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THE DIMENSION OF ORBITS AND BUNDLES OF HERMITIAN PENCILS UNDER $*$-CONGRUENCE[*]

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Abstract. We provide an explicit formula for the dimension of the $*$-congruence orbits and bundles of Hermitian matrix pencils over the field of complex numbers. The formula is given in terms of the sizes of the canonical blocks in the Hermitian Kronecker canonical form of the pencils. This extends the formulas provided only for the generic orbits and bundles in a previous work.

Key words. Matrix pencils, Hermitian, $*$-Congruence, Orbit, Bundle, Dimension, Codimension, Differentiable manifold, Tangent space.

AMS subject classifications. 15A18, 15A21, 15A22, 14J60.

1. Introduction. The purpose of this paper is to provide an explicit formula for the (co)dimension of the $*$-congruence orbits and bundles of $n \times n$ Hermitian matrix pencils.

Matrix pencils (or just "pencils" for short) in this work are seen as matrix polynomials of degree 1 in the variable λ, namely of the form $A + \lambda B$, with $A, B \in \mathbb{C}^{n\times n}$. Two pencils $A + \lambda B$ and $\widetilde{A} + \lambda\widetilde{B}$ are said to be $*$-*congruent* if there is an invertible matrix $P \in \mathbb{C}^{n\times n}$ such that $\widetilde{A} + \lambda\widetilde{B} = P^*(A + \lambda B)P$, namely $\widetilde{A} = P^*AP$ and $\widetilde{B} = P^*BP$ (where $*$ denotes the conjugate transpose).

The $*$-*congruence orbit* of a pencil $A+\lambda B$ is the set of all pencils that are $*$-congruent to $A+\lambda B$, namely $\mathcal{O}^*(A + \lambda B) := \{P^*(A + \lambda B)P : \; P \text{ invertible}\}$. When $\mathcal{H}(\lambda) := A + \lambda B$ is *Hermitian* (that is, $A^* = A$ and $B^* = B$), this orbit is known as the *Hermitian orbit of* $\mathcal{H}$ (see [5]) and is denoted by $\mathcal{O}^H(\mathcal{H})$. Note that all pencils in $\mathcal{O}^H(\mathcal{H})$ are Hermitian.

One of the motivations for dealing with Hermitian orbits is because of their connection with the *eigenstructure*. The eigenstructure of matrix pencils is the relevant information of the pencils in most of the applied problems where they arise, in particular those related with the system of ODEs $Bx'(t)+Ax(t) = f(t)$, where $x(t)$ is a vector of unknown functions and $f(t)$ is some vector function. More precisely, the eigenstructure is key to analyze the solvability of the system, and it provides an explicit expression of the solution when it exists [22]. The eigenstructure of a Hermitian pencil, $\mathcal{H}(\lambda)$, comprises the set of invariants of $\mathcal{H}(\lambda)$ under $*$-congruence (see Section 2.1). Therefore, all pencils in the Hermitian orbit $\mathcal{O}^H(\mathcal{H})$ have the same eigenstructure; moreover, any Hermitian pencil of the same size having the same eigenstructure as $\mathcal{H}(\lambda)$ belongs to this orbit. As a consequence, there is a one-to-one correspondence between eigenstructures of Hermitian pencils and Hermitian orbits. The eigenstructure is encoded in the *Hermitian Kronecker canonical form*, denoted HKCF (see Theorem 2.2), which is the canonical form under $*$-congruence of Hermitian pencils, and it is the same for every pencil in $\mathcal{O}^H(\mathcal{H})$.

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It is natural to pose the following questions:

(Q1) Which is the most likely eigenstructure (or HKCF) for arbitrary $n \times n$ Hermitian pencils?
(Q2) Given two particular eigenstructures (or HKCFs), which one is more likely to happen?

