Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 42, pp. 499-521, June 2026.

NULL DECOMPOSITION AND A COMBINATORIAL-BLOCK GENERALIZED INVERSION OF TREES*

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Abstract. We study the Drazin inverse of the adjacency matrix of a tree T through its null decomposition. This decomposition, which partitions the vertex set of T, reveals a close connection between the Drazin inverse and the matching and independence structures of the tree. In addition, we use a Bjerhammar-type condition for the Drazin inverse of a matrix.

Key words. Trees, Drazin inverse, Maximum matchings, Null decomposition, Gallai–Edmonds decomposition.

AMS subject classifications. 15A09, 05C38.

1. Introduction. Since 1977, it has been known that if T is a tree with an invertible adjacency matrix $A(T)$, then $A(T)^{-1}$ is a $\{-1, 0, 1\}$-matrix; see [2]. In 1985, Godsil showed that $A(T)^{-1}$ is diagonally similar to a symmetric $\{0, 1\}$-matrix, meaning that $A(T)^{-1}$ can be viewed as the adjacency matrix of a graph; see [4]. Godsil used this fact to obtain sharp bounds on the smallest positive eigenvalue of $A(T)$ and to prove a conjecture by Gutman: among all trees of order n with a unique perfect matching, the path P_n minimizes the smallest positive eigenvalue.

In 1990, Pavlíková and Krč-Jediný provided a geometric description of the inverse of a nonsingular tree; see [6]. As emphasized by Sander in [14], this is an extraordinary property, as it allows for the construction of the inverse directly from the structure of the tree.

In 2006, Barik, Neumann, and Pati gave a combinatorial description of the inverse of bipartite graphs with a unique perfect matching; see [10]. Additionally, in 2005, Britz, Olesky, and Van Den Driessche showed that the Moore–Penrose inverse of any real matrix without square submatrices with more than one diagonal entry can be expressed in terms of bipartite graphs; see [8]. In particular, Theorem 2.6 in [8] gives the Moore–Penrose inverse of any (weighted) forest.

This article is organized as follows. In Section 2, we present the combinatorial description of the Drazin inverse of the adjacency matrix of a tree. In Section 3, we summarize some results from [16] and [20]. In Section 4, we show that the zero pattern of the Drazin inverse of a tree depends on its null decomposition. In Section 5, we study some relations between the Drazin inverse of a tree and its null decomposition. In Section 6, we prove that the combinatorial Drazin inverse and the adjacency matrix of any tree commute. In Section 7, we show that for any tree, the combinatorial Drazin inverse is an internal pseudo-inverse of its adjacency matrix. Finally, in Section 8, we use a Bjerhammar-type condition to prove that the combinatorial Drazin inverse is also an external inverse of the adjacency matrix.

*Received by the editors on October 29, 2025. Accepted for publication on May 12, 2026. Handling Editor: Zejun Huang. Corresponding Author: Cristian Panelo.

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2. Preliminaries. We work exclusively with simple graphs. Let G be a graph. The vertex set and edge set of G are denoted by $V(G)$ and $E(G)$, respectively. If $e \in E(G)$ and $e = \{u, v\}$, we write $e = uv$.

Let $v \in V(G)$. The *neighborhood* of v, denoted by $N_T(v)$, is the set $\{u \in V(G) : u \sim v\}$, where $u \sim v$ means u and v are adjacent. The *neighborhood* of a subset $S \subseteq V(G)$ is defined as $N_T(S) := \bigcup_{v \in S} N_T(v)$. The *closed neighborhood* of a vertex v is $N_T[v] := N_T(v) \cup \{v\}$, and for a subset $S \subseteq V(G)$, the closed neighborhood is $N_T[S] := S \cup N_T(S)$. If T is clear from the context, we simply write $N(S)$ or $N[S]$. If H is a subtree of T, we define $N_H(v) := N(v) \cap V(H)$. We denote by $\deg(v)$, the number $|N(v)|$.

An *independent set* $I \subseteq V(G)$ is a set of pairwise nonadjacent vertices. The *independence number* of G, denoted by $\alpha(G)$, is the maximum cardinality of an independent set. The set of all maximum independent sets is denoted by $\mathcal{I}(G)$ and its cardinality by $a(G)$.

Given $U \subseteq V(G)$, the subgraph of G induced by U is denoted by $G[U]$. The collection of connected components of G is denoted by $\mathcal{K}(G)$.

Let $\mathbb{R}^G$ denote the space of real-valued functions on $V(G)$. For $\vec{x} \in \mathbb{R}^G$ and $v \in V(G)$, we write $\vec{x}_v$ instead of $\vec{x}(v)$. The canonical vector e_v is defined by $e_v(u) = 1$ if $u = v$, and $e_v(u) = 0$ otherwise. The null vector is denoted by $\vec{0}$. The null space of G is denoted by $\mathcal{N}(G)$, and its dimension (nullity) is denoted by $\mathrm{nl}(G)$. For trees, this null space induces a partition of the vertex set.

Definition 2.1. *Let T be a tree. The* support *of T is*

$$\mathrm{Supp}(T) := \{v \in V(T) : \exists\, \vec{x} \in \mathcal{N}(T) \text{ such that } \vec{x}_v \neq 0\}.$$

The core *of T, denoted by $\mathrm{Core}(T)$, is defined as $N(\mathrm{Supp}(T)) - \mathrm{Supp}(T)$. The* invertible part *of T, denoted by $\mathrm{Inv}(T)$, is $V(T) - N[\mathrm{Supp}(T)]$.*

By definition, we understand that the vertex set of tree T is partitioned into three pairwise disjoint subsets: $\mathrm{Supp}(T)$, $\mathrm{Core}(T)$, and $\mathrm{Inv}(T)$. The elements of $\mathrm{Supp}(T)$, $\mathrm{Core}(T)$, and $\mathrm{Inv}(T)$ are called *supported*, *core*, and *invertible* vertices, respectively. Their cardinalities are denoted by $\mathrm{supp}(T)$, $\mathrm{core}(T)$, and $\mathrm{inv}(T)$, respectively.

A *matching* $M \subseteq E(G)$ is a set of pairwise nonadjacent edges. Vertices incident to edges in M are *matched*; otherwise, they are *unmatched*. The *matching number* $\mu(G)$ is the size of a maximum matching. The set of all maximum matchings is denoted by $\mathcal{M}(G)$ and its cardinality by $m(G)$. The vertex set of a matching is

$$V(M) := \{v \in V(G) : v \text{ is matched by } M\}.$$

Let P be a path in G and M a matching. The path P is *M-alternating* if its edges alternate between matched and unmatched by M. An M-alternating path $v_0 v_1 \ldots v_k v_{k+1}$ is called:

- *mm-alternating* if v_0v_1 and v_kv_{k+1} are matched by M;
- *nn-alternating* if v_0v_1 and v_kv_{k+1} are unmatched by M;
- *nm-alternating* if $v_0v_1 \notin M$ and $v_kv_{k+1} \in M$ (and *mn-alternating* if the opposite holds).

The set of all mm-alternating paths with respect to M is denoted by $\mathrm{P}_{\mathrm{mm}}(G, M)$.

Let T be a tree and $i, j \in V(T)$. Denote by iP_Tj (or simply iPj) the unique path between i and j in T. Define:

$$\mathcal{M}(T, i, j) := \{M \in \mathcal{M}(T) : iPj \in \mathrm{P}_{\mathrm{mm}}(T, M)\},$$

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
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and let $m(T,i,j) := |\mathcal{M}(T,i,j)|$. Note that $m(T,i,j) = 0$ if $d(i,j)$ is even, where $d(i,j)$ denotes the distance between i and j, i.e., the number of edges in iPj.

Definition 2.2. *Let T be a tree of order n. The* combinatorial inverse *of T is the matrix $R(T) := (r_{ij})_{1\le i,j\le n}$, where*

$$r_{ij} = (-1)^{\lfloor d(i,j)/2 \rfloor} \cdot \frac{m(T,i,j)}{m(T)}.$$

The *Drazin inverse* of a symmetric matrix A, see [3], is the unique matrix D satisfying

$$AD = DA, \quad ADA = A, \quad \text{and} \quad DAD = D. \tag{2.1}$$

For symmetric matrices, the Drazin inverse coincides with the Moore–Penrose and group inverses. For linear algebra concepts not defined here, we refer to [7].

Theorem 2.3. *If T is a tree, then $R(T)$ is the Drazin inverse of its adjacency matrix $A(T)$.*

Proof. Let $A = A(T)$ and $R = R(T)$. By Theorem 6.5, we have $AR = RA$. By Theorem 7.7, $ARA = A$. Finally, by Corollary 8.13, $RAR = R$. Hence, R is the Drazin inverse of A. □

Each step of this proof corresponds to a section of the article. Although this result follows from Theorem 2.6 in [8], our graph-theoretic approach provides deeper insight into the interplay between matchings, independent sets, and the linear-algebraic structure of trees. Notably, the null decomposition of a tree can be computed in linear time [17], making this factorization practically useful. An analogous formula for weighted trees was given in [18].

3. Matching and independence structures of trees. On trees, the decomposition given by the null space coincides with the Gallai–Edmonds decomposition, see [13]:

$$\mathrm{Supp}(T) = \{v \in V(T) : \exists\, M \in \mathcal{M}(T) \text{ such that } v \notin V(M)\}.$$

Definition 3.1 ([20]). *The set of bond edges of a tree T, denoted by $BE(T)$, is the set of all edges between core vertices:*

$$BE(T) := \{uv \in E(T) : u, v \in \mathrm{Core}(T)\}.$$

Definition 3.2 ([16]). *The set of connection edges of T, denoted by $CE(T)$, is the set of all edges between a core vertex and an invertible vertex:*

$$CE(T) := \{uv \in E(T) : u \in \mathrm{Core}(T) \text{ and } v \in \mathrm{Inv}(T)\}.$$

By the Gallai–Edmonds Structure Theorem (see Theorem 3.2.1 in [13]), we have the following result.

Theorem 3.3. *If T is a tree, then*

1. $\mathrm{Supp}(T)$ *is an independent set of T;*
2. $T[\mathrm{Inv}(T)]$ *has a perfect matching;*
3. *the connection and bond edges are never in any maximum matching:*

$$M \cap CE(T) = M \cap BE(T) = \emptyset \quad \forall M \in \mathcal{M}(T);$$

4. *for any $e \in M$ where $M \in \mathcal{M}(T)$, exactly one holds:*

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(a) $e \in E(T\,[\mathrm{Inv}(T)])$;
(b) $|e \cap \mathrm{Core}(T)| \cdot |e \cap \mathrm{Supp}(T)| = 1$.

The null decomposition of trees breaks any tree into two forests: a forest of trees with unique perfect matching and a forest of trees with unique maximum independent set. Trees with a unique perfect matching have a nonsingular adjacency matrix, see [15]. They are called *invertible trees*. On the other hand, trees with a unique maximum independent set were characterized in [5]. Here,we give another characterization using the support.

THEOREM 3.4 ([16, 20]). *Let T be a tree. The following are equivalent:*

1. $N[\mathrm{Supp}(T)] = V(T)$;
2. $a(T) = 1$.

