2}$? We prove that this is not true in general by proving that $F_k(K_n \otimes K_2)$ is not Laplacian integral for $n \geq 3$ and $2 \leq k \leq n$, even though $K_n \otimes K_2$ itself is Laplacian integral. Here, $K_n \otimes K_2$ denotes the Kronecker product of graphs K_n and K_2. Interestingly, we prove that the token graphs of cographs are Laplacian integrals, thereby showing that the token graph operation can generate new Laplacian integral graphs.

Key words. Token graph, Cograph, Laplacian eigenvalue, Laplacian integral graph, Laplacian spectral radius.

AMS subject classifications. 05C50, 05C76.

1. Introduction. Let $G = (V(G), E(G))$ be a simple, finite, and undirected graph with vertex set $V(G) = \{v_1, \ldots, v_n\}$ and edge set $E(G)$. If two vertices v_i and v_j are adjacent in G, we write $v_i \sim v_j$ and say $v_iv_j \in E(G)$. The degree of a vertex v is denoted by $d(v)$, and its *neighborhood* $N(v) = \{u : uv \in E(G)\}$. The maximum degree of G is $\Delta(G) = \max_{v \in V(G)} |N(v)|$. A graph is called *r-regular* if every vertex has degree r. A graph G is *bipartite* if its vertex set can be partitioned into two independent sets. If $G = (V_1, V_2, E)$ is bipartite and every vertex in V_1 has degree r while every vertex in V_2 has degree s, then G is called a *semiregular bipartite graph*. We use S_n, C_n, $K_{p,q}$ $(p + q = n)$, and K_n to denote the star, cycle, complete bipartite graph, and complete graph on n vertices, respectively. The double star graph S_{n_1,n_2} is obtained by taking two stars S_{n_1} and S_{n_2} and then joining their central vertices by a new edge.

The *Laplacian matrix* of G is defined as $L(G) = D(G) - A(G)$, where $A(G)$ is the $(0, 1)$ *adjacency matrix* of G and $D(G)$ is the diagonal matrix of the vertex degrees. Since $L(G)$ is a symmetric and positive semidefinite matrix, all its eigenvalues are real and nonnegative, and 0 is always an eigenvalue with eigenvector **1**. The Laplacian spectrum of G is denoted by $S(G) = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, where $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. In [13], Fiedler proved that $\lambda_2 > 0$ *if and only if* G *is connected.* This led Fiedler to define λ_2 as the *algebraic connectivity* of G, viewing it as a quantitative measure of connectivity. We denote the algebraic connectivity of G by $a(G)$. The largest eigenvalue of $L(G)$ is called the *Laplacian spectral radius*, and it is denoted by $\rho_L(G) = \lambda_n$. A graph is called *Laplacian integral* if all the Laplacian eigenvalues of G are integers.

Let G be a graph on n vertices and let k be an integer such that $1 \leq k \leq n$. The *k-token graph* $F_k(G)$ of G is defined as the graph whose vertices are the $\binom{n}{k}$ k-subsets of $V(G)$. Two vertices $A, B \in V(F_k(G))$ are adjacent whenever their symmetric difference $A \triangle B = \{a, b\}$, such that $a \in A, b \in B$, and $ab \in E(G)$ (for an

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example, see Fig. 2). Since $F_1(G) \cong G$ and $F_k(G) \cong F_{n-k}(G)$, we usually restrict to $2 \leq k \leq \frac{n}{2}$. It is easy to verify that $|E(F_k(G))|=\binom{n-2}{k-1}\ |E(G)|$ and that $F_k(G)$ is connected if and only if G is connected.

Fabila-Monroy, Flores-Peñaloza, Huemer, Hurtado, Urrutia, and Wood [12] introduced token graphs as a way to model configurations of k indistinguishable tokens placed on distinct vertices of G, where two configurations are adjacent if one configuration can be obtained from the other by moving a single token along an edge from its current position to an unoccupied vertex. In the same article, the authors conducted a detailed study of several combinatorial parameters of token graphs, such as connectivity, diameter, cliques, chromatic number, Hamiltonian paths, and Cartesian products. They also proved that G is bipartite if and only if $F_k(G)$ is bipartite.

In [2], Audenaert, Godsil, Royle, and Rudolph defined k-token graphs as the k-th symmetric powers of graphs and showed a connection between them and the exchange of Hamiltonian operators in quantum mechanics. It has been proved that the interaction Hamiltonian of n qubit system, H_{int}, can be expressed as a direct sum of the adjacency matrices of token graphs, that is, $H_{int} = \bigoplus_{k=1}^{n} A(F_k(G))$. Similarly, for the Heisenberg ferromagnetic Hamiltonian, Ouyang [18] showed that the normalized Heisenberg Hamiltonian $\hat{H}_1$ can be written as a direct sum of the Laplacian matrices of token graphs. Thus, token graphs provide a combinatorial framework that links graph theory with quantum Hamiltonians.