In order to answer these questions, we first need to clarify what we mean by "likely". For this, there is a prior issue that deserves to be considered. The eigenstructure of a Hermitian pencil includes the so-called *eigenvalues*. These are complex numbers, together with ∞ and (up to some extent related to the difference between real and pairs of non-real conjugate eigenvalues) they are equally likely to happen. Hence, we should give all of them the same likelihood and, as a consequence, their exact value becomes irrelevant. This leads to the notion of *Hermitian bundle*, which is the set containing all Hermitian pencils with the same eigenstructure up to the specific values of the eigenvalues (as long as the eigenvalues which are different in one pencil stay different in any other pencil). This notion was first introduced by Arnold in 1971 [1] for matrices (the eigenstructure being in this case the set of invariants under similarity).

Using the notion of bundle, an answer to question (Q1) has been provided in [5]. In that work, the set of $n\times n$ Hermitian pencils is described as the union of a finite number of bundle closures, none of them including any other. This means that there is a finite number of HKCFs of $n \times n$ Hermitian pencils (up to the specific values of the eigenvalues) such that, in any neighborhood of every $n\times n$ Hermitian pencil, there are infinitely many Hermitian pencils having one of these HKCFs. Hence, the eigenstructures corresponding to these HKCFs, which are termed *generic*, are the most likely ones among all eigenstructures of $n \times n$ Hermitian pencils. The corresponding bundles are also termed "generic" bundles. Later in [5], the codimension of these bundles is shown to be equal to 0, which supports referring to them as generic.

Following the ideas mentioned in the preceding paragraph, the answer to (Q2) could come using closure inclusions between bundles as well. More precisely, the eigenstructure $\mathcal{K}_1$ is more likely to occur than the eigenstructure $\mathcal{K}_2$ if $\mathcal{K}_2$ belongs to the closure of the bundle associated with $\mathcal{K}_1$. However, describing such relationships between bundle closures is a tough problem. Instead, we could get an idea of how "large" the orbit (or bundle) is by means of its dimension (considering the orbit as a differentiable manifold over $\mathbb{R}$, see Section 2.1). In particular, if the dimension of the bundle associated with $\mathcal{K}_1$ is larger than that of the bundle associated with $\mathcal{K}_2$, then $\mathcal{K}_2$ cannot be more likely than $\mathcal{K}_1$. Then, computing the dimension of the orbits (and bundles) can be seen as a first step toward the complete answer to (Q2).

Instead of the dimension of orbits, we deal with their codimension, since the codimension is more straightforward to compute. In Theorem 3.3 and Corollary 5.1, which are the main results of this paper, we provide an explicit formula for the codimension of the Hermitian orbit and the Hermitian bundle of any Hermitian pencil, respectively, which is given in terms of the HKCF of the pencil.

This work lies within a research field devoted to the analysis of orbits of matrices and matrix pencils under particular equivalence relations. In particular, the (co)dimension of orbits and bundles, which is the main topic of the present work, has been considered in the following references:

- The seminal reference [21], where the codimension of the set singular $n \times n$ matrix pencils was obtained.
- Some years later, in [11], the (co)dimension of orbits of matrices under similarity and of matrix pencils under strict equivalence was obtained. In the recent reference [10], it is proved, by direct methods, that if a pencil M belongs to the closure of the orbit of another pencil L, then the dimension

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of the orbit of M is smaller than or equal to the dimension of the orbit of L, and the equality is attained if and only if M belongs to the orbit of L.
- In [15], the authors obtained the codimension of the so-called *controllability pairs* of matrices.
- The dimension of the orbits of triples of matrices associated with linear systems, under a suitable action of a suitable group, was obtained in [17].
- Matrix quadruples (E, A, B, C), with $E, A \in \mathbb{C}^{n\times n}, B \in \mathbb{C}^{n\times m}$, and $C \in \mathbb{C}^{p\times n}$, associated with linear time-invariant systems of the form $Ex'(t) = Ax(t) + Bu(t),\ y(t) = Cx(t)$, were considered in [3]. A suitable action of the group $\mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$ on these quadruples was introduced, and lower and upper bounds for the dimension of the associated orbits were obtained.
- The codimension of the generic orbits under strict equivalence for the set of matrix pencils with bounded rank was obtained in [6].
- In [19], the authors considered the so-called *generalized matrix products*. They introduced an appropriate similarity transformation on these matrices and presented a suitable canonical form. Then, they provided a formula for the dimension of the associated orbits in terms of this canonical form.
- The work [8], where the authors computed the dimension of the congruence orbit of square matrices.
- The subsequent paper [7], which is a natural continuation of [8] for orbits of square matrices under $*$-congruence. Related to this work, in [2], the authors deal with the Lie algebra $\{X \in \mathbb{C}^{n\times n} : SX^* = -XS\}$, with $S \in \mathbb{C}^{n\times n}$ being invertible, and obtain its real dimension.
- The papers [13] and [14] present the codimension of the orbits of, respectively, skew-symmetric and symmetric matrix pencils under congruence.
- In [4], the dimension of the sets of structured matrix pencils (palindromic and alternating ones) was obtained by analyzing the dimension of congruence orbits of matrices.
- In the recent paper [5], mentioned above, a general approach for computing the codimension of orbits and bundles of $n \times n$ Hermitian pencils has been presented, and, using this approach, the codimension of the generic bundles has been proved to be 0.