A tree that satisfies any of the conditions in Theorem 3.4 is called an *independent tree*. Consider the subtree $T_1 := T[\{v_4, \ldots, v_7\}]$ from Fig. 1, $\mathcal{N}(T_1) = [e_5 - e_7, e_6 - e_7]$; $\mathrm{Supp}(T_1) = \{v_5, v_6, v_7\}$; $\alpha(T_1) = 3$; and $N\,[\mathrm{Supp}(T_1)] = V(T_1)$. Thus, T_1 is an independent tree.

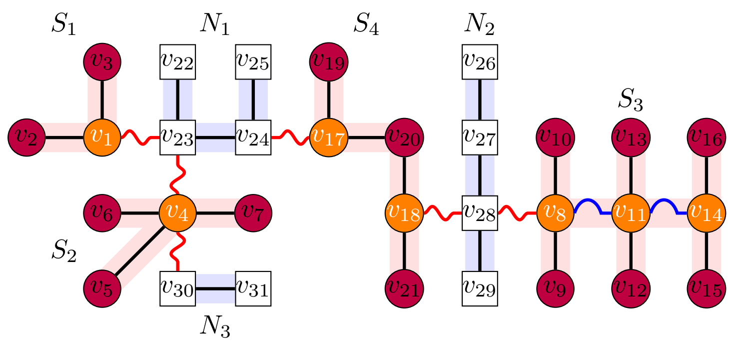

Figure 1: A tree T and its null decomposition. Support vertices are represented by red circles, core vertices by orange circles, and invertible vertices by white squares. Connection edges are in red "snakes" and bond edges in blue "coils".

Let T be a tree. We define

$$\mathcal{F}_{\mathrm{indep}}(T) := T[N[\mathrm{Supp}(T)]] \quad \text{and} \quad \mathcal{F}_{\mathrm{match}}(T) := T[\mathrm{Inv}(T)].$$

The forest $\mathcal{F}_{\mathrm{indep}}(T)$ is called the *independence forest* of T, while $\mathcal{F}_{\mathrm{match}}(T)$ is the *invertible forest* of T. The empty set is simultaneously an independence forest of T and an invertible forest of T.

THEOREM 3.5 ([16]). *Let T be a tree. $\mathcal{F}_{\mathrm{indep}}(T)$ is a forest of independent trees, and $\mathcal{F}_{\mathrm{match}}(T)$ is a forest of invertible trees.*

Note that K_1 (the complete graph of order 1) is an independent tree. In fact, K_1 is the only independent tree without a core.

An independent subtree of $\mathcal{F}_{\mathrm{indep}}(T)$ is called an *independent part* of T, while an invertible subtree of $\mathcal{F}_{\mathrm{match}}(T)$ is called an *invertible part* of T. The *null forest* of T is the forest $\mathcal{F}_{\mathrm{null}}(T) := \mathcal{F}_{\mathrm{indep}}(T) \cup \mathcal{F}_{\mathrm{match}}(T)$. Each subtree of $\mathcal{F}_{\mathrm{null}}(T)$ is called a *part* of T.

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For T in Fig. 1, the independent parts are $\mathcal{F}_{\text{indep}}(T) = \{S_1, S_2, S_3, S_4\}$, where $S_1 = T[\{v_1, v_2, v_3\}]$; $S_2 = T[\{v_4, \ldots, v_7\}]$; $S_3 = T[\{v_8, \ldots, v_{16}\}]$; and $S_4 = T[\{v_{17}, \ldots, v_{21}\}]$. Its invertible forest is $\mathcal{F}_{\text{match}}(T) = \{N_1, N_2, N_3\}$, where $N_1 = T[\{v_{22}, \ldots, v_{25}\}]$; $N_2 = T[\{v_{26}, \ldots, v_{29}\}]$; and $N_3 = T[\{v_{30}, v_{31}\}]$. Therefore, $\mathcal{F}_{\text{null}}(T) = \{S_1, S_2, S_3, S_4, N_1, N_2, N_3\}$.

Theorem 3.6 ([16]). *If T is a tree, then*

$$\begin{aligned}
\operatorname{Supp}(T) &= \bigcup_{S \in \mathcal{F}_{\text{indep}}(T)} \operatorname{Supp}(S);\\
\operatorname{Core}(T) &= \bigcup_{S \in \mathcal{F}_{\text{indep}}(T)} \operatorname{Core}(S);\\
\operatorname{Inv}(T) &= \bigcup_{N \in \mathcal{F}_{\text{match}}(T)} \operatorname{Inv}(N).
\end{aligned}$$

The *rank* of a graph G, denoted by $\operatorname{rk}(G)$, is the rank of $A(G)$.

Theorem 3.7 ([16]). *If T is a tree, then*

$$\begin{aligned}
\operatorname{nl}(T) &= \operatorname{supp}(T) - \operatorname{core}(T);\\
\operatorname{rk}(T) &= 2\operatorname{core}(T) + |V(\mathcal{F}_{\text{match}}(T))|;\\
\mu(T) &= \operatorname{core}(T) + \frac{|V(\mathcal{F}_{\text{match}}(T))|}{2};\\
m(T) &= \prod_{S \in \mathcal{F}_{\text{indep}}(T)} m(S);\\
\alpha(T) &= \operatorname{supp}(T) + \frac{|V(\mathcal{F}_{\text{match}}(T))|}{2};\\
a(T) &= \prod_{N \in \mathcal{F}_{\text{match}}(T)} a(N).
\end{aligned}$$

Let $\mathcal{F}$ be a forest. We write $H \in \mathcal{F}$ to mean $H \in \mathcal{K}(\mathcal{F})$.

Theorem 3.8 ([13]). *Let T be a tree.*

1. *If $M \in \mathcal{M}(T)$, then for each $H \in \mathcal{F}_{\text{null}}(T)$*

$$M \cap E(H) \in \mathcal{M}(H).$$

2. *For each $H \in \mathcal{F}_{\text{null}}(T)$, let $M_H \in \mathcal{M}(H)$. Then,*

$$\bigcup_{H \in \mathcal{F}_{\text{null}}(T)} M_H \in \mathcal{M}(T).$$

Independent subtrees without bond edges play a key role in the matching structure of trees. They were first introduced in [5] via *strong maximum independent sets* (maximum independent sets whose complements are also independent).

Theorem 3.9 ([5, 20]). *Let T be a tree. The following are equivalent.*

1. *T has a unique strong maximum independent set.*

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A publication of the International Linear Algebra Society
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2. *The distance between any two pendant vertices of T is even.*
3. *T is an independent tree with no bond edges.*

A tree that satisfies any (and hence all) conditions in Theorem 3.9 is called an *atom tree*, typically denoted by $\mathfrak{A}$.

Definition 3.10 ([20]). *Let T be a tree. The forest of atom trees of T, denoted by $\mathcal{F}_{\text{atom}}(T)$, is the forest induced by the set of edges of T that belong to some but not all maximum matchings of T.*

4. Zeros of $R(T)$. In this section, we use the structure provided by the null decomposition to characterize the zero blocks of the matrix $R(T)$.

The set of all mm-alternating paths of a graph G is

$$\mathrm{P}_{\text{mm}}(G) := \bigcup_{M\in\mathcal{M}(G)} \mathrm{P}_{\text{mm}}(G, M).$$

Let P be a path in a tree T, with initial and final vertices u and v. We write $\mathcal{M}(T, P)$ instead of $\mathcal{M}(T, u, v)$, and $m(T, P)$ instead of $m(T, u, v)$.

Remark 4.1. *Let T be a tree and P be a path in T. Then $m(T, P) = 0$ if and only if $P \notin \mathrm{P}_{\text{mm}}(T)$.*

Lemma 4.2. *Let G be a graph, $M \in \mathcal{M}(G)$, and $P \in \mathrm{P}_{\text{mm}}(G, M)$. If Q is an mm-alternating subpath of P, then $Q \in \mathrm{P}_{\text{mm}}(G, M)$.*

Observe that, by Theorem 3.3, the parts of T are matched independently and that none of the connection or bond edges are used in any maximum matching of T. Therefore, the mm-alternating property becomes a local requirement on the different parts. The following results formalize this observation.

Let G and H be two graphs. By $G \cap H$, we denote the graph with set of vertices $V(G) \cap V(H)$ and set of edges $E(G) \cap E(H)$.

Lemma 4.3. *Let T be a tree and let P be a path in T. If $H \in \mathcal{F}_{\text{null}}(T)$ such that $|V(P \cap H)| = 1$, then $P \notin \mathrm{P}_{\text{mm}}(T)$.*

Lemma 4.4. *Let T be a tree. The path $P \in \mathrm{P}_{\text{mm}}(T)$ if and only if the subpaths $P \cap H \in \mathrm{P}_{\text{mm}}(H)$ for each $H \in \mathcal{F}_{\text{null}}(T)$ such that $P \cap H \neq \emptyset$.*

Proof. Let $P = v_1 \dots v_k \in \mathrm{P}_{\text{mm}}(T)$. By Lemma 4.3, for every $H \in \mathcal{F}_{\text{null}}(T)$ such that $P \cap H \neq \emptyset$, we have $|V(P \cap H)| > 1$. Let v_i and v_j be, respectively, the first and last vertices of P belonging to H, with $i < j$.

Let $M \in \mathcal{M}(T)$ be a maximum matching such that $P \in \mathrm{P}_{\text{mm}}(T, M)$. By Theorem 3.3, the edges v_iv_{i+1} and $v_{j-1}v_j$ belong to M, and moreover, $M \cap E(H)$ is a maximum matching of H. Therefore, the subpath $P \cap H$ is mm-alternating with respect to a maximum matching of H, that is, $P \cap H \in \mathrm{P}_{\text{mm}}(H)$.

Conversely, assume that P is a path in T such that for every $H \in \mathcal{F}_{\text{null}}(T)$ with $P \cap H \neq \emptyset$, the subpath $P \cap H$ belongs to $\mathrm{P}_{\text{mm}}(H)$. For each such H, let $M(H)$ be a maximum matching of H with respect to which $P \cap H$ is mm-alternating.

By Theorem 3.8, the matching

$$M = \bigcup_{H\in\mathcal{F}_{\text{null}}(T)} M(H),$$

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A publication of the International Linear Algebra Society
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is a maximum matching of T. Since P is mm-alternating with respect to $M(H)$ on each component H, it follows that P is mm-alternating with respect to M in T. Hence, $P \in \mathrm{P}_{\mathrm{mm}}(T)$. ∎

Let G be a graph. By $H \leq G$, we mean that H is a subgraph of G.

LEMMA 4.5. *Let S be an independent part of a tree T. If $u, v \in \mathrm{Core}(S)$, then $m(T, u, v) = 0$.*

Proof. Assume that the path $uPv = x_0x_1 \ldots x_k$, where $u = x_0$ and $v = x_k$, is mm-alternating path with respect to a maximum matching M in T, and clearly $uPv \leq S \leq T$. Since uPv is a mm-alternating path with respect to M, we know that $x_0x_1 \in M$ and $x_1x_2 \notin M$. Furthermore, since the vertex x_0 is a core vertex in T, by Theorem 3.3, we can conclude that $x_1 \in \mathrm{Supp}(T)$. By Theorems 3.3 and 3.6, we have that $x_2 \in \mathrm{Core}(S)$. This implies that $x_2 \neq x_k = v$, because $x_{k-1}x_k \in M$. Applying this argument again, with x_0 replaced by x_2, it follows that $x_3 \in \mathrm{Supp}(S)$ and $x_4 \in \mathrm{Core}(S)$, with $x_4 \neq v$. We can continue in this fashion, obtaining an infinite sequence of vertices in uPv none of which is v, which is impossible. ∎

The following Lemmas, which are implicit in [16], show that it is only possible to enter or exit an independent part (or atom tree) through core vertices.