Dalfó, Duque, Fabila-Monroy, Fiol, Huemer, Trujillo-Negrete, and Zaragoza Martínez [9] studied the Laplacian spectra of token graphs and conjectured that $a(F_k(G)) = a(G)$ for all k. This conjecture is a special case of Aldous' spectral gap conjecture, which was proved by Caputo, Liggett, and Richthammer [7]. Aldous' spectral gap conjecture was originally motivated by random walks and the interchange process on graphs. More recently, Barik and Verma [4] and Reyes, Dalfó, and Fiol [19] studied the adjacency and Laplacian spectra of $F_k(G)$ in relation to the spectra of G. In particular, Barik and Verma [4] posed the problem of characterizing all graphs for which $\rho_L(F_k(G)) = \rho_L(G)$. In this article, we present a complete solution to this problem.

We also consider cographs and graph products, as they play an important role in our analysis. The *complement* of a graph G, denoted by G^c, is the graph with the same vertex set where two vertices are adjacent in G^c if and only if they are not adjacent in G. If G_1 and G_2 are graphs with disjoint vertex sets, their union $G_1 \cup G_2$ has vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. Their join $G_1 \vee G_2$ is obtained from $G_1 \cup G_2$ by adding all edges between $V(G_1)$ and $V(G_2)$. A graph G is a *cograph* if it contains no induced path on four vertices (see Corneil, Lerchs, and Burlingham [8]). Alternatively, cographs can be constructed from isolated vertices by a sequence of operations of union and join. Since connected cographs are formed through the join operation, their complements are disconnected. Merris [16] proved that every cograph is Laplacian integral. If $G \ncong C_4$ is a connected graph, then for $k \geq 2$, $F_k(G)$ is not a cograph (see Barik and Verma [4]). This raises the question of whether $F_k(G)$ remains Laplacian integral when G is a cograph.

Graph products provide a powerful tool for constructing new classes of graphs. For graphs G and H with disjoint vertex sets $V(G) = \{u_1, \ldots, u_m\}$ and $V(H) = \{v_1, \ldots, v_n\}$, respectively. The *Cartesian product* $G \square H$ has vertex set $V(G) \times V(H)$, where (u_i, v_j) is adjacent to (u_r, v_s) whenever either $u_i = u_r$ and $v_j \sim v_s$ in H or $u_i \sim u_r$ in G and $v_j = v_s$. The *Kronecker product* (or direct product) $G \otimes H$ also has vertex set $V(G) \times V(H)$, with (u_i, v_j) adjacent to (u_r, v_s) if $u_i \sim u_r$ in G and $v_j \sim v_s$ in H. Weichsel [21] proved that $G \otimes H$ is connected if and only if both G and H are connected and at least one of them is nonbipartite.

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Merris [17] characterized the Laplacian spectrum of $G\Box H$ in terms of the Laplacian spectra of G and H. Barik, Bapat, and Pati [3] expressed the Laplacian spectrum of $G \otimes H$ in terms of the Laplacian spectra of G and H, but only when the graphs G and H are regular. Let G and H be the graphs on n_1 and n_2 vertices, respectively. Suppose their Laplacian spectra are given by $S(G) = (\lambda_1, \lambda_2, \ldots, \lambda_{n_1})$ and $S(H) = (\mu_1, \mu_2, \ldots, \mu_{n_2})$. Then, the Laplacian spectrum of $G_1\Box G_2$ is $\{\lambda_i + \mu_j\}$, for $i \in \{1, \ldots, n_1\}$ and $j \in \{1, \ldots, n_2\}$. If G and H are regular graphs with degrees r and s, respectively, then the Laplacian spectrum of $G \otimes H$ is $\{s\lambda_i + r\mu_j - \lambda_i\mu_j\}$, for $i \in \{1, \ldots, n_1\}$ and $j \in \{1, \ldots, n_2\}$. These results imply that if G and H are Laplacian integrals, then $G\Box H$ is Laplacian integral; moreover, if G and H are regular, then $G \otimes H$ is also Laplacian integral.

Grone and Merris [14] observed that the Laplacian matrices are more likely to have integer eigenvalues than the adjacency matrices, motivating the study of Laplacian integer graphs (see, for instance, Kirkland, De Freitas, Del Vecchio, and De Abreu [15] and the references therein). In this article, we investigate whether the k-token graph of a Laplacian integral graph must also be Laplacian integral. We prove that this property holds for cographs but fails in general by exhibiting a family of counterexamples.

The main objective of this work is to establish a connection between the Laplacian spectrum of G and that of its k-token graphs. In particular, we study the Laplacian spectral radius and Laplacian integrality of $F_k(G)$. The article is organized as follows. In Section 2, we list some known results that will be used in our proofs. Section 3 contains our main results: we prove that $\rho_L(F_k(G)) = \rho_L(G)$ if and only if $G \cong S_n$. Further, we show that $F_k(G)$ is Laplacian integral whenever G is a cograph and prove that for $n \geq 3$ and $2 \leq k \leq n$, $F_k(K_n \otimes K_2)$ is not Laplacian integral.

2. Preliminaries. This section contains several well-known results that will be used in the sequel. We begin with the classical bounds on the Laplacian spectral radius $\rho_L(G)$ of a graph G in terms of its vertex degrees.

Theorem 2.1. (Anderson and Morley [1], Theorem 2) *Let G be a connected graph on n vertices. Then,*

$$\rho_L(G) \leq \max\{d(u) + d(v) \colon uv \in E(G)\},$$

with equality if and only if G is semiregular bipartite graph.