Besides the previous works, the codimensions of orbits of linearizations of full rank matrix polynomials and general matrix polynomials have been studied, respectively, in [18] and [12].

Moreover, the main approach and techniques used in the present work resemble very much those of the previous works [9, 8, 7, 5, 14, 13]. In all these works, a $\star$-Sylvester equation $AX + X^\star B = 0$ or a system of two $\star$-Sylvester equations must be solved at a certain moment. After appropriate transformations, the equation or the system of equations is reduced to another one in which A and B have suitable canonical form, for which the corresponding equation is easier to solve. We will take advantage of the solution of some of these equations in our developments.

The paper is organized as follows. In Section 2, we introduce the basic notions and notation that will be used throughout the manuscript, and we also present, in Section 2.2, the general approach followed to get the main goal. Section 3, which is the main section of the manuscript, is devoted to computing the dimension of the solution space of all systems of equations arising when the approach mentioned before is carried out. Plugging all this information, we state the main result of the paper, namely Theorem 3.3, which displays the codimension of the Hermitian orbits of Hermitian matrix pencils in terms of their associated HKCF. In Section 4, we illustrate Theorem 3.3 by explicitly displaying the codimension of all possible HKCFs of 3×3 Hermitian pencils. In Section 5, we deal with bundles instead of orbits, and present, in Corollary 5.1, the codimension count for the Hermitian bundles. Finally, in Section 6, we summarize the main contribution of the paper and outline some future research.

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2. Basic definitions and notation. We denote the fields of real and complex numbers by $\mathbb{R}$ and $\mathbb{C}$, respectively. For a complex number μ, we use the standard notations $\operatorname{Re}(\mu)$ and $\operatorname{Im}(\mu)$ to denote its real and imaginary parts, respectively. The transpose and the conjugate transpose of a matrix A are denoted by $A^\top$ and A^*, respectively.

Two $n \times n$ pencils $\mathcal{H}_1(\lambda)$ and $\mathcal{H}_2(\lambda)$ are said to be $*$-congruent if there exists an invertible matrix $Q \in \mathbb{C}^{n\times n}$ such that $\mathcal{H}_2(\lambda) = Q^*\mathcal{H}_1(\lambda)Q$. This equivalence preserves the Hermitian structure, namely: $\mathcal{H}_2(\lambda)$ is Hermitian if and only if $\mathcal{H}_1(\lambda)$ is Hermitian.

Sometimes, and for the sake of brevity, we will omit the variable λ when referring to a pencil and write, for instance, $\mathcal{H}$ instead of $\mathcal{H}(\lambda)$.

The set of complex $n \times n$ Hermitian pencils is denoted by $\mathrm{PENCIL}^H_{n\times n}$.

The *direct sum* of pencils $\mathcal{P}_1, \dots, \mathcal{P}_k$ is the block diagonal pencil with diagonal blocks $\mathcal{P}_1, \dots, \mathcal{P}_k$ in this order, and it is written as $\bigoplus_{i=1}^k \mathcal{P}_i$ (or $\mathcal{P}_1 \oplus \cdots \oplus \mathcal{P}_k$).