LEMMA 4.6. *Let T be a tree, and $S \in \mathcal{F}_{\mathrm{indep}}(T)$. If $v \in \mathrm{Supp}(S)$, then $N_S(v) = N_T(v)$.*

LEMMA 4.7. *Let T be a tree, and $\mathfrak{A} \in \mathcal{F}_{\mathrm{atom}}(T)$. If $v \in \mathrm{Supp}(\mathfrak{A})$, then $N_{\mathfrak{A}}(v) = N_T(v)$.*

Let P be a path in T. We say that P *traverses* a subtree H of T if the endpoints of P lie outside H, but $V(P) \cap V(H) \neq \emptyset$. If P traverses an independent part S, by Lemma 4.6 the endpoints of the subpath $P \cap S$ must be core vertices. Similarly, if P traverses an atom tree $\mathfrak{A}$, then, by Lemma 4.7, the endpoints of the subpath $P \cap \mathfrak{A}$ must be core vertices.

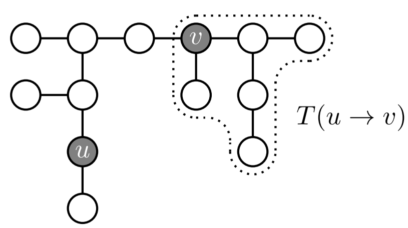

Figure 2: T and $T(u \to v)$.

LEMMA 4.8. *Let T be a tree, let P be a path in T, and let $\mathfrak{A}$ be an atom tree of T. If P traverses $\mathfrak{A}$, then $P \notin \mathrm{P}_{\mathrm{mm}}(T)$.*

Proof. Assume that $P \in \mathrm{P}_{\mathrm{mm}}(T)$. Since P traverses $\mathfrak{A}$, it must enter and leave $\mathfrak{A}$, and hence $|V(P)| \geq 4$. Moreover, by Lemma 4.3, $|V(P) \cap V(\mathfrak{A})| \neq 1$. Let $P' = P \cap \mathfrak{A} = v_iv_{i+1} \ldots v_j$. By Lemma 4.7, v_i and v_j are core vertices of T. Hence, by Theorem 3.3, the edges $v_{i-1}v_i$ and v_jv_{j+1} are in $CE(T) \cup BE(T)$. Therefore, these edges are unmatched in any maximum matching of T. Combined with the assumption that $P \in P_{mm}(T)$, it follows that the subpath P' is mm-alternating with respect to M, for a maximum matching of T. However, by Lemmas 4.5 and 4.7, this is impossible. Hence, $P \notin \mathrm{P}_{\mathrm{mm}}(T)$. ∎

COROLLARY 4.9. *Let T be a tree, P be a path in T, and S be an independent part of T. If P traverses S, then $P \notin \mathrm{P}_{\mathrm{mm}}(T)$.*

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We need the following notation from [16]. Let T be a tree and let $u, v \in V(T)$. We define $T(u \to v)$ as the subtree of T induced by the set of vertices $x \in V(T)$ such that the unique path from u to x in T contains the vertex v, see Fig. 2. That is,

$$T(u \to v) = T\left[\{x \in V(T) : v \in V(uP_T x)\}\right].$$

Let H_1, H_2 be two parts of T, the distance between H_1 and H_2 is

$$d(H_1, H_2) := |CE(T) \cap E(P)|,$$

where P is any path between a vertex in H_1 and a vertex in H_2.

Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be two atom trees of some independent part of T, the distance between $\mathfrak{A}_1$ and $\mathfrak{A}_2$ is defined as

$$d(\mathfrak{A}_1, \mathfrak{A}_2) := |BE(T) \cap E(P)|,$$

where P is any path between a vertex in $\mathfrak{A}_1$ and a vertex in $\mathfrak{A}_2$. We say that $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are *adjacent atom trees*, denoted by $\mathfrak{A}_1 \sim \mathfrak{A}_2$, if $d(\mathfrak{A}_1, \mathfrak{A}_2) = 1$.

Now we set the rules to find zero entries and zero blocks in the combinatorial Drazin inverse of any tree.

Lemma 4.10. *Let T be a tree, $R(T) = (r_{ij})_{1 \le i,j \le n}$ be its combinatorial Drazin inverse, and $i, j \in V(T)$. Then, $r_{ij} = 0$ if (at least) one of the following statements holds:*

1. *if $d(i, j)$ is even;*
2. *if $i, j \in \mathrm{Core}(T)$;*
3. *if there exists $u \in V(T)$ such that $iu \in CE(T)$, and $j \in V(T(i \to u))$;*
4. *if $i \in V(S_1)$, $j \in V(S_2)$, and $d(S_1, S_2) > 2$, where $S_1, S_2 \in \mathcal{F}_{\text{indep}}(T)$ are two different independent parts of the tree T;*
5. *if $i \in V(N_1)$ and $j \in V(N_2)$, where $N_1, N_2 \in \mathcal{F}_{\text{match}}(T)$ are two different invertible parts of the tree T;*
6. *if $i \in V(S)$, $j \in V(N)$, and $d(S, N) > 1$, where $S \in \mathcal{F}_{\text{indep}}(T)$, and $N \in \mathcal{F}_{\text{match}}(T)$;*
7. *if $i \in V(\mathfrak{A}_1)$, $j \in V(\mathfrak{A}_2)$, and $d(\mathfrak{A}_1, \mathfrak{A}_2) > 1$, where $\mathfrak{A}_1, \mathfrak{A}_2 \in \mathcal{F}_{\text{atom}}(S)$, and $S \in \mathcal{F}_{\text{indep}}(T)$;*
8. *if there exists a part H of T such that $|V(iPj \cap H)| = 1$.*

Proof. We verify each statement:

1. This follows from the definition of $R(T)$.

2. If there exist $S \in \mathcal{F}_{\text{indep}}(T)$ such that $i, j \in \mathrm{Core}(S)$, then, by Lemma 4.5, $m(T, i, j) = 0$. Hence, $r_{ij} = 0$. Therefore, we can assume that i and j are core vertices in different independent parts of T. Let iPj be the unique path in T that connects i with j. Assume that there exists a maximum matching M such that iPj is mm-alternating with respect to M. Let S be the independent part of T where i lives. Let $u \in V(P) \cap V(S)$ be the other final vertex of the subpath $P \cap S$. Let w_0 be the vertex immediately before u on P, and w_1 the vertex immediately after u. Hence, $uw_1 \in CE(T) \cup BE(T)$. Therefore, by Theorem 3.3, $uw_1 \notin M$. But iPj is an alternating path with respect to M, so $w_0u \in M$. Hence, iPu is a mm-alternating path in S with respect to M, which is impossible by Lemma 4.5.

3. Follow directly from Theorem 3.3.

4. $-$ 7. The hypothesis of each of these statements implies that the path iPj traverses either an independent part or an atom tree of T. Therefore, by Lemma 4.8 and Corollary 4.9, we have that $m(T, i, j) = 0$.

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8. Follow from Lemma 4.3. ◻

The adjacency matrix of the tree T in Fig. 1, using the null decomposition, is

$$A(T)=\begin{bmatrix} A_{S_1} & & & & A_{S_1N_1} & & \\ & A_{S_2} & & & A_{S_2N_1} & & A_{S_2N_3} \\ & & A_{S_3} & & & A_{S_3N_2} & \\ & & & A_{S_4} & A_{S_4N_1} & A_{S_4N_2} & \\ A_{N_1S_1} & A_{N_1S_2} & & A_{N_1S_4} & A_{N_1} & & \\ & & A_{N_2S_3} & A_{N_2S_4} & & A_{N_2} & \\ & A_{N_3S_2} & & & & & A_{N_3} \end{bmatrix}, \tag{4.2}$$

where all the omitted blocks are zero submatrices. For any $H, K \in \mathcal{F}_{\text{null}}(T)$, $(A_H)_{ij} = (A(T))_{ij}$ for all $i, j \in V(H)$ and $(A_{HK})_{ij} = (A(T))_{ij}$ for all $i \in V(H), j \in V(K)$. Moreover, by Lemma 4.10, we have that the combinatorial Drazin inverse of T is

$$R(T)=\begin{bmatrix} R_{S_1} & & & & R_{S_1N_1} & & \\ & R_{S_2} & & & R_{S_2N_1} & & R_{S_2N_3} \\ & & R_{S_3} & & & R_{S_3N_2} & \\ & & & R_{S_4} & R_{S_4N_1} & R_{S_4N_2} & \\ R_{N_1S_1} & R_{N_1S_2} & & R_{N_1S_4} & R_{N_1} & & \\ & & R_{N_2S_3} & R_{N_2S_4} & & R_{N_2} & \\ & R_{N_3S_2} & & & & & R_{N_3} \end{bmatrix}, \tag{4.3}$$

where all the omitted blocks are zero submatrices. For any $H, K \in \mathcal{F}_{\text{null}}(T)$, $(R_H)_{ij} = (R(T))_{ij}$ for all $i, j \in V(H)$ and $(R_{HK})_{ij} = (R(T))_{ij}$ for all $i \in V(H), j \in V(K)$.

5. $R(T)$ and the null decomposition of T.

In this section, we show that the combinatorial Drazin inverse of trees is essentially local in each part of the null decomposition. In order to understand this phenomenon, we need to know more about mm-alternating paths in trees.

The next proposition is a direct consequence of Theorem 3.7 and Lemma 4.4. It shows that a path is mm-alternating in a tree if and only if it is mm-alternating in each part of the null decomposition of the tree.

PROPOSITION 5.1. *Let T be a tree and P be a path in T. Then,*

$$m(T,P) = \prod_{\substack{H\in\mathcal{F}_{\text{null}}(T)\\ H\cap P\neq\emptyset}} m(H, P\cap H) \prod_{\substack{S\in\mathcal{F}_{\text{indep}}(T)\\ S\cap P=\emptyset}} m(S).$$

Usually, we apply Proposition 5.1 with $i, j \in V(H) \in \mathcal{F}_{\text{null}}(T)$, thus the formula takes the form

$$m(T,i,j) = m(H,i,j) \prod_{\substack{S\in\mathcal{F}_{\text{indep}}(T)\\ S\neq H}} m(S).$$

Let G be a graph, $U \subset V(G)$, and $H = G[U]$. Let $X(G)$ be a matrix whose rows and columns are indexed by the vertices of G. The H-*induced* submatrix of $X(G)$, denoted by $X(G)\downharpoonright^{G}_{H}$, is defined to be the submatrix of $X(G)$ whose rows and columns are indexed by the vertices of H.