Das [11] improved the bound as follows.

Theorem 2.2. (Das [11], Theorem 2.1) *Let G be a connected graph on n vertices. Then,*

$$\rho_L(G) \leq \max\{d(u) + d(v) - |N(u) \cap N(v)| \colon uv \in E(G)\}.$$

Grone and Merris [14] obtained the following lower bound, and Zhang and Luo [22] later characterized the case of equality.

Lemma 2.3. (Grone and Merris [14], Zhang and Luo [22]) *Let G be a connected graph on n vertices. Then,*

$$\rho_L(G) \geq \Delta(G) + 1,$$

with equality if and only if $\Delta(G) = n - 1$.

A subgraph of G is called a spanning subgraph if it contains all the vertices of G but possibly only some of the edges. The following result is an immediate consequence of Theorem 4.1 of Grone and Merris [14].

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Lemma 2.4. (Grone and Merris [14]) *Let G be a graph and G' be a spanning subgraph of G. Then,*

$$\rho_L(G') \leq \rho_L(G).$$

Dalfó, Duque, Fabila-Monroy, Fiol, Huemer, Trujillo-Negrete, and Zaragoza Martínez [9] studied the Laplacian spectra of token graphs and proved the following result, which shows an inclusion property of Laplacian eigenvalues of token graphs.

Theorem 2.5. (Dalfó, Duque, Fabila-Monroy, Fiol, Huemer, Trujillo-Negrete, and Zaragoza Martínez [9], Theorem 4.1) Let G be a graph on n vertices and h, k be integers such that $1 \leq h \leq k \leq \frac{n}{2}$. If λ is an eigenvalue of $L(F_h(G))$, then λ is also an eigenvalue of $L(F_k(G))$.

For any graph G on n vertices, there exists a well-known relationship between Laplacian eigenvalues of G and G^c, given by $\lambda_i(G) + \lambda_{n+2-i}(G^c) = n$, for $i = 2, \ldots, n$. Similarly, Dalfó, Duque, Fabila-Monroy, Fiol, Huemer, Trujillo-Negrete, and Zaragoza Martínez [9] also related the Laplacian spectra of $F_k(G)$ and $F_k(G^c)$.

Theorem 2.6. (Dalfó, Duque, Fabila-Monroy, Fiol, Huemer, Trujillo-Negrete, and Zaragoza Martínez [9], Theorem 6.2) Let G be a connected graph on n vertices and G^c be its complement. For any integer k such that $1 \leq k \leq \frac{n}{2}$, consider the token graphs $F_k(G)$ and $F_k(G^c)$. Then, the Laplacian spectrum of $F_k(G^c)$ is the complement of the Laplacian spectrum of $F_k(G)$ with respect to the Laplacian spectrum of $F_k(K_n)$. That is, every eigenvalue $\lambda(F_k(K_n))$ of $F_k(K_n)$ is the sum of one eigenvalue $\lambda(F_k(G))$ of $F_k(G)$ and one eigenvalue $\lambda(F_k(G^c))$ of $F_k(G^c)$, where each $\lambda(F_k(G))$ and $\lambda(F_k(G^c))$ is used once:

$$\lambda(F_k(G)) + \lambda(F_k(G^c)) = \lambda(F_k(K_n)).$$

The following result follows from Lemmas 6.4 and 6.5 of Barik and Verma [4].

Lemma 2.7. (Barik and Verma [4]) *Let G be a connected graph on $n \geq 4$ vertices and let v be a vertex in G such that $d(v) = n - 1$. Then, for any integer k such that $2 \leq k \leq \frac{n}{2}$,*

$$\rho_L(F_k(G)) \geq \rho_L(G) = n,$$

with equality if and only if $G \cong S_n$.

The degree of a vertex in $F_k(G)$ can be expressed in terms of degrees of vertices in G as follows (see Barik and Verma [4] and Dalfó, Fiol, and Messegué [10]).

Lemma 2.8. *Let G be a graph on n vertices. Then, for any vertex $A = \{v_1, v_2, \ldots, v_k\} \in V(F_k(G))$,*

$$d(A) = \sum_{i=1}^{k} d(v_i) - 2t,$$

where t is the number of adjacent pairs among $\{v_1, v_2, \ldots, v_k\}$ in G.

The following remark can be found in [4].

Remark 2.9. *Let G be a connected graph on n vertices. Then, for any integer k such that $2 \leq k \leq \frac{n}{2}$, $\Delta(F_k(G)) < \binom{n}{k} - 1$. Thus, $F_k(G)$ cannot be a complete graph.*

We will also need the well-known Bolzano theorem.