We denote the $k \times k$ *reverse identity matrix* by R_k, namely

$$R_k := \begin{bmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{bmatrix}_{k\times k}.$$

A $k \times k$ Jordan block associated with the eigenvalue a is denoted by $J_k(a)$, namely

$$J_k(a) := \begin{bmatrix} a & 1 & & \\ & \ddots & \ddots & \\ & & a & 1 \\ & & & a \end{bmatrix}_{k\times k}.$$

We also introduce the matrices

$$F_d := \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \end{bmatrix}_{d\times(d+1)}, \qquad G_d := \begin{bmatrix} 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix}_{d\times(d+1)},$$

$$H_k(a) := \begin{bmatrix} R_{k-1} & \\ & 0 \end{bmatrix} + aR_k = \begin{bmatrix} 0 & & 1 & a \\ & \iddots & \iddots & \\ 1 & a & & \\ a & & & \end{bmatrix}_{k\times k}, \quad \text{and} \quad L_k(a) := \begin{bmatrix} a & & & \\ 1 & a & & \\ & \ddots & \ddots & \\ & & 1 & a \end{bmatrix}_{k\times k}.$$

The following straightforward identities will be used several times throughout this work:

$$R_k = R_k^\top = R_k^* = R_k^{-1}, \tag{2.1}$$

$$R_{d+1}F_d^\top R_d = G_d^\top, \tag{2.2}$$

$$L_k(a) = R_k J_k(a) R_k^{-1}, \tag{2.3}$$

$$R_k H_k(a) = L_k(a), \quad \text{and} \tag{2.4}$$

$$H_k(a) R_k = J_k(a). \tag{2.5}$$

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Also, the following matrix pencils, constructed from the previous matrices, will be used (we borrow the notation from [5]):

$$\mathcal{L}_d(\lambda) := F_d + \lambda G_d = \begin{bmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \lambda & 1 \end{bmatrix}_{d\times(d+1)}, \quad \mathcal{M}_d(\lambda) := \begin{bmatrix} 0 & \mathcal{L}_d(\lambda)^\top \\ \mathcal{L}_d(\lambda) & 0 \end{bmatrix}_{(2d+1)\times(2d+1)}, \tag{2.6}$$

$$\mathcal{J}_k^H(\mu) := H_k(-\mu) + \lambda R_k := \begin{bmatrix} & & 1 & \lambda-\mu \\ & \iddots & \iddots & \\ 1 & \lambda-\mu & & \\ \lambda-\mu & & & \end{bmatrix}_{k\times k} \quad (\mu\in\mathbb{C}), \tag{2.7}$$

$$\mathcal{J}_k^H(\infty) := R_k + \lambda H_k(0) = \begin{bmatrix} & & \lambda & 1 \\ & \iddots & \iddots & \\ \lambda & 1 & & \\ 1 & & & \end{bmatrix}_{k\times k}, \tag{2.8}$$

$$\mathcal{J}_k^H(\mu,\bar{\mu}) := \begin{bmatrix} 0 & \mathcal{J}_k^H(\bar{\mu}) \\ \mathcal{J}_k^H(\mu) & 0 \end{bmatrix}_{(2k)\times(2k)}. \tag{2.9}$$

We set $\mathcal{M}_0(\lambda) := 0$ (namely, a null 1×1 matrix).

2.1. Hermitian orbit and Hermitian Kronecker canonical form. The *Hermitian orbit* of a given $n\times n$ Hermitian pencil $\mathcal{H}(\lambda) = A + \lambda B$ is defined as the set of pencils that are $*$-congruent to $\mathcal{H}$, namely

$$\mathcal{O}^H(\mathcal{H}) := \{\, Q^*\mathcal{H}(\lambda)Q : Q\in\mathbb{C}^{n\times n} \text{ invertible}\,\}.$$

Note that if $\mathcal{H}$ is Hermitian, then every pencil in $\mathcal{O}^H(\mathcal{H})$ is Hermitian as well.