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Theorem 5.2. *Let T be a tree and let $H \in \mathcal{F}_{\text{null}}(T)$. Then,*

$$R(H) = R(T)\downharpoonleft^{T}_{H},$$

Proof. Given $i, j \in V(H)$, by Proposition 5.1 and the fact that $d_T(i,j) = d_H(i,j)$, we observe that

$$\begin{aligned} r(T)_{ij} &= (-1)^{\lfloor \frac{d_T(i,j)}{2} \rfloor} \frac{m(T,i,j)}{m(T)} \\ &= (-1)^{\lfloor \frac{d_T(i,j)}{2} \rfloor} \frac{m(H,i,j)\prod_{H \neq S \in \mathcal{F}_{\text{indep}}(T)} m(S)}{m(T)} \\ &= (-1)^{\lfloor \frac{d_H(i,j)}{2} \rfloor} \frac{m(H,i,j)}{m(H)} \\ &= r(H)_{ij}. \end{aligned}$$

∎

Consider the tree T in Fig. 1. We have that $m(T) = 144$ and $m(S_1) = 2$. Following (4.3),

$$R(T)\downharpoonleft^{T}_{S_1} = R_{S_1} = \begin{bmatrix} 0 & \frac{72}{144} & \frac{72}{144} \\ \frac{72}{144} & 0 & 0 \\ \frac{72}{144} & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 \end{bmatrix} = R(S_1).$$

In order to study the null space of the combinatorial Drazin inverse of a tree, we will need versions of the previous results for atom trees. The proofs are similar.

Lemma 5.3. *Let S be an independent tree. Then, $P \in \mathrm{P}_{\text{mm}}(S)$ if and only if $P \cap \mathfrak{A} \in \mathrm{P}_{\text{mm}}(\mathfrak{A})$ for all $\mathfrak{A} \in \mathcal{F}_{\text{atom}}(S)$ such that $P \cap \mathfrak{A} \neq \emptyset$.*

Corollary 5.4. *Let S be an independent tree and P a path in S. Then,*

$$m(S,P) = \prod_{\substack{\mathfrak{A} \in \mathcal{F}_{\text{atom}}(T) \\ \mathfrak{A} \cap P \neq \emptyset}} m(\mathfrak{A}, P \cap \mathfrak{A}) \prod_{\substack{\mathfrak{A} \in \mathcal{F}_{\text{atom}}(T) \\ \mathfrak{A} \cap P = \emptyset}} m(\mathfrak{A}).$$

Let G be a connected graph, and F and H be disjoint subgraphs of G. With $\mathcal{P}(G,F,H)$, we denote the set of all paths in G between a vertex of F and a vertex of H.

We now define the notions of "in" and "out" of a subtree with respect to another subtree, see Fig. 3. Let U and W be subtrees of a tree T such that $|V(U) \cap V(W)| \leq 1$. Then,

$$V(U) \cap \left(\bigcap_{P \in \mathcal{P}(T,U,W)} V(P) \right),$$

is a singleton containing the vertex that we denoted by $\text{in}(U \leftarrow W)$. If U and W are disjoint, then

$$N(\text{in}(U \leftarrow W)) \cap \left(\bigcap_{P \in \mathcal{P}(T,U,W)} V(P) \right),$$

is a singleton whose element is denoted by $\text{out}(U \to W)$. When either $U = K_1$ or $W = K_1$, we write $\text{in}(U \leftarrow z)$ or $\text{out}(z \to W)$ instead of $\text{in}(U \leftarrow K_1)$ or $\text{out}(K_1 \to W)$, respectively. Note that if $v \in U$, then $\text{in}(U \leftarrow v) = v$. In this case, we define $\text{out}(U \to v) := \emptyset$.

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Corollary 5.5. *Let T be a tree and $\mathfrak{A}$ be an atom tree of T. If $u \in V(\mathfrak{A})$ and $v \in V(T) - V(\mathfrak{A})$, then*

$$m(T, u, v) = m(\mathfrak{A}, u, \mathrm{in}(\mathfrak{A} \leftarrow v)) \cdot m(T - \mathfrak{A}, \mathrm{out}(\mathfrak{A} \rightarrow v), v),$$

where $T - \mathfrak{A}$ is the forest that remains after taking away from T the atom tree $\mathfrak{A}$.

Let T be a tree and i, j be two vertices of T at odd distance. Then, for all $v \in V(iPj)$,

$$\left\lfloor \frac{d_T(i,j)}{2} \right\rfloor = \left\lfloor \frac{d_T(i,v)}{2} \right\rfloor + \left\lfloor \frac{d_T(v,j)}{2} \right\rfloor .$$

This will be used in the proof of Theorem 5.6. Theorem 5.6 allows us to extract information about the null space of a tree from the null space of its atom trees.

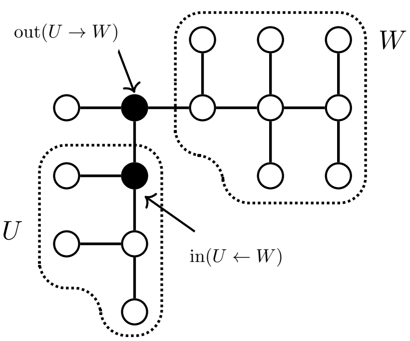

Figure 3: The in and out of subtree U with respect to the subtree W.

We need the following notation introduced in [16]. Given a graph G, let $\vec{x}$ be a vector of $\mathbb{R}^G$. Let H be a subgraph of G. Then, the vector obtained when we restrict $\vec{x}$ to the coordinates (vertices) associated with H is denoted by $\vec{x}\downharpoonright^G_H$. The vector obtained when we lift a vector $\vec{y} \in \mathbb{R}^H$ to a vector of $\mathbb{R}^G$ is denoted by $\vec{y}\upharpoonright^G_H$. Thus, for $u \in V(G) - V(H)$, $(\vec{y}\upharpoonright^G_H)_u := 0$, and for $u \in V(H)$, $(\vec{y}\upharpoonright^G_H)_u := \vec{y}_u$.

Theorem 5.6. *Let T be a tree and $\mathfrak{A}$ be an atom tree of T. If $j \in V(\mathfrak{A})$ and $i \in V(T)$, then*

$$r(T)_{ij} = c(i, \mathfrak{A})\, r(\mathfrak{A})_{\mathrm{in}(\mathfrak{A} \leftarrow i)\, j},$$

where

$$c(i, \mathfrak{A}) := \begin{cases} 1 & , \text{ if } i \in V(\mathfrak{A}), \\ (-1)^{\left\lfloor \frac{d_T(i, \mathrm{in}(\mathfrak{A} \leftarrow i))}{2} \right\rfloor} \dfrac{m(T - \mathfrak{A}, i, \mathrm{out}(\mathfrak{A} \rightarrow i))}{\prod_{\mathfrak{A} \neq \hat{\mathfrak{A}} \in \mathcal{F}_{\mathrm{atom}}(T)} m(\hat{\mathfrak{A}})} & , \text{ otherwise.} \end{cases}$$

Proof. Let $v = \mathrm{in}(\mathfrak{A} \leftarrow i)$ and $w = \mathrm{out}(\mathfrak{A} \rightarrow i)$. Assume that $i \in V(T) - V(\mathfrak{A})$. By Corollary 5.5, we have

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A publication of the International Linear Algebra Society
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$$\begin{aligned} r_{ij} &= (-1)^{\lfloor \frac{d_T(i,j)}{2} \rfloor} \frac{m(T,i,j)}{m(T)} \\ &= (-1)^{\lfloor \frac{d_T(i,v)}{2} \rfloor + \lfloor \frac{d_T(v,j)}{2} \rfloor} \frac{m(\mathfrak{A},j,v) \cdot m(T-\mathfrak{A},w,i)}{m(\mathfrak{A}) \cdot \prod_{\mathfrak{A} \neq \hat{\mathfrak{A}} \in \mathcal{F}_{\text{atom}}(T)} m(\hat{\mathfrak{A}})} \\ &= (-1)^{\lfloor \frac{d_T(i,v)}{2} \rfloor} \frac{m(T-\mathfrak{A},w,i)}{\prod_{\mathfrak{A} \neq \hat{\mathfrak{A}} \in \mathcal{F}_{\text{atom}}(T)} m(\hat{\mathfrak{A}})} \, r(\mathfrak{A})_{vj}. \end{aligned}$$ ∎

For example, in T in Fig. 1, the subtree S_2 is an atom tree. As we know $m(T) = 144$, and $m(S_2) = 3$. Thus,

$$r(T)_{v_{22}v_5} = (-1)^{\lfloor \frac{d_T(v_{22},v_5)}{2} \rfloor} \frac{m(T,v_{22},v_5)}{m(T)} = -\frac{48}{144} = -\frac{1}{3},$$

and

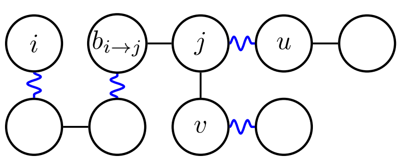
$$c(v_{22},S_2) \cdot r(S_2)_{v_4v_5} = (-1)^{\lfloor \frac{d_T(v_{22},v_4)}{2} \rfloor} \frac{48}{48} \cdot \left(\frac{1}{3}\right) = -\frac{1}{3}.$$

6. $AR = RA$. From now on, T denotes a tree, and if the tree T is clear from context, we just write A, R and r_{ij}, instead of $A(T)$, $R(T)$, and $(R(T))_{ij}$, respectively. In this section, we prove that $AR = RA$.

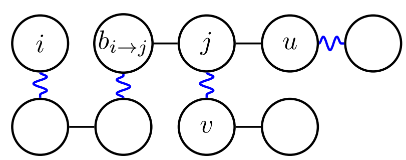
Let $i, j \in V(T)$. The (i,j)-*flower* of T, denoted by $\mathfrak{F}_T(i,j)$, is

$$\mathfrak{F}_T(i,j) := \sum_{v \sim j} r_{iv}.$$

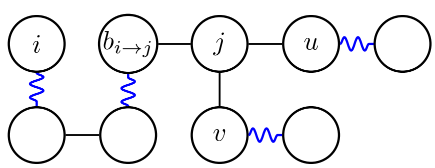
Note that $\mathfrak{F}_T(i,j)$ is the inner product of the row i of A and the column j of R. If $d(i,j)$ is odd, then for each $v \in N(j)$, we have that $d(i,v)$ is even. Hence, $r_{iv} = 0$ for all $v \sim j$. Therefore, $\mathfrak{F}_T(i,j) = 0$.



(a) $M_1 \in \mathcal{M}(T,i,u) \subset \mathcal{M}(T,i,b_{i\to j})$



(b) $M_3 \in \mathcal{M}(T,i,v) \subseteq \mathcal{M}(T,i,b_{i\to j})$



(c) $M_2 \in \mathcal{M}(T,i,b_{i\to j})$, but $M_2 \notin \mathcal{M}(T,i,x)$ for all $x \in N(j \leftarrow i)$

Figure 4: Inclusion relation among sets of maximum matchings.

In order to understand the flowers of a given tree, we introduce the following notation. Let T be a tree and let $i, j \in V(T)$ be two distinct vertices. The *i-far-neighborhood* of j, denoted by $N(j \leftarrow i)$, is the set of neighbors of j that are strictly further from i than j is

$$N(j \leftarrow i) := \{u \in N(j) : d(u,i) = d(i,j) + 1\}.$$

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A publication of the International Linear Algebra Society
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The *i-bystander* of j, denoted by $b_{i\to j}$, is the unique neighbor of j that lies on the path from i to j. In other words, $b_{i\to j}$ satisfies: $d(i, b_{i\to j}) = d(i,j) - 1$. Note that $N(j) = N(j \leftarrow i) \cup \{b_{i\to j}\}$.

The proof of each statement in the following result is simple; see Fig. 4.