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THEOREM 2.10. (Bolzano [5]) *Let $f(x)$ be a continuous function on $[a,b]$ such that $f(a)$ and $f(b)$ have opposite signs. Then, there exists at least one real number c in (a,b) such that $f(c)=0$.*

Let S be an $n\times n$ matrix of the form

$$S=\begin{pmatrix} S_{11} & S_{12} & \cdots & S_{1m}\\ S_{21} & S_{22} & \cdots & S_{2m}\\ \vdots & \vdots & \ddots & \vdots\\ S_{m1} & S_{m2} & \dots & S_{mm}\end{pmatrix},$$

where rows and columns are partitioned into subsets $\pi=\{Y_1,\dots,Y_m\}$ of $\{1,\dots,n\}$. The *quotient matrix* Q associated with π is the $m\times m$ matrix whose entries are the average row sum of the blocks S_{ij} of S. More precisely, the (i,j)-the entry of Q is

$$q_{ij}=\frac{\mathbf{1}^T S_{ij}\mathbf{1}}{|Y_i|}.$$

The partition is called an *equitable* partition if each block S_{ij} of S has constant row sum (see Brouwer and Haemers [6]). Equitable partitions are significant in spectral graph theory because they allow us to estimate the eigenvalues of large matrices via a smaller quotient matrix.

If $L(G)$ is the Laplacian matrix of G and Q_L is its quotient Laplacian matrix with respect to an equitable partition π, then the characteristic polynomial of Q_L divides the characteristic polynomial of $L(G)$. Furthermore, if G is bipartite, then the spectral radius of Q_L equals the spectral radius of $L(G)$ (see Reyes, Dalfó, Fiol, and Messegué [20]).

3. Main results. In this section, we characterize all graphs for which the equality $\rho_L(F_k(G))=\rho_L(G)$ holds, thus resolving a problem posed by Barik and Verma [4].

From Theorem 2.5, we have

$$S(G)\subset S(F_2(G))\subset\cdots\subset S(F_{\lfloor\frac{n}{2}\rfloor}(G)).$$

Consequently, the Laplacian spectral radii satisfy

$$\rho_L(G)\le\rho_L(F_2(G))\le\cdots\le\rho_L(F_{\lfloor\frac{n}{2}\rfloor}(G)).$$

We begin with a lemma that provides a lower bound for $\rho_L(F_2(G))$ in terms of the maximum degree sum of an edge of G under a specific condition.

LEMMA 3.1. *Let $G\ncong S_n$ be a connected graph on n vertices and let $u,v\in V(G)$ be adjacent vertices such that $N(u)\cap N(v)=\phi$ and $d(u)+d(v)=\max\{d(r)+d(s)\colon rs\in E(G)\}$. Then,*

$$\rho_L(F_2(G))>d(u)+d(v).$$

Proof. Let $d(u)=t+1$ and $d(v)=s+1$ with neighborhoods

$$N(u)=\{v,u_1,\dots,u_t\}\text{ and }N(v)=\{u,v_1,\dots,v_s\}.$$

Since $G\ncong S_n$ is a connected graph and $N(u)\cap N(v)=\phi$, the double star graph $S_{t+1,s+1}$ (possibly with isolated vertices) is a spanning subgraph of G, where $t,s\ge 1$. Hence, $F_2(S_{t+1,s+1})$ along with isolated vertices is a spanning subgraph of $F_2(G)$.

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If $t = 1$ and $s = 1$, then $F_2(S_{2,2})$ together with isolated vertices is a spanning subgraph of $F_2(G)$. By direct calculation,

$$\rho_L(F_2(S_{2,2})) \approx 4.7321 > 2 + 2 = d(u) + d(v).$$

Similarly, if $t = 2$ and $s = 1$, then $F_2(S_{3,2})$ together with isolated vertices is a spanning subgraph of $F_2(G)$. A direct calculation gives

$$\rho_L(F_2(S_{3,2})) \approx 5.7093 > 3 + 2 = d(u) + d(v).$$

Now assume $t, s \geq 2$. Let $\pi = \{S_1, S_2, S_3, S_4, S_5, S_6, S_7, S_8\}$ be a partition of $V(F_2(S_{t+1,s+1}))$ defined by

$$\begin{aligned}
&S_1 = \{\{u, v\}\}, \qquad S_2 = \{\{u, v_i\} \colon i \in \{1, \dots, s\}\}, \qquad S_3 = \{\{v, u_i\} \colon i \in \{1, \dots, t\}\},\\
&S_4 = \{\{u, u_i\} \colon i \in \{1, \dots, t\}\}, \qquad S_5 = \{\{v, v_i\} \colon i \in \{1, \dots, s\}\}, \qquad S_6 = \{\{u_i, u_j\} \colon i, j \in \{1, \dots, t\}, i < j\}\\
&S_7 = \{\{v_i, v_j\} \colon i, j \in \{1, \dots, s\}, i < j\}, \qquad S_8 = \{\{u_i, v_j\} \colon i \in \{1, \dots, t\},\ j \in \{1, \dots, s\}\}.
\end{aligned}$$

Each set S_i is nonempty for $i = 1, \dots, 8$. By Lemma 2.8, all vertices within each set S_i have the same degree. From Table 1, one can observe that a vertex in S_1 has s neighbors in S_2 and t neighbors in S_3. Similarly, a vertex in S_2 has t neighbors in S_8 along with one neighbor in both S_1 and S_2. The adjacency structure for vertices in the remaining sets $S_3, \dots, S_8$ follows analogously.