The Hermitian orbit of a Hermitian pencil $\mathcal{H}$ is a differentiable manifold over $\mathbb{R}$, and the tangent space at the point $\mathcal{H}$ is (see [5, Section 6]):

$$T^H(\mathcal{H}) := \{P^*\mathcal{H}(\lambda) + \mathcal{H}(\lambda)P \;:\; P\in\mathbb{C}^{n\times n}\}.$$

The dimension (over $\mathbb{R}$) of $T^H(\mathcal{H})$ at $\mathcal{H}$ is the *dimension of the Hermitian orbit* of $\mathcal{H}$, denoted by $\dim_{\mathbb{R}}\mathcal{O}^H(\mathcal{H})$. Using the Frobenius inner product $\langle A+\lambda B, C+\lambda D\rangle = \operatorname{trace}(AC^* + BD^*)$ to define orthogonality over the set of $n\times n$ matrix pencils, the *codimension of the Hermitian orbit* of $\mathcal{H}$, denoted by $\operatorname{codim}_{\mathbb{R}}\mathcal{O}^{\rm H}(\mathcal{H})$, is the dimension of the normal space to $T^H(\mathcal{H})$. The tangent space $T^H(\mathcal{H})$ and the normal space here are all considered in the vector space $\text{PENCIL}^H_{n\times n}$.

We are interested in the dimension of $\mathcal{O}^H(\mathcal{H})$. The general approach that we are going to follow to obtain this dimension was introduced in [5, page 277], and we reproduce it here for completeness and for the ease of reading. Since the dimension and codimension can be obtained from each other by virtue of the identity

$$\operatorname{codim}_{\mathbb{R}}\mathcal{O}^{\rm H}(\mathcal{H}) = \dim_{\mathbb{R}}(\text{PENCIL}^{\rm H}_{\rm n\times n}) - \dim_{\mathbb{R}}\mathcal{O}^{\rm H}(\mathcal{H}), \tag{2.10}$$

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we will consider instead the codimension of $\mathcal{O}^H(\mathcal{H})$. The following theorem in [5] reduces the problem of computing the codimension of $\mathcal{O}^H(A+\lambda B)$ to computing the dimension of the solution space over $\mathbb{R}$ of a system of equations associated with the Hermitian pencil $A+\lambda B$.

Theorem 2.1. [5, Theorem 6.1] *The (real) codimension of the Hermitian orbit of an $n\times n$ Hermitian pencil $A+\lambda B$ is equal to the (real) dimension of the solution space of the system*

$$X^*A + AX = 0, \qquad X^*B + BX = 0. \tag{2.11}$$

Since (2.11) is a system of equations associated with the pencil $\mathcal{H}(\lambda) = A+\lambda B$, to obtain the codimension of the Hermitian orbit of a given Hermitian pencil $\mathcal{H}$, one can choose a proper Hermitian pencil in $\mathcal{O}^H(\mathcal{H})$ to get a suitable system (2.11), which is easy to handle. The pencil in $\mathcal{O}^H(\mathcal{H})$ that we are going to use is presented in the following result, which was obtained in [20, Theorem 6.1], though we use the version from [5, Theorem 2.1].

Theorem 2.2 (Hermitian Kronecker canonical form). *Every $n\times n$ Hermitian pencil, $\mathcal{H}$, is $*$-congruent to a direct sum of blocks of the following types:*

(i) *blocks $\sigma\mathcal{J}_k^H(a)$, with $a\in\mathbb{R}$ and $\sigma\in\{+1,-1\}$;*
(ii) *blocks $\sigma\mathcal{J}_k^H(\infty)$, with $\sigma\in\{+1,-1\}$;*
(iii) *blocks $\mathcal{J}_k^H(\mu,\overline{\mu})$, with $\mu\in\mathbb{C}$ having positive imaginary part;*
(iv) *blocks $\mathcal{M}_k(\lambda)$.*

- *The parameters a, k, σ, and μ may be distinct in different blocks.*
- *The number of blocks of each type and the corresponding parameters are uniquely determined by $\mathcal{H}$, and they are the invariants of $\mathcal{H}$ under $*$-congruence.*
- *The direct sum is unique up to permutation of blocks.*

The direct sum in Theorem 2.2 is called the *Hermitian Kronecker canonical form (HKCF)* of $\mathcal{H}$, and we denote it by HKCF($\mathcal{H}$).