Lemma 6.1. *Let T be a tree and $i, j \in V(T)$ be such that $d(i,j)$ is even and greater than 1. The following statements are true:*

1. *if $u, v \in N(i)$, then $\mathcal{M}(T,i,u) \cap \mathcal{M}(T,i,v) = \emptyset$;*
2. *if $u, v \in N(j \leftarrow i)$, then $\mathcal{M}(T,i,u) \cap \mathcal{M}(T,i,v) = \emptyset$;*
3. *if $u \in N(j \leftarrow i)$, then $\mathcal{M}(T,i,u) \subset \mathcal{M}(T,i,b_{i\to j})$;*
4. *$\bigcup_{u\in N(j\leftarrow i)} \mathcal{M}(T,i,u) \subset \mathcal{M}(T,i,b_{i\to j})$.*

The set of all edges incident to the vertex $i \in V(T)$ is denoted by $E_i(T)$. Thus, $E_i(T) = \{e \in E(T) : i \in e\}$. We need to work with some sets of matchings associated with flowers. The set of all maximum matchings of T using an incident edge to i is denoted by $\mathcal{M}_{\text{int}}(T,i)$. Then,

$$\mathcal{M}_{\text{int}}(T,i) = \{M \in \mathcal{M}(T) \ : \ M \cap E_i(T) \neq \emptyset\}. \tag{6.4}$$

Similarly, let i and j be two different vertices of the tree T, the set of all maximum matchings of T such that they do not have any edge incident to j, but the path from i to $b_{i\to j}$ is mm-alternating in T with respect to them, is denoted by $\mathcal{M}_{\text{ex}}(T,i,j)$. Then,

$$\mathcal{M}_{\text{ex}}(T,i,j) = \{M \in \mathcal{M}(T) : M \cap E_j(T) = \emptyset \wedge iPb_{i\to j} \in \mathrm{P}_{\text{mm}}(T,M)\}. \tag{6.5}$$

Let $i, j \in V(T)$ such that $i \neq j$. From the definition of i-bystander of j, it follows that

$$\mathcal{M}(T,i,b_{i\to j}) = \mathcal{M}_{\text{ex}}(T,i,j) \dot\cup \left(\dot{\bigcup_{v\,\in\, N(j\leftarrow i)}} \mathcal{M}(T,i,v) \right),$$

where $\dot\cup$ denotes disjoint union. This equality is used in the second part of the proof of the following result, where we make explicit the connections among these sets of maximum matchings and flowers.

Lemma 6.2. *Let T be a tree and $i, j \in V(T)$ be such that $d(i,j)$ is even. Then,*

1. *$m(T)\, \mathfrak{F}_T(i,i) = |\mathcal{M}_{\text{int}}(T,i)|$;*
2. *$m(T)\, \mathfrak{F}_T(i,j) = (-1)^{1+\lfloor \frac{d(i,j)}{2} \rfloor} |\mathcal{M}_{\text{ex}}(T,i,j)|$.*

Proof. 1. Note that by Lemma 6.1, part 1.,

$$\begin{aligned} m(T)\, \mathfrak{F}_T(i,i) &= m(T) \sum_{v\sim i} r_{iv} \\ &= m(T) \sum_{v\sim i} \frac{m(T,i,v)}{m(T)} \\ &= \sum_{v\sim i} |\mathcal{M}(T,i,v)| \\ &= \left| \dot{\bigcup_{v\sim i}} \mathcal{M}(T,i,v) \right| \\ &= \left| \{M \in \mathcal{M}(T) : M \cap E_i(T) \neq \emptyset\} \right|. \end{aligned}$$

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2. We first note that for all $u, v \in N(j \leftarrow i)$, the numbers r_{iu} and r_{iv} have the same sign $(-1)^{\lfloor \frac{d(i,j)}{2} \rfloor}$, and $r_{ib_{i\to j}}$ has the opposite sign. Thus,

$$\begin{aligned} m(T)\,\mathfrak{F}_T(i,j) &= m(T)\sum_{v\sim j} r_{iv} \\ &= (-1)^{\lfloor \frac{d(i,j)}{2} \rfloor -1} m(T,i,b_{i\to j}) + \sum_{v\,\in\, N(j\leftarrow i)} (-1)^{\lfloor \frac{d(i,j)}{2} \rfloor} m(T,i,v) \\ &= (-1)^{\lfloor \frac{d(i,j)}{2} \rfloor} \left(-\left|\mathcal{M}(T,i,b_{i\to j})\right| + \left| \dot{\bigcup_{v\,\in\, N(j\leftarrow i)}} \mathcal{M}(T,i,v) \right| \right) \\ &= (-1)^{\lfloor \frac{d(i,j)}{2} \rfloor -1} \left|\mathcal{M}_{\text{ex}}(T,i,j)\right|. \end{aligned}$$

∎

The proof of the following result follows from (6.5) and is illustrated in Fig. 5.

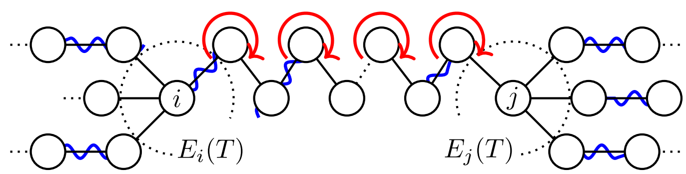

Figure 5: $|\mathcal{M}_{\text{ex}}(T,i,j)| = |\mathcal{M}_{\text{ex}}(T,j,i)|$.

Corollary 6.3. *Let T be a tree. For any $i, j \in V(T)$ such that $i \neq j$, $|\mathcal{M}_{\text{ex}}(T,i,j)| = |\mathcal{M}_{\text{ex}}(T,j,i)|$.*

By Lemma 6.2 and Corollary 6.3, we have the following result.

Corollary 6.4. *Let T be a tree, and $i, j \in V(T)$. Then, $\mathfrak{F}_T(i,j) = \mathfrak{F}_T(j,i)$.*

We now present the main result of this section.

Theorem 6.5. *If T is a tree, then $A(T)R(T) = R(T)A(T)$.*

Proof. By Corollary 6.4, we have

$$(AR)_{ij} = \sum_{v\sim i} r_{vj} = \mathfrak{F}_T(i,j) = \mathfrak{F}_T(j,i) = \sum_{v\sim j} r_{vi} = (AR)_{ji}.$$

This shows that $AR = (AR)^t$. Therefore, $AR = (AR)^t = RA$. ∎

Lemma 6.6. *Let T be a tree and $i, j \in V(T)$ with $i \neq j$ be such that $d(i,j)$ is even. Then,*

1. *$\mathcal{M}_{\text{int}}(T,i) = \mathcal{M}(T)$ if $i \in \text{Core}(T) \cup \text{Inv}(T)$;*
2. *$\mathcal{M}_{\text{ex}}(T,i,j) = \emptyset$ if i or j belongs to $\text{Core}(T) \cup \text{Inv}(T)$;*
3. *$\mathcal{M}_{\text{ex}}(T,i,j) = \emptyset$ if $j \in \text{Supp}(\mathfrak{A})$, $\mathfrak{A} \in \mathcal{F}_{\text{atom}}(T)$, and $i \in V(T - \mathfrak{A})$.*

Proof. 1. and 2. follow from the fact that all core and invertible vertices of a tree are always matched in any maximum matching. Thus, if $i \in \text{Core}(T) \cup \text{Inv}(T)$, the internal maximum matchings at i coincide with all maximum matchings of T. Similarly, if either i or j is in $\text{Core}(T) \cup \text{Inv}(T)$, no external maximum matching between them exists because they are already internally matched.

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A publication of the International Linear Algebra Society
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For 3. we have that if $j \in \mathrm{Supp}(\mathfrak{A})$, then $b_{i\to j} \in \mathrm{Core}(T)$, and, as $\mathfrak{A}$ is an atom tree of T, $d(j, \mathrm{in}(\mathfrak{A} \leftarrow i))$ is odd. Hence, $d(b_{i\to j}, \mathrm{out}(\mathfrak{A} \to i))$ is odd too. So in any mm-alternating path from i to $b_{i\to j}$, the edge between $\mathrm{in}(\mathfrak{A} \leftarrow i)$ and $\mathrm{out}(\mathfrak{A} \to i)$ must be matched, but this edge does not belong to any maximum matching. ∎

Corollary 6.7. *Let T be a tree, and $i, j \in V(T)$, with $i \neq j$, such that $d(i, j)$ is even. Then,*

1. $\mathfrak{F}_T(i, i) = 1$, *if* $i \in \mathrm{Core}(T) \cup \mathrm{Inv}(T)$;
2. $\mathfrak{F}_T(i, j) = 0$ *if* i *or* j *is in* $\mathrm{Core}(T) \cup \mathrm{Inv}(T)$;
3. $\mathfrak{F}_T(i, j) = 0$ *if* $i \in \mathrm{Supp}(\mathfrak{A})$, $\mathfrak{A} \in \mathcal{F}_{\mathrm{atom}}(T)$, *and* $j \in V(T - \mathfrak{A})$.

Consider the tree T in Fig. 1. We have that $m(S_1) = 2$. Following (4.2), (4.3), and Theorem 5.2,

$$A_{S_1} R_{S_1} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

7. $ARA = A$. In this section, we prove that $ARA = A$. We do this in a coordinate-wise form, see Corollary 7.2, and Lemmas 7.4 and 7.6.

Proposition 7.1. *Let T be a tree and $i, j \in V(T)$. Then,*

$$(A(T)R(T)A(T))_{ij} = \sum_{v\sim i} \mathfrak{F}_T(v, j).$$

Proof. By Theorem 6.5, it is clear that $ARA = A^2R$. Hence,

$$\begin{aligned}(ARA)_{ij} &= \sum_{u\sim i}\sum_{v\sim j} r_{uv} \\ &= \deg(i) r_{ij} + \sum_{\substack{v\in V(T)\\ d(v,i)=2}} r_{vj}.\end{aligned}$$

By the proof of Theorem 6.5, it is clear that

$$(ARA)_{ij} = \sum_{v\sim i} \mathfrak{F}_T(v, j).$$

∎

For example, for T in Fig. 1, we have that $m(S_4) = 3$. Following (4.2), (4.3), and Theorems 5.2 and 6.5,

$$\begin{aligned} A_{S_4}^2 R_{S_4} &= \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & \frac{2}{3} & \frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & -\frac{1}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 \\ -\frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{3} & \frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & \frac{1}{3} & \frac{2}{3} & \frac{1}{3} \\ 0 & 0 & -\frac{1}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix}. \end{aligned}$$

By Lemma 4.10 and Proposition 7.1, we have the following result.

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A publication of the International Linear Algebra Society
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Corollary 7.2. *Let T be a tree. If $i, j \in V(T)$ and $d(i,j)$ is even, then*

$$(A(T)R(T)A(T))_{ij} = 0.$$

The following result is key in order to prove that $(ARA)_{ij} = 0$ when $i, j \in V(T)$ and $d(i,j) \geq 3$ is odd. Its proof follows from (6.5) and is illustrated in Fig. 6.