Table 1
Adjacency relation between vertex pairs in the partition π of $V(F_2(S_{t+1,s+1}))$

S.No.	**Vertex** A	$N(A)$
1	$\{u, v\} \in S_1$	$S_2 \cup S_3$
2	$\{u, v_r\} \in S_2$	$\{\{u, v\}, \{v, v_r\}, \{u_1, v_r\}, \dots, \{u_t, v_r\}\}$
3	$\{v, u_r\} \in S_3$	$\{\{u, v\}, \{u, u_r\}, \{v_1, u_r\}, \dots, \{v_s, u_r\}\}$
4	$\{u, u_r\} \in S_4$	$\{\{v, u_r\}, \{u_1, u_r\}, \dots, \{u_{r-1}, u_r\}, \{u_{r+1}, u_r\}, \dots, \{u_t, u_r\}\}$
5	$\{v, v_r\} \in S_5$	$\{\{u, v_r\}, \{v_1, v_r\}, \dots, \{v_{r-1}, v_r\}, \{v_{r+1}, v_r\}, \dots, \{v_s, v_r\}\}$
6	$\{u_p, u_r\} \in S_6$	$\{\{u, u_r\}, \{u_p, u\}\}$
7	$\{v_p, v_r\} \in S_7$	$\{\{v, v_r\}, \{v_p, v\}\}$
8	$\{u_p, v_r\} \in S_8$	$\{\{u, v_r\}, \{u_p, v\}\}$

The quotient Laplacian matrix Q_L for the partition π is

$$Q_L = \begin{pmatrix}
t+s & -s & -t & 0 & 0 & 0 & 0 & 0\\
-1 & t+2 & 0 & 0 & -1 & 0 & 0 & -t\\
-1 & 0 & s+2 & -1 & 0 & 0 & 0 & -s\\
0 & 0 & -1 & t & 0 & -t+1 & 0 & 0\\
0 & -1 & 0 & 0 & s & 0 & -s+1 & 0\\
0 & 0 & 0 & -2 & 0 & 2 & 0 & 0\\
0 & 0 & 0 & 0 & -2 & 0 & 2 & 0\\
0 & -1 & -1 & 0 & 0 & 0 & 0 & 2
\end{pmatrix}.$$

Since π is equitable and $F_2(S_{t+1,s+1})$ is bipartite, the largest eigenvalue of Q_L is $\rho_L(F_2(S_{t+1,s+1}))$. By direct calculation, the characteristic polynomial of Q_L is $\phi(x) = |xI - Q_L| = x(x^3 - (t+s+4)x^2 + (2t + 2s + st +$

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$5)x - t - s - 2)(x^4 - (2t + 2s + 6)x^3 + (t^2 + s^2 + 13 + 3st + 8t + 8s)x^2 - (2t^2 + 2s^2 + 12 + 6st + 12t + 12s + st^2 + s^2t)x + 2t^2 + 2s^2 + 4 + 4st + 6t + 6s)$. Evaluating $\phi(x)$ at specific points gives

$$\phi(t + s + 2) = -st(s + t)(s + t + 2)^3 < 0,$$

$$\phi(t + s + 3) = (t + s + 3)((s + t + 2)^2 + st(s + t + 3))(st(s + t + 3) + 4) > 0.$$

Hence, by Theorem 2.10, there exists a root of $\phi(x)$ in $(s + t + 2, s + t + 3)$, implying

$$\rho_L(F_2(S_{t+1,s+1})) > s + t + 2 = d(u) + d(v).$$

Since $F_2(S_{t+1,s+1})$ along with isolated vertices is a spanning subgraph of $F_2(G)$, Lemma 2.4 yields, for all $s, t \geq 1$,

$$\rho_L(F_2(G)) \geq \rho_L(F_2(S_{t+1,s+1})) > d(u) + d(v).$$

This completes the proof. ∎

Theorem 3.2. *Let $G \ncong S_n$ be a connected graph on $n \geq 4$ vertices. Then, for any integer k such that $2 \leq k \leq \frac{n}{2}$,*

$$\rho_L(F_k(G)) > \rho_L(G).$$

Proof. Let $\mathcal{G}_n$ be the set of connected semiregular bipartite graphs on n vertices. Since $\rho_L(G) \leq \rho_L(F_2(G)) \leq \cdots \leq \rho_L(F_{\lfloor \frac{n}{2} \rfloor}(G))$, it suffices to prove $\rho_L(F_2(G)) > \rho_L(G)$.

We consider the main cases.

Case 1. If $G \in \mathcal{G}_n \setminus S_n$. Let V_1 and V_2 be two partite sets of G, with degrees r and s, respectively ($r \geq s$). By Theorem 2.1,

$$\rho_L(G) = r + s.$$

Since $G \ncong S_n$, we have $|V_1|, |V_2| \geq 2$. Choosing $u, v \in V_1$, Lemma 2.8 gives

$$\Delta(F_2(G)) = d(\{u, v\}) = d(u) + d(v) - 2 \times 0 = 2r$$

and from Remark 2.9, we have $\Delta(F_2(G)) \neq \binom{n}{2} - 1$. Applying Lemma 2.3,

$$\rho_L(F_2(G)) > \Delta(F_2(G)) + 1 = 2r + 1 \geq r + s + 1.$$

Thus, we have $\rho_L(F_2(G)) > \rho_L(G)$.