The values a associated with the blocks $\sigma\mathcal{J}_k^H(a)$ in Theorem 2.2 are the *real eigenvalues* of $\mathcal{H}$, and the pair of values $(\mu,\overline{\mu})$ associated with the blocks $\mathcal{J}_k^H(\mu,\overline{\mu})$ are *pairs of complex conjugate eigenvalues* of $\mathcal{H}$. All together they form the set of *finite eigenvalues* of $\mathcal{H}$. If there is at least one block of the form $\sigma\mathcal{J}_k^H(\infty)$ in HKCF($\mathcal{H}$), then $\mathcal{H}$ has an *infinite* eigenvalue. As for the blocks $\mathcal{M}_k(\lambda)$, they correspond to a pair of *left and right minimal indices* equal to k [5]. The eigenvalues, together with the sizes of the associated Jordan blocks and the parameters σ, as well as the minimal indices, form the so-called *eigenstructure* of $\mathcal{H}$.

2.2. The block partition approach. Assume that the Hermitian pencil $\mathcal{H}(\lambda) = A+\lambda B$ in Theorem 2.1 is the direct sum of m blocks, namely $A+\lambda B = (A_1+\lambda B_1)\oplus(A_2+\lambda B_2)\oplus\cdots\oplus(A_m+\lambda B_m)$. We replace this direct sum into the system (2.11) and partition the unknown matrix X accordingly:

$$\begin{bmatrix} X_{11}^* & X_{21}^* & \cdots & X_{m1}^* \\ X_{12}^* & X_{22}^* & \cdots & X_{m2}^* \\ \vdots & \vdots & \ddots & \vdots \\ X_{1m}^* & X_{1m}^* & \cdots & X_{mm}^* \end{bmatrix} \begin{bmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_m \end{bmatrix} + \begin{bmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_m \end{bmatrix} \begin{bmatrix} X_{11} & X_{12} & \cdots & X_{1m} \\ X_{21} & X_{22} & \cdots & X_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ X_{m1} & X_{m2} & \cdots & X_{mm} \end{bmatrix} = 0,$$

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$$\begin{bmatrix} X_{11}^* & X_{21}^* & \cdots & X_{m1}^* \\ X_{12}^* & X_{22}^* & \cdots & X_{m2}^* \\ \vdots & \vdots & \ddots & \vdots \\ X_{1m}^* & X_{2m}^* & \cdots & X_{mm}^* \end{bmatrix} \begin{bmatrix} B_1 & & & \\ & B_2 & & \\ & & \ddots & \\ & & & B_m \end{bmatrix} + \begin{bmatrix} B_1 & & & \\ & B_2 & & \\ & & \ddots & \\ & & & B_m \end{bmatrix} \begin{bmatrix} X_{11} & X_{12} & \cdots & X_{1m} \\ X_{21} & X_{22} & \cdots & X_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ X_{m1} & X_{m2} & \cdots & X_{mm} \end{bmatrix} = 0.$$

Operating on the left-hand side of the above identities yields the equivalent system

$$\begin{bmatrix} X_{11}^*A_1 + A_1X_{11} & X_{21}^*A_2 + A_1X_{12} & \cdots & X_{m1}^*A_m + A_1X_{1m} \\ X_{12}^*A_1 + A_2X_{21} & X_{22}^*A_2 + A_2X_{22} & \cdots & X_{m2}^*A_m + A_2X_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ X_{1m}^*A_1 + A_mX_{m1} & X_{2m}^*A_2 + A_mX_{m2} & \cdots & X_{mm}^*A_m + A_mX_{mm} \end{bmatrix} = 0,$$

$$\begin{bmatrix} X_{11}^*B_1 + B_1X_{11} & X_{21}^*B_2 + B_1X_{12} & \cdots & X_{m1}^*B_m + B_1X_{1m} \\ X_{12}^*B_1 + B_2X_{21} & X_{22}^*B_2 + B_2X_{22} & \cdots & X_{m2}^*B_m + B_2X_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ X_{1m}^*B_1 + B_mX_{m1} & X_{2m}^*B_2 + B_mX_{m2} & \cdots & X_{mm}^*B_m + B_mX_{mm} \end{bmatrix} = 0.$$