Lemma 7.3. *Let T be a tree. If $i, j \in V(T)$ and $d(i,j) \geq 3$ is odd, then*

$$|\mathcal{M}_{\text{ex}}(T, i, b_{i\to j})| = \sum_{v \,\in\, N(j\leftarrow i)} |\mathcal{M}_{\text{ex}}(T, i, v)|\,.$$

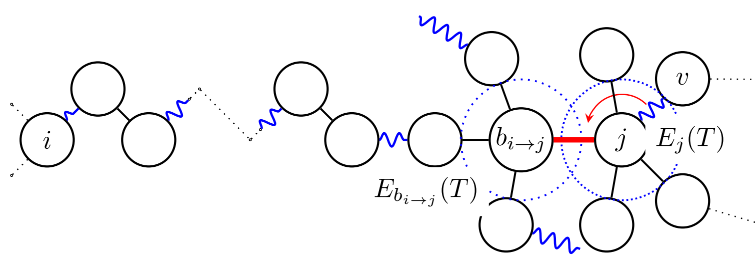

Figure 6: $M \in \mathcal{M}_{\text{ex}}(T, i, b_{i\to j})$ and $M' = M - vj + jb_{i\to j} \in \mathcal{M}_{\text{ex}}(T, i, v)$.

Lemma 7.4. *Let T be a tree. If $i, j \in V(T)$ and $d(i,j) \geq 3$, then*

$$(A(T)R(T)A(T))_{ij} = 0.$$

Proof. By Corollary 7.2, we can assume that $d(i,j)$ is odd. The sign of all flowers $\mathfrak{F}_T(i,v)$, with $v \in N(j \leftarrow i)$ is the same number,

$$(-1)^{1+\lfloor \frac{d(i,j)}{2} \rfloor} = (-1)^{\lceil \frac{d(i,j)}{2} \rceil},$$

while the sign of the flower $\mathfrak{F}_T(i, b_{i\to j})$ is the opposite, namely $(-1)^{\lfloor \frac{d(i,j)}{2} \rfloor}$.

By Proposition 7.1 and Lemma 7.3,

$$\begin{aligned}
(ARA)_{ij} &= \sum_{v\sim i} \mathfrak{F}_T(v,j) \\
&= \sum_{v\sim i} \mathfrak{F}_T(j,v) \\
&= \mathfrak{F}_T(j, b_{j\to i}) + \sum_{v \,\in\, N(i\leftarrow j)} \mathfrak{F}_T(j,v) \\
&= (-1)^{\lfloor \frac{d(i,j)}{2} \rfloor} \frac{|\mathcal{M}_{\text{ex}}(T,j,b_{j\to i})|}{m(T)} + (-1)^{\lceil \frac{d(i,j)}{2} \rceil} \sum_{v \,\in\, N(i\leftarrow j)} \frac{|\mathcal{M}_{\text{ex}}(T,j,v)|}{m(T)} \\
&= 0.
\end{aligned}$$

∎

Let $v \in V(T)$. We define the following set

(7.6) $$\mathcal{M}^{\text{c}}_{\text{int}}(T,v) = \mathcal{M}(T) - \mathcal{M}_{\text{int}}(T,v),$$

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A publication of the International Linear Algebra Society
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see (6.4). Note that $\mathcal{M}(T) = \mathcal{M}_{\text{int}}(T,v) \,\dot\cup\, \mathcal{M}^{\text{c}}_{\text{int}}(T,v)$ and

$$|\mathcal{M}(T)| = |\mathcal{M}_{\text{int}}(T,v)| + |\mathcal{M}^{\text{c}}_{\text{int}}(T,v)|\,.$$

The proof of the following result follows from (6.5), (7.6) and is illustrated in Fig. 7.

LEMMA 7.5. *Let T be a tree and $i,j \in V(T)$ be such that $i \sim j$. Then,*

$$|\mathcal{M}^{\text{c}}_{\text{int}}(T,j)| = \sum_{v \,\in\, N(i \leftarrow j)} |\mathcal{M}_{\text{ex}}(T,j,v)|\,.$$

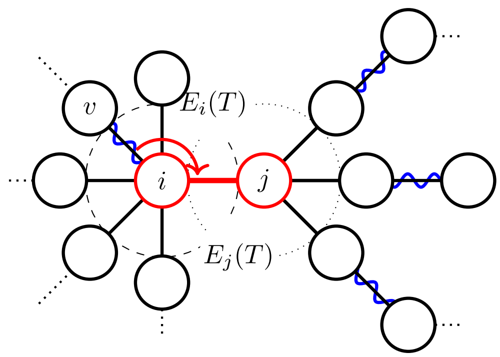

Figure 7: $M \in \mathcal{M}^{\text{c}}_{\text{int}}(T,j)$ and $M = M - iv + ij \in \mathcal{M}_{\text{ex}}(T,j,v)$.

LEMMA 7.6. *Let T be a tree, and $i,j \in V(T)$ such that $i \sim j$. Then,*

$$(A(T)R(T)A(T))_{ij} = 1.$$

Proof. Since $i \sim j$, for each $v \in N(i \leftarrow j)$, we have $d(j,v) = 2$. Hence, by Proposition 7.1 and Lemmas 6.2, 7.4, and 7.5,

$$\begin{aligned}
(ARA)_{ij} &= \sum_{v \sim i} \mathfrak{F}_T(j,v) \\
&= \mathfrak{F}_T(j,j) + \sum_{v \in N(i \leftarrow j)} \mathfrak{F}_T(j,v) \\
&= \frac{1}{m(T)} \left(|\mathcal{M}_{\text{int}}(T,j)| + \sum_{v \in N(i \leftarrow j)} |\mathcal{M}_{\text{ex}}(T,j,v)| \right) \\
&= \frac{1}{m(T)} \left(|\mathcal{M}_{\text{int}}(T,j)| + |\mathcal{M}^{\text{c}}_{\text{int}}(T,j)| \right) \\
&= \frac{1}{m(T)} |\mathcal{M}(T)| \\
&= 1.
\end{aligned}$$

∎

By Corollary 7.2 and Lemmas 7.4, and 7.6, we have the following result.

THEOREM 7.7. *Let T be a tree, then $A(T)R(T)A(T) = A(T)$.*

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8. $RAR = R$**.** In the setting of Moore–Penrose theory, Bjerhammar was the first to notice that if you have additional information about the fundamental spaces of the matrix and its candidate for the generalized inverse, then you do not need to check all the Moore–Penrose conditions, see [1]. In this section, we use a Bjerhammar-type result for the Drazin inverse, which we proved in 2020. We do this because, for some combinatorial problems, the condition $DAD = D$ in the definition of Drazin inverse of symmetric matrices is usually more difficult to verify than the other conditions, see (2.1).

The *index* of a square matrix A, denoted by $\text{Ind}(A)$, is the smallest nonnegative integer k for which $\mathcal{R}(A^k) = \mathcal{R}(A^{k+1})$. The Drazin inverse of A, denoted by D, satisfies

$$AD = DA, \quad A^{k+1}D = A^k, \quad DAD = D.$$

For a symmetric matrix A, we have that $\text{Ind}(A) = 1$. Thus, its Drazin inverse must satisfy the conditions in (2.1). We have the following result.

Theorem 8.1 ([19]). *Let A and D be square matrices, with $\text{Ind}(A) = k$, such that $\mathcal{N}(A^k) = \mathcal{N}(D)$ and $AD = DA$. Then, $A^{k+1}D = A^k$ if and only if $D^2A = D$.*

Note that, if $AD = DA$, $ADA = A$, and $y \in \mathcal{N}(D)$, then $Ay = ADAy = A^2Dy = \vec{0}$. Hence, $\mathcal{N}(D) \subset \mathcal{N}(A)$. We have the following result.

Lemma 8.2. *Let A and D be two $n \times n$ matrices such that $AD = DA$ and $ADA = A$. Then, $\mathcal{N}(D) \subset \mathcal{N}(A)$.*

In Sections 6 and 7, we proved that $RA = AR$ and $ARA = A$. Therefore, if we prove that $RAR = R$, then we have proved that R is the Drazin inverse of A. It turns out if $RA = AR$ and $ARA = A$, then, by Theorem 8.1 and Lemma 8.2, it suffices to show that $\mathcal{N}(A) \subset \mathcal{N}(R)$. We want to prove that $R\vec{x} = \vec{0}$ for $\vec{x} \in \mathcal{N}(T)$. It is enough to work with a basis of $\mathcal{N}(T)$. Instead of working with an arbitrary basis, it turns out it is more suitable to work with some special bases of $\mathcal{N}(T)$, which were introduced in many articles, see [9, 11, 12, 17, 20]. These bases are composed of special vectors, which are nonzero only on a particular atom tree.

On a per-entry basis, we need to show that $(R\vec{x})_i = 0$ for all $i \in V(T)$; that is, the inner product of the ith row of $R(T)$ with $\vec{x}$ vanishes. Since $\vec{x}$ is nonzero only on a fixed atom tree $\mathfrak{A} \in \mathcal{F}_{\text{atom}}(T)$, only the entries of the ith row of $R(T)$ corresponding to vertices of $\mathfrak{A}$ may contribute to this product. By Theorem 5.6, these entries form a scalar multiple of a row of $R(\mathfrak{A})$. Hence, $(R\vec{x})_i = 0$ follow whenever $\vec{x}$ belongs to the null space of $R(\mathfrak{A})$. Therefore, to conclude that $R\vec{x} = 0$, it suffices to prove that for any atom tree $\mathfrak{A}$, we have

$$\mathcal{N}(\mathfrak{A}) \subseteq \mathcal{N}(R(\mathfrak{A})).$$

In order to introduce the special bases of $\mathcal{N}(T)$, we need some definitions and results from [20].

Theorem 8.3 ([20]). *Let $\mathfrak{B}$ be an atom tree of order n. The following are equivalent:*

1. *If $v \in \text{Core}(\mathfrak{B})$, then $\deg(v) = 2$.*
2. $\text{supp}(\mathfrak{B}) = \text{core}(\mathfrak{B}) + 1$.
3. $\text{nl}(\mathfrak{B}) = 1$.
4. $\mu(\mathfrak{B}) = \dfrac{n-1}{2}$.
5. $\alpha(\mathfrak{B}) = \dfrac{n+1}{2}$.

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A publication of the International Linear Algebra Society
Volume 42, pp. 499-521, June 2026.

An atom tree that satisfies any (and hence all) of these conditions is called a basic tree.

The basic trees were first characterized, in terms of the FOX algorithm, in [9], where they are called S_w trees.

Definition 8.4. *Let $\mathfrak{A}$ be an atom tree and $\mathfrak{B}$ a subtree of $\mathfrak{A}$. We say that $\mathfrak{B}$ is a null basis subtree of $\mathfrak{A}$ if the following statements are simultaneously true.*

1. *If $v \in V(\mathfrak{B}) \cap \mathrm{Core}(\mathfrak{A})$, then $\deg_{\mathfrak{B}}(v) = 2$.*
2. *If $v \in V(\mathfrak{B}) \cap \mathrm{Supp}(\mathfrak{A})$, then $N_{\mathfrak{B}}(v) = N_{\mathfrak{A}}(v)$.*

Lemma 8.5 ([20]). *If $\mathfrak{B}$ is a null basis subtree of an atom tree $\mathfrak{A}$, then*

1. $\mathrm{Supp}(\mathfrak{B}) = \mathrm{Supp}(\mathfrak{A}) \cap V(\mathfrak{B})$;
2. $\mathrm{Core}(\mathfrak{B}) = \mathrm{Core}(\mathfrak{A}) \cap V(\mathfrak{B})$.