Case 2. If $G \notin \mathcal{G}_n$. Let $u, v \in E(G)$ such that $d(u) + d(v)$ is the maximum among all edges.

Subcase 2.1. If $N(u) \cap N(v) \neq \phi$, then from Theorem 2.2,

$$\rho_L(G) \leq d(u) + d(v) - 1.$$

Lemma 2.8 gives $\Delta(F_2(G)) \geq d(\{u, v\}) = d(u) + d(v) - 2$. Now, from Remark 2.9 and Lemma 2.3, we get

$$\rho_L(F_2(G)) > \Delta(F_2(G)) + 1 \geq d(u) + d(v) - 1.$$

Thus, $\rho_L(F_2(G)) > \rho_L(G)$.

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Subcase 2.2. If $N(u) \cap N(v) = \phi$, then from Theorem 2.1, we have

$$\rho_L(G) < d(u) + d(v).$$

Since $d(u) + d(v)$ is maximum among all adjacent vertices in G and $N(u) \cap N(v) = \phi$, from Lemma 3.1,

$$\rho_L(F_2(G)) > d(u) + d(v).$$

Hence, $\rho_L(F_2(G)) > \rho_L(G)$.

Thus, in all cases, $\rho_L(F_k(G)) > \rho_L(G)$, and hence $\rho_L(F_k(G)) > \rho_L(G)$, for any integer k such that $2 \leq k \leq \frac{n}{2}$. □

The following result characterizes graphs that satisfy $\rho_L(F_k(G)) = \rho_L(G)$, thereby resolving the problem posed in [4].

Corollary 3.3. *Let G be a connected graph on n vertices. Then, for any integer k such that $2 \leq k \leq \frac{n}{2}$,*

$$\rho_L(F_k(G)) = \rho_L(G),$$

if and only if $G \cong S_n$.

Proof. By Theorem 3.2, for any connected graph $G \ncong S_n$, we have

$$\rho_L(F_k(G)) > \rho_L(G).$$

Moreover, by Lemma 2.7, if G is a connected graph on n vertices with at least one vertex of degree $n-1$, then $\rho_L(F_k(G)) \geq \rho_L(G)$ and equality holds if and only if $G \cong S_n$. Hence, the proof is complete. □

Since $a(F_k(S_n)) = a(S_n)$ (see Dalfó, Duque, Fabila-Monroy, Fiol, Huemer, Trujillo-Negrete, and Zaragoza Martínez [9]), we have the following result that follows from Corollary 3.3.

Corollary 3.4. *Let G be a connected graph on n vertices. Then, for any integer k such that $2 \leq k \leq \frac{n}{2}$,*

$$a(F_k(G)) = a(G) \quad \text{and} \quad \rho_L(F_k(G)) = \rho_L(G),$$

if and only if $G \cong S_n$.

The following observation follows from Theorem 10 in Fabila-Monroy, Flores Peñaloza, Huemer, Hurtado, Urrutia, and Wood [12].

Observation 3.5. *Let G have t connected components $H_1, \ldots, H_t$. Then, $F_k(G)$ has at least t connected components. In fact, each component is a Cartesian product of the form*

$$F_{k_1}(H_1) \square \cdots \square F_{k_t}(H_t),$$

where $\sum_{i=1}^{t} k_i \leq k$ and $k_i \leq |V(H_i)|$ for $i \in \{1, \ldots, t\}$.

The graph $K_4 - e$ is defined as the graph obtained by deleting an edge e from K_4. In Table 2, we list the Laplacian spectra of all connected cographs on 4 vertices together with the Laplacian spectra of their 2-token graph. This verifies that the 2-token graph of any connected cograph on 4 vertices is Laplacian integral.

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TABLE 2
Laplacian spectra of all connected cographs on $n = 4$ vertices and their 2-token graphs

S.No.	**Graph** G	$S(G)$	$S(F_2(G))$
1	S_4	$(0, 1, 1, 4)$	$(0, 1, 1, 3, 3, 4)$
2	C_4	$(0, 2, 2, 4)$	$(0, 2, 2, 2, 4, 6)$
3	U (Fig. 1)	$(0, 1, 3, 4)$	$(0, 1, 3, 3, 4, 5)$
4	$K_4 - e$	$(0, 2, 4, 4)$	$(0, 2, 4, 4, 4, 6)$
5	K_4	$(0, 4, 4, 4)$	$(0, 4, 4, 4, 6, 6)$

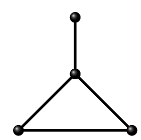

FIGURE 1. *Graph U.*

Characterizing Laplacian integral graphs is a fundamental problem in spectral graph theory. This motivates the following natural question: If G is Laplacian integral on n vertices, is $F_k(G)$ also Laplacian integral for any integer k such that $2 \leq k \leq \frac{n}{2}$? We answer this question affirmatively for the class of cographs. Moreover, we prove that the k-token graph of $K_n \otimes K_2$ is not Laplacian integral.