Note that the (i,j)-block in the off-diagonal of the block-partitioned matrices in the left-hand side of the previous identities is the conjugate transpose of the (j,i)-block, since the matrices A_i and B_i are Hermitian, for $i = 1, \ldots, m$. Therefore, we only need to consider the identities coming from the upper triangular blocks (including the diagonal ones). Moreover, the equations in the above system can be divided into two classes according to the block positions: those on the diagonal, of the form $X^*M + MX = 0$, with $M = A_i$ or B_i, for $i = 1, \ldots, m$; and those in the off-diagonal, of the form $ZM + NY = 0$, with $M = A_i$ or B_i and $N = A_j$ or B_j, for $i \neq j$ and $i,j = 1, \ldots, m$. Accordingly, for the Hermitian pencils $A_i + \lambda B_i$ and $A_j + \lambda B_j$, we introduce the following systems, as defined in [5, page 277]:

$$\text{syst}(A_i + \lambda B_i): \quad X^*A_i + A_iX = 0, \qquad X^*B_i + B_iX = 0; \tag{2.12}$$

$$\text{syst}(A_i + \lambda B_i, A_j + \lambda B_j): \quad ZA_j + A_iY = 0, \qquad ZB_j + B_iY = 0. \tag{2.13}$$

If we want to obtain the codimension of $\mathcal{O}^H(A + \lambda B)$, we can compute the dimension of the solution space of each $\text{syst}(A_i + \lambda B_i)$, for $i = 1, \ldots, m$, and $\text{syst}(A_i + \lambda B_i, A_j + \lambda B_j)$, for $i,j = 1, \ldots, m$ and $i < j$, and then sum the dimension of the solution spaces of all these systems.

When $A + \lambda B$ is given in HKCF, namely every $A_i + \lambda B_i$ is of one of the four types of blocks mentioned in Theorem 2.2, we need to compute the dimension of the solution spaces for the following 14 systems:

$$\begin{array}{lll} \text{syst}\left(\mathcal{J}_k(\infty)\right), & \text{syst}\left(\sigma_1\mathcal{J}_{k_1}(\infty), \sigma_2\mathcal{J}_{k_2}(\infty)\right), & \text{syst}\left(\sigma_1\mathcal{J}_{k_1}(a), \sigma_2\mathcal{J}_{k_2}(\infty)\right), \\ \text{syst}\left(\mathcal{J}_k(a)\right), & \text{syst}\left(\sigma_1\mathcal{J}_{k_1}(a), \sigma_2\mathcal{J}_{k_2}(b)\right), & \text{syst}\left(\sigma\mathcal{J}_{k_1}(\infty), \mathcal{J}_{k_2}(\mu,\bar{\mu})\right), \\ \text{syst}\left(\mathcal{J}_k(\mu,\bar{\mu})\right), & \text{syst}\left(\mathcal{J}_{k_1}(\mu_1,\bar{\mu}_1), \mathcal{J}_{k_2}(\mu_2,\bar{\mu}_2)\right), & \text{syst}\left(\sigma\mathcal{J}_{k_1}(a), \mathcal{J}_{k_2}(\mu,\bar{\mu})\right), \\ \text{syst}\left(\mathcal{M}_d\right), & \text{syst}\left(\mathcal{M}_{d_1}, \mathcal{M}_{d_2}\right), & \text{syst}\left(\sigma\mathcal{J}_k(\infty), \mathcal{M}_d\right), \\ & & \text{syst}\left(\sigma\mathcal{J}_k(a), \mathcal{M}_d\right), \\ & & \text{syst}\left(\mathcal{J}_k(\mu,\bar{\mu}), \mathcal{M}_d\right). \end{array} \tag{2.14}$$

For a general Hermitian pencil $\mathcal{H}$, Theorem 2.2 guarantees that there exists a Hermitian pencil in $\mathcal{O}^H(\mathcal{H})$, which is in HKCF, namely HKCF($\mathcal{H}$). Therefore, we can restrict ourselves to $\mathcal{H}$ being in HKCF and solve the corresponding system in (2.14), depending on its canonical blocks. We then apply the results of Section 3 and sum the dimension of the solution spaces of those systems to obtain $\text{codim}_{\mathbb{R}}\mathcal{O}^{\text{H}}(\mathcal{H})$.