Proposition 8.6 ([20]). *If $\mathfrak{A}$ is an atom tree, then $\mathfrak{A}$ is $(\mathrm{Core}(\mathfrak{A}), \mathrm{Supp}(\mathfrak{A}))$-bipartite.*

Let $\mathfrak{A}$ be an atom tree, $\mathfrak{B}$ a null basis subtree of $\mathfrak{A}$, and s a supported vertex of $\mathfrak{B}$. The *null basis vector* associated with $\mathfrak{B}$ and s, denoted by $\overrightarrow{\mathfrak{B}(s)}$, is a vector in $\mathbb{R}^{\mathfrak{A}}$, such that for each $v \in V(\mathfrak{B})$

$$\overrightarrow{\mathfrak{B}(s)}_v := \begin{cases} (-1)^{\frac{d(v,s)}{2}} & \text{if } v \in \mathrm{Supp}(\mathfrak{B}), \\ 0 & \text{otherwise.} \end{cases}$$

In fact, the actual supported vertex chosen does not matter: if $u, v \in \mathrm{Supp}(\mathfrak{B})$, then either $\overrightarrow{\mathfrak{B}(u)} = \overrightarrow{\mathfrak{B}(v)}$ or $\overrightarrow{\mathfrak{B}(u)} = -\overrightarrow{\mathfrak{B}(v)}$. Therefore, we just write $\overrightarrow{\mathfrak{B}}$, and we say that $\overrightarrow{\mathfrak{B}}$ is the *null basis vector of* $\mathfrak{A}$ *associated with* the null basis subtree $\mathfrak{B}$.

Theorem 8.7 ([9, 11, 12, 16, 17, 20]). *Let $\mathfrak{A}$ be an atom tree. There exists a basis of $\mathcal{N}(\mathfrak{A})$ whose vectors are all null basis vectors of $\mathfrak{A}$.*

By $\mathcal{SBB}(\mathfrak{A})$, we denote an arbitrary but fixed basis of an atom tree $\mathfrak{A}$ whose vectors are all basic vectors of $\mathfrak{A}$. We can obtain a basis for the null space of T from different bases of its atom subtrees, see [20]:

$$\mathcal{SBB}(T) := \bigcup_{\mathfrak{A} \in \mathcal{F}_{\mathrm{atom}}(T)} \mathcal{SBB}(\mathfrak{A}) \upharpoonright_{\mathfrak{A}}^{T}.$$

For example, for T in Fig. 1, we have that the following atom subtrees:

- $\mathfrak{A}_1 := T[\{v_1, v_2, v_3\}]$, and $\mathcal{SBB}(\mathfrak{A}_1) = \{e_{v_2} - e_{v_3}\}$;
- $\mathfrak{A}_2 := T[\{v_5, v_6, v_7\}]$, and $\mathcal{SBB}(\mathfrak{A}_2) = \{e_{v_5} - e_{v_6}, e_{v_5} - e_{v_7}\}$;
- $\mathfrak{A}_3 := T[\{v_8, v_9, v_{10}\}]$, and $\mathcal{SBB}(\mathfrak{A}_3) = \{e_{v_9} - e_{v_{10}}\}$;
- $\mathfrak{A}_4 := T[\{v_{11}, v_{12}, v_{13}\}]$, and $\mathcal{SBB}(\mathfrak{A}_4) = \{e_{v_{12}} - e_{v_{13}}\}$;
- $\mathfrak{A}_5 := T[\{v_{14}, v_{15}, v_{16}\}]$, and $\mathcal{SBB}(\mathfrak{A}_5) = \{e_{v_{15}} - e_{v_{16}}\}$; and
- $\mathfrak{A}_6 := T[\{v_{17}, v_{18}, v_{19}, v_{20}, v_{21}\}]$, and $\mathcal{SBB}(\mathfrak{A}_6) = \{e_{v_{19}} - e_{v_{20}} + e_{v_{21}}\}$.

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A publication of the International Linear Algebra Society
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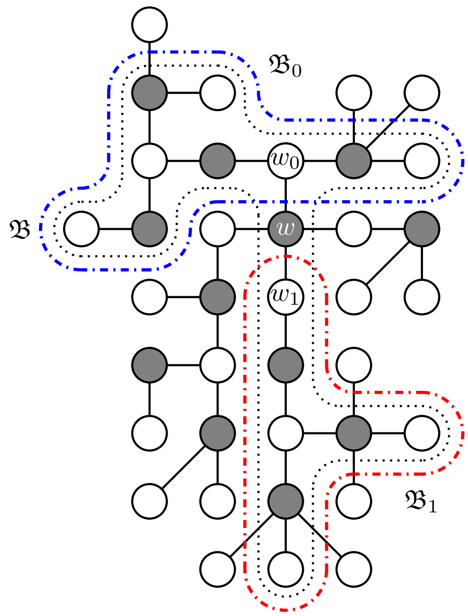
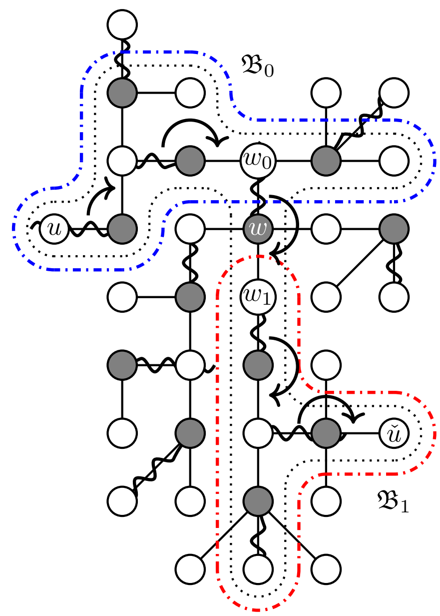


(a) Atom tree and its subtree $\mathfrak{B}$



(b) S-shift algorithm

Figure 8: A null basis subtree $\mathfrak{B}$ and its two subtrees, $\mathfrak{B}_0$ and $\mathfrak{B}_1$, determined by w.

Let $\mathfrak{B}$ be a null basis subtree of an atom tree $\mathfrak{A}$. The core vertices of $\mathfrak{A}$ divide $\mathfrak{B}$ into two subtrees. Let $v \in \text{Core}(\mathfrak{A})$ and $w := \text{in}(\mathfrak{B} \leftarrow v)$. By Definition 8.4 and Lemma 8.5, the neighborhood of any supported vertex of $\mathfrak{B}$ is equal in both trees $\mathfrak{B}$ and $\mathfrak{A}$. Therefore, w is a core vertex of $\mathfrak{B}$. Let w_0, w_1 be the two vertices of the neighborhood of the core vertex w in $\mathfrak{B}$. Then, $\{w_0, w_1\} = N_{\mathfrak{B}}(w)$. Therefore, w divides $\mathfrak{B}$ in two subtrees: $\mathfrak{B}_0 := \mathfrak{B}(w \to w_0)$ and $\mathfrak{B}_1 := \mathfrak{B}(w \to w_1)$, see Fig. 8a. These two subtrees have "unanonymous" matching structures, which are, in some sense, "mirrored through w".

Lemma 8.8. *Let $\mathfrak{A}$ be an atom tree, and $\mathfrak{B}$ a null basis subtree of $\mathfrak{A}$. If $v \in \text{Core}(\mathfrak{B})$ and $u \in N_{\mathfrak{B}}(v)$, then $\mathfrak{B}(v \to u)$ is an atom tree satisfying*

1. *$\mathfrak{B}(v \to u)$ is a subatom tree of $\mathfrak{B}$;*
2. *$\text{Supp}(\mathfrak{B}(v \to u)) = \text{Supp}(\mathfrak{B}) \cap V(\mathfrak{B}(v \to u))$;*
3. *$\text{Core}(\mathfrak{B}(v \to u)) = \text{Core}(\mathfrak{B}) \cap V(\mathfrak{B}(v \to u))$.*

Note that both $\mathfrak{B}_0$ and $\mathfrak{B}_1$ are subatom trees of $\mathfrak{A}$, such that $\text{Supp}(\mathfrak{B}_i) = \text{Supp}(\mathfrak{B}) \cap V(\mathfrak{B}_i)$, for $i = 0, 1$, and $\text{Supp}(\mathfrak{B}_0) \cup \text{Supp}(\mathfrak{B}_1) = \text{Supp}(\mathfrak{B})$.

For $k = 0, 1$, let

$$\mathcal{M}_k(\mathfrak{A}, \mathfrak{B}, v) := \bigcup_{u \in \text{Supp}(\mathfrak{B}) \cap V(\mathfrak{B}_k)} \{(M, u) \ : \ M \in \mathcal{M}(\mathfrak{A}, v, u)\}.$$

The proof of the following result is illustrated in Fig. 8b.

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A publication of the International Linear Algebra Society
Volume 42, pp. 499-521, June 2026.

THEOREM 8.9. *Let* $\mathfrak{B}$ *be a null basis subtree of an atom tree* $\mathfrak{A}$*, and* $v \in \mathrm{Core}(\mathfrak{A})$*. Then,*

$$|\mathcal{M}_0(\mathfrak{A}, \mathfrak{B}, v)| = |\mathcal{M}_1(\mathfrak{A}, \mathfrak{B}, v)|.$$

The following theorem proves that, for atom tree, the null space of the adjacency matrix and the null space of the combinatorial inverse are the same.

THEOREM 8.10. *Let* $\mathfrak{A}$ *be an atom tree and* $\vec{\mathfrak{B}}$ *be a null basis vector of* $\mathfrak{A}$ *associated with a null basic subtree* $\mathfrak{B}$*. Then,* $\vec{\mathfrak{B}} \in \mathcal{N}(R(\mathfrak{A}))$*.*

We describe the strategy to prove the theorem. For each vertex $i \in V(\mathfrak{A})$, we would like to prove that the entry $(R(\mathfrak{A})\,\vec{\mathfrak{B}})_i$ is zero. But this entry is the inner product between the row i of $R(\mathfrak{A})$, denoted by $R(\mathfrak{A})_{i*}$, and the vector $\vec{\mathfrak{B}}$. We take advantage of the fact that i breaks $\mathfrak{B}$ into two subtrees with mirrored "unanonymous" matching structures: $\mathfrak{B}_0$ and $\mathfrak{B}_1$. This also breaks the sum $R(\mathfrak{A})_{i*}$ into two parts, one for each of the subtrees $\mathfrak{B}_0$ and $\mathfrak{B}_1$. These sums, by Theorem 8.9, are equal, but of opposite sign. Hence, $(R(\mathfrak{A})\,\vec{\mathfrak{B}})_i$ would be zero.

Proof. We start with some observations about $\vec{\mathfrak{B}}$:

1. $\vec{\mathfrak{B}} \in \mathbb{R}^{\mathfrak{A}}$.
2. If $u, v \in \mathrm{Supp}(\mathfrak{B})$, then $\vec{\mathfrak{B}}_u = (-1)^{\frac{d(u,v)}{2}}\ \vec{\mathfrak{B}}_v$.
3. If $u \notin \mathrm{Supp}(\mathfrak{B})$, then $\vec{\mathfrak{B}}_u = 0$.

For $i \in V(\mathfrak{A})$

$$(R\,\vec{\mathfrak{B}})_i = \sum_{v \in V(\mathfrak{A})} r_{iv}\vec{\mathfrak{B}}_v = \sum_{\substack{v \in V(\mathfrak{B}),\\ d(i,v) \equiv 1 (\mathrm{mod}\ 2)}} r_{iv}\vec{\mathfrak{B}}_v.$$

We want to prove that $(R\,\vec{\mathfrak{B}})_i = 0$, for all $i \in V(\mathfrak{A})$.