For a complete graph K_n, the Laplacian eigenvalues of $F_k(K_n)$ are of the form $j(n-j+1)$ with multiplicity $\binom{n}{j} - \binom{n}{j-1}$, for $j \in \{0, \ldots, k\}$ (see Dalfó, Duque, Fabila-Monroy, Fiol, Huemer, Trujillo-Negrete, and Zaragoza Martínez [9]). Hence, $F_k(K_n)$ is Laplacian integral. Since cographs are themselves Laplacian integrals, the following result shows that their token graphs retain this property.

THEOREM 3.6. *Let G be a connected cograph on n vertices. Then, for any integer k such that $2 \leq k \leq \frac{n}{2}$, $F_k(G)$ is Laplacian integral.*

Proof. We proceed by strong induction on $n = |V(G)|$.

Base case: For $n = 4$, the connected cographs are S_4, C_4, U (see Fig. 1), $K_4 - e$, and K_4. From Table 2, it is straightforward to verify that their 2-token graphs are Laplacian integral.

Induction hypothesis: Assume that for every connected cograph H with fewer than n vertices, the k-token graph $F_k(H)$ is Laplacian integral for all $2 \leq k \leq \frac{|V(H)|}{2}$.

Induction step: Let G be a connected cograph on n vertices. Since G is a connected cograph, its complement G^c is disconnected. By Observation 3.5, $F_k(G^c)$ is the Cartesian product of the token graphs of the connected components of G^c. Each of these components has fewer than n vertices, and hence by the induction hypothesis, the corresponding token graphs are Laplacian integral. Since the Cartesian product of Laplacian integral graphs is also Laplacian integral, it follows that $F_k(G^c)$ is Laplacian integral.

By Theorem 2.6, each Laplacian eigenvalue of $F_k(G)$ and $F_k(G^c)$ satisfies

$$\lambda(F_k(G)) + \lambda(F_k(G^c)) = \lambda(F_k(K_n)),$$

where $\lambda(H)$ is a Laplacian eigenvalue of graph H. Since both $\lambda(F_k(G^c))$ and $\lambda(F_k(K_n))$ are integers, it follows that all Laplacian eigenvalues of $F_k(G)$ are integers. Hence, $F_k(G)$ is Laplacian integral. This completes the proof. ∎

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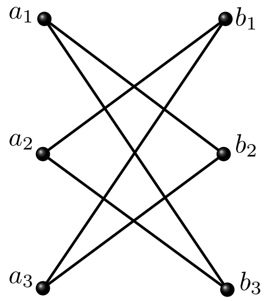

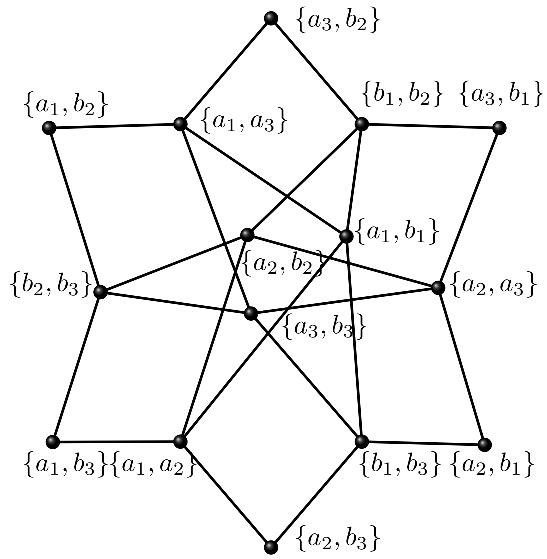

FIGURE 2. *Graph $K_3 \otimes K_2$ (left) and its 2-token graph (right).*

The Kronecker product $G \otimes K_2$ is a bipartite graph, also called the *bipartite double cover* of G. If $V(G) = \{u_1, \ldots, u_n\}$ and $V(K_2) = \{v_1, v_2\}$, then

$$V(G \otimes K_2) = \{(u_i, v_j) \colon i \in \{1, \ldots, n\}, j \in \{1, 2\}\} \quad \text{and} \quad E(G \otimes K_2) = \{(u_i, v_1)(u_k, v_2) \colon u_i u_k \in E(G)\}.$$

Since every edge of G is mapped into two edges in $G \otimes K_2$, both the number of vertices and the number of edges are doubled compared to G.

As a special case, $K_n \otimes K_2$ (also known as the *Crown graph*) is a bipartite double cover of K_n; equivalently, it is the complete bipartite graph $K_{n,n}$ minus a perfect matching. Hence, $K_n \otimes K_2$ is $(n-1)$-regular graph with $|V(K_n \otimes K_2)| = 2n$ and $|E(K_n \otimes K_2)| = n(n-1)$. Since K_n is a regular graph, the Laplacian spectrum of $K_n \otimes K_2$ is

$$S(K_n \otimes K_2) = \left(0, (n-2)^{(n-1)}, n^{(n-1)}, 2(n-1)\right),$$

where the superscripts denote multiplicities. Thus, $K_n \otimes K_2$ is Laplacian integral.