By Proposition 8.6, for $u, v \in V(\mathfrak{A})$, $d(u, v)$ is even if and only if either $u, v \in \mathrm{Core}(\mathfrak{A})$ or $u, v \in \mathrm{Supp}(\mathfrak{A})$.

Case 1: $i \in \mathrm{Supp}(\mathfrak{A})$: if $i \in \mathrm{Supp}(\mathfrak{A})$, then $r_{i,v} = 0$ for all $v \in \mathrm{Supp}(\mathfrak{A})$. For all $v \in \mathrm{Core}(\mathfrak{A})$, by observation 3, we have that $\vec{\mathfrak{B}}_v = 0$. Therefore, for all $i \in \mathrm{Supp}(\mathfrak{A})$, we have that $(R\,\vec{\mathfrak{B}})_i = 0$.

Case 2: $i \in \mathrm{Core}(\mathfrak{A})$: the neighborhood of $w = \mathrm{in}(\mathfrak{B} \leftarrow i)$ in $\mathfrak{B}$ has only two vertices: $N_{\mathfrak{B}}(w) = \{w_0, w_1\}$. We note that:

1. $d(i, w_0) = d(i, w_1)$;
2. $\mathrm{sgn}\, r_{iw_0} = \mathrm{sgn}\, r_{iw_1}$;
3. $\mathrm{sgn}\, \vec{\mathfrak{B}}_{w_0} = -\,\mathrm{sgn}\, \vec{\mathfrak{B}}_{w_1}$.

Therefore, $\mathrm{sgn}\, r_{iw_0}\vec{\mathfrak{B}}_{w_0} = -\,\mathrm{sgn}\, r_{iw_1}\vec{\mathfrak{B}}_{w_1}$, and this situation is translated to all supported vertices of the corresponding subtrees $\mathfrak{B}_0 := \mathfrak{B}(w \to w_0)$, and $\mathfrak{B}_1 := \mathfrak{B}(w \to w_1)$. Thus,

- For $j \in \mathfrak{B}_0 \cap \mathrm{Supp}(\mathfrak{B})$:

  $$\mathrm{sgn}\, r_{ij} = (-1)^{\frac{d(w_0,j)}{2}}\,\mathrm{sgn}\, r_{iw_0}, \quad \mathrm{sgn}\, \vec{\mathfrak{B}}_j = (-1)^{\frac{d(w_0,j)}{2}}\,\mathrm{sgn}\, \vec{\mathfrak{B}}_{w_0}.$$

  Thus, $\mathrm{sgn}\, r_{ij}\vec{\mathfrak{B}}_j = \mathrm{sgn}\, r_{iw_0}\vec{\mathfrak{B}}_{w_0}$ for all $j \in V(\mathfrak{B}_0)$.
- For $j \in \mathfrak{B}_1 \cap \mathrm{Supp}(\mathfrak{B})$:

  $$\mathrm{sgn}\, r_{ij} = (-1)^{\frac{d(w_1,j)}{2}}\,\mathrm{sgn}\, r_{iw_1}, \quad \mathrm{sgn}\, \vec{\mathfrak{B}}_j = (-1)^{\frac{d(w_1,j)}{2}}\,\mathrm{sgn}\, \vec{\mathfrak{B}}_{w_1}.$$

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A publication of the International Linear Algebra Society
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Thus, $\operatorname{sgn} r_{ij}\vec{\mathfrak{B}}_j = -\operatorname{sgn} r_{iw_0}\vec{\mathfrak{B}}_{w_0}$ for all $j \in V(\mathfrak{B}_1)$.

Note that for any tree T

$$\begin{aligned} m(T,i,j) &= |\mathcal{M}(T,i,j)| \\ &= |\{(M,j) : M \in \mathcal{M}(T), iPj \in \mathrm{P}_{\mathrm{mm}}(T,M)\}| . \end{aligned}$$

The second expression has the advantage that for different j's these sets are all disjoint. This allows us to write

$$r_{ij}\,\vec{\mathfrak{B}}_j = \frac{1}{m(\mathfrak{A})}\operatorname{sgn} r_{ij}\,\vec{\mathfrak{B}}_j\ |\{(M,j) : M \in \mathcal{M}(T), iPj \in \mathrm{P}_{\mathrm{mm}}(\mathfrak{A},M)\}|.$$

Therefore, if $i \in \mathrm{Core}(\mathfrak{A})$, then

$$\begin{aligned} (R\,\vec{\mathfrak{B}})_i &= \sum_{d(i,j)\,\equiv\,1\,(\mathrm{mod}\ 2)} r_{ij}\,\vec{\mathfrak{B}}_j \\ &= \sum_{j\,\in\,\mathrm{Supp}(\mathfrak{B})} r_{ij}\vec{\mathfrak{B}}_j \\ &= \sum_{j\,\in\,\mathrm{Supp}(\mathfrak{B}_0)} r_{ij}\,\vec{\mathfrak{B}}_j + \sum_{j\,\in\,\mathrm{Supp}(\mathfrak{B}_1)} r_{ij}\,\vec{\mathfrak{B}}_j . \end{aligned}$$

Claim:

$$\sum_{j\,\in\,\mathrm{Supp}(\mathfrak{B}_0)} r_{ij}\,\vec{\mathfrak{B}}_j = -\sum_{j\,\in\,\mathrm{Supp}(\mathfrak{B}_1)} r_{ij}\,\vec{\mathfrak{B}}_j .$$

Hence, if $i \in \mathrm{Core}(\mathfrak{A})$, then $(R\,\vec{\mathfrak{B}})_i = 0$.

Proof of the claim: Each sum can be expressed as:

$$\begin{aligned} \sum_{j\in\mathrm{Supp}(\mathfrak{B}_0)} r_{ij}\vec{\mathfrak{B}}_j &= \frac{1}{m(\mathfrak{A})}\operatorname{sgn} r_{iw_0}\vec{\mathfrak{B}}_{w_0}\big|\mathcal{M}_0(\mathfrak{A},\mathfrak{B},i)\big|; \\ \sum_{j\in\mathrm{Supp}(\mathfrak{B}_1)} r_{ij}\vec{\mathfrak{B}}_j &= -\frac{1}{m(\mathfrak{A})}\operatorname{sgn} r_{iw_0}\vec{\mathfrak{B}}_{w_0}\big|\mathcal{M}_1(\mathfrak{A},\mathfrak{B},i)\big|. \end{aligned}$$

By Theorem 8.9, $\big|\mathcal{M}_0\big| = \big|\mathcal{M}_1\big|$, proving the claim. □

We need the following result from [20].

Theorem 8.11 ([20]). *Let T be a tree. Then,*

1. $\mathcal{N}(T) = \bigoplus_{\mathfrak{A}\in\mathcal{F}_{\mathrm{atom}}(T)} \mathcal{N}(\mathfrak{A})\!\uparrow_{\mathfrak{A}}^{T}$;
2. $\mathcal{SBB}(T) := \bigcup_{\mathfrak{A}\in\mathcal{F}_{\mathrm{atom}}(T)} \mathcal{SBB}(\mathfrak{A})\!\uparrow_{\mathfrak{A}}^{T}$ *is a basis of* $\mathcal{N}(T)$.

The following result follows from Lemma 8.2 and Theorems 5.6, 8.10, and 8.11.

Theorem 8.12. *Let T be a tree. Then, $\mathcal{N}(A(T)) = \mathcal{N}(R(T))$.*

Finally, by Theorems 6.5, 7.7, 8.1, and 8.11, we obtain the main result of this section.

Theorem 8.13. *Let T be a tree, then $R(T)A(T)R(T) = R(T)$.*

Electronic Journal of Linear Algebra, ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 42, pp. 499-521, June 2026.

Acknowledgment. We are grateful to the anonymous referee for additional helpful comments. This work was partially supported by Universidad Nacional de San Luis (Arg), PROICO 03-0723, Agencia I+D+i (Arg), grants PICT-2020-Serie A-00549, and CONICET (Arg) grant PIP 11220220100068CO.

REFERENCES

[1] A. Bjerhammar. A generalized matrix algebra. *Kungl. Tekniska Högskol. Handlingar*, 124:1–32, 1958.
[2] D.M. Cvetković, I. Gutman, and S.K. Simić. On self pseudo-inverse graphs. *Publ. Elektroteh. fak., Ser. Mat. fiz.*, (602/633):111–117, 1978.
[3] R.E. Cline. *Elements of the Theory of Generalized Inverses for Matrices*. Springer, New York, 1979.
[4] C.D. Godsil. Inverses of trees. *Combinatorica*, 5(1):33–39, 1985.
[5] G. Hopkins and W. Staton. Graphs with unique maximum independent sets. *Discrete Math.*, 57(3):245–251, 1985.
[6] S. Pavlíková and J. Krč-Jediný. On the inverse and the dual index of a tree. *Linear Multilinear Algebra*, 28(1–2):93–109, 1990.
[7] C.D. Meyer. *Matrix Analysis and Applied Linear Algebra*. Society for Industrial and Applied Mathematics, Philadelphia, 2000.
[8] T. Britz, D.D. Olesky, and P. van den Driessche. The Moore–Penrose inverse of matrices with an acyclic bipartite graph. *Linear Algebra Appl.*, 390:47–60, 2004.
[9] J. W. Sander and T. Sander. On simply structured bases of tree kernels. *AKCE J. Graphs Comb.*, 2(1):45–56, 2005.
[10] S. Barik, M. Neumann, and S. Pati. On nonsingular trees and a reciprocal eigenvalue property. *Linear Multilinear Algebra*, 54(6):453–465, 2006.
[11] S. Akbari, A. Alipour, E. Ghorbani, and G.B. Khosrovshahi. $\{-1,0,1\}$-basis for the null space of a forest. *Linear Algebra Appl.*, 414(2–3):506–511, 2006.
[12] M. Nath and B. K. Sarma. On the null-spaces of acyclic and unicyclic singular graphs. *Linear Algebra Appl.*, 427(1):42–54, 2007.
[13] L. Lovász and M.D. Plummer. *Matching Theory*. AMS Chelsea Publishing, Providence, RI, 2009.
[14] T. Sander. *On Simply Structured Bases of Graph Eigenspaces*. Ph.D. Thesis, TU Clausthal, 2009.
[15] R.B. Bapat. *Graphs and Matrices*. Springer, London, 2014.
[16] D.A. Jaume and G. Molina. Null decomposition of trees. *Discrete Math.*, 341(3):836–850, 2018.
[17] D.A. Jaume, G. Molina, A.D. Pastine, and M.D. Safe. A $\{-1,0,1\}$- and sparsest basis for the null space of a forest in optimal time. *Linear Algebra Appl.*, 549:53–66, 2018.
[18] S. Pavlíková and J. Širáň. Inverting non-invertible weighted trees. *Australas. J. Comb.*, 75(2):246–255, 2019.
[19] A.M. Encinas, D.A. Jaume, C. Panelo, and A.D. Pastine. Drazin inverse of singular adjacency matrices of directed weighted cycles. *Rev. Unión Mat. Argent.*, 61(2):209–227, 2020.
[20] D.A. Jaume, G. Molina, and R. Sota. Trees with a unique maximum independent set and their linear properties. *Rev. Unión Mat. Argent.*, 63(1):223–238, 2022.