EXAMPLE 3.7. Let $V(K_3) = \{u_1, u_2, u_3\}$ and $V(K_2) = \{v_1, v_2\}$. Then,

$$V(K_3 \otimes K_2) = \{a_1, a_2, a_3, b_1, b_2, b_3\},$$

where $a_i = (u_i, v_1)$ and $b_i = (u_i, v_2)$, for $i \in \{1, 2, 3\}$. The graph $K_3 \otimes K_2$ (see Fig. 2 (left)) is a complete bipartite graph $K_{3,3}$ minus a perfect matching. The Laplacian eigenvalues of $K_3 \otimes K_2$ and $F_2(K_3 \otimes K_2)$ are

$$S(K_3 \otimes K_2) = (0, 1, 1, 3, 3, 4),$$

and

$$S(F_2(K_3 \otimes K_2)) = \left(0, 1^{(2)}, \left(\frac{7 - \sqrt{17}}{2}\right)^{(2)}, 2, 5 - \sqrt{5}, 3^{(2)}, 4, 5^{(2)}, \left(\frac{7 + \sqrt{17}}{2}\right)^{(2)}, 5 + \sqrt{5}\right).$$

Thus, while $K_3 \otimes K_2$ is Laplacian integral, $F_2(K_3 \otimes K_2)$ is not.

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In the following result, we show that for $2 \leq k \leq n$, none of the k-token graphs of $K_n \otimes K_2$ is Laplacian integral.

THEOREM 3.8. *For any integer k and $n \geq 3$ such that $2 \leq k \leq n$. Then, $F_k(K_n \otimes K_2)$ is not Laplacian integral. Moreover, $\rho_L(F_2(K_n \otimes K_2)) = 3n - 4 + \sqrt{n^2 - 4n + 8}$.*

Proof. Let $V(K_n \otimes K_2) = \{a_1, \ldots, a_n, b_1, \ldots, b_n\}$ with edges between a_i to b_j, whenever $i \neq j$. Let (V_1, V_2) be the bipartition of $K_n \otimes K_2$, where $V_1 = \{a_1, \ldots, a_n\}$ and $V_2 = \{b_1, \ldots, b_n\}$. From Theorem 2.5, we have

$$S(K_n \otimes K_2) \subset S(F_2(K_n \otimes K_2)) \subset \cdots \subset S(F_n(K_n \otimes K_2)),$$

so it suffices to show that $S(F_2(K_n \otimes K_2))$ is not Laplacian integral.

For a vertex $\{i, j\} \in V(F_2(K_n \otimes K_2))$, Lemma 2.8 gives

$$d(\{i,j\}) = \begin{cases} 2(n-2), & \text{if } ij \in E(K_n \otimes K_2), \\ 2(n-1), & \text{otherwise.} \end{cases}$$

Let the partite sets of $F_2(K_n \otimes K_2)$ be $U_1 = \{\{i, j\} \colon \text{ either } i, j \in V_1 \text{ or } i, j \in V_2\}$ and $U_2 = V(F_2(K_n \otimes K_2)) \setminus U_1$. Thus, the total number of vertices in U_1 is $n(n-1)$, and each vertex in U_1 has degree $2(n-1)$. The set U_2 has n^2 vertices, of which $n(n-1)$ vertices have degree $2(n-2)$ and n have degree $2(n-1)$. Let $U_2 = A \cup B$, where $A = \{\{i, j\} \colon \{i, j\} \in U_2 \text{ and } d(\{i, j\}) = 2(n-2)\}$ and $B = U_2 \setminus A$.

Each vertex in U_1 has $2(n-2)$ neighbors in A and 2 neighbors in B. Let $\pi = \{A, B, U_1\}$ be the partition set of $V(F_2(K_n \otimes K_2))$, then the quotient Laplacian matrix of $L(F_2(K_n \otimes K_2))$ with partition π is

$$Q_L = \begin{pmatrix} 2(n-2) & 0 & -2(n-2) \\ 0 & 2(n-1) & -2(n-1) \\ -2(n-2) & -2 & 2(n-1) \end{pmatrix}.$$

Since each block has a constant row sum, π is an equitable partition. By simple calculations, the eigenvalues of Q_L are 0 and $3n - 4 \pm \sqrt{n^2 - 4n + 8}$. Since $K_n \otimes K_2$ is a bipartite graph, it follows that $F_2(K_n \otimes K_2)$ is bipartite as well. Thus, the spectral radius of Q_L equals the Laplacian spectral radius of $F_2(K_n \otimes K_2)$, that is,

$$\rho_L(F_2(K_n \otimes K_2)) = 3n - 4 + \sqrt{n^2 - 4n + 8}.$$

Since $n^2 - 4n + 8$ is not a perfect square for $n \geq 3$, then $F_2(K_n \otimes K_2)$ is not Laplacian integral. □

REMARK 3.9. *If $G \ncong C_4$, then for $k \geq 2$ $F_k(G)$ is not a cograph. Theorem 3.6 shows that k-token graphs of cographs are Laplacian integral. This result allows us to construct new graphs with integer Laplacian eigenvalues using the token graph operation. However, Theorem 3.8 shows that the k-token graph of Laplacian integral graphs is not necessarily Laplacian integral. Thus, it leads to the following interesting open problem.*

Open problem. Characterize all Laplacian integral graphs whose token graphs are also Laplacian integrals.

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Data availability.

No data were used for the research described in the article.

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