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2.3. The Sylvester equation. As we will see in Section 3, some of the systems in (2.14) will end up with a Sylvester equation, so here we present some results on Sylvester equations to be used in Section 3.

The *Sylvester equation* is the linear matrix equation $AX - XB = C$, where A, B, and C are given matrices, and X is the unknown. We focus on the homogeneous case $C = 0$, i.e., the equation

$$AX = XB, \tag{2.15}$$

where A and B are square matrices (not necessarily of the same size).

The solution of (2.15) is a vector space over $\mathbb{C}$. Chapter 8.1 in [16] describes the solution of (2.15) and its dimension. The relevant information for our purposes is that the dimension of the solution space depends only on the *Jordan canonical form* of A and B. In particular, when A and B are Jordan blocks of sizes $m \times m$ and $n \times n$, respectively, then $X = 0$ is the only solution of (2.15) if the eigenvalues of A and B are different; however, if they have the same eigenvalue, the dimension of the solution space is $\min\{m, n\}$, and the structure of X is the following, depending on the sign of $m - n$ (see [16, page 218]):

$$X = \begin{bmatrix} x_1 & x_2 & \cdots & x_m \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & x_1 & x_2 \\ 0 & \cdots & 0 & x_1 \end{bmatrix}, \quad X = \begin{bmatrix} 0 & \cdots & 0 & x_1 & x_2 & \cdots & x_m \\ \vdots & & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & x_1 & x_2 \\ 0 & \cdots & 0 & \cdots & 0 & 0 & x_1 \end{bmatrix}, \quad X = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & x_2 \\ 0 & \cdots & 0 & x_1 \\ 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \end{bmatrix}, \tag{2.16}$$

$$(m = n) \qquad\qquad (m < n) \qquad\qquad (m > n)$$

where $x_1, x_2, \ldots, x_{\min\{m,n\}}$ are free variables. More in general, if the elementary divisors of A and B are

$$\begin{aligned} &(A): \quad (\lambda - \lambda_1)^{p_1}, \ (\lambda - \lambda_2)^{p_2}, \ \ldots, \ (\lambda - \lambda_u)^{p_u}, \quad (p_1 + p_2 + \cdots + p_u = m), \\ &(B): \quad (\lambda - \mu_1)^{q_1}, \ (\lambda - \mu_2)^{q_2}, \ \ldots, \ (\lambda - \mu_v)^{q_v}, \quad (q_1 + q_2 + \cdots + q_v = n), \end{aligned}$$

then the dimension of the solution space of (2.15) is

$$N = \sum_{\alpha=1}^{u} \sum_{\beta=1}^{v} \delta_{\alpha\beta}, \tag{2.17}$$

where $\delta_{\alpha\beta}$ denotes the degree of the greatest common divisor of $(\lambda - \lambda_\alpha)^{p_\alpha}$ and $(\lambda - \mu_\beta)^{q_\beta}$ (see Theorem 1 of [16, Chapter 8]). Regarding the structure of the solution of (2.15), since the only case where we need is precisely when both A and B in (2.15) are similar to Jordan blocks, here we only reproduce this case from [16, Chapter 8].

Let U and V be nonsingular matrices of sizes $m \times m$ and $n \times n$, respectively, such that $J_A = U^{-1}AU$ and $J_B = V^{-1}BV$ are the Jordan canonical forms of A and B, respectively. Then, the general solution of (2.15) is given by

$$X = U\widetilde{X}V^{-1}, \tag{2.18}$$

where $\widetilde{X}$ is the general solution of the equation $J_A X = X J_B$, described above. Hence, when J_A and J_B are both Jordan blocks, we can first obtain $\widetilde{X}$ from (2.16) and then use (2.18) to obtain the general solution X.

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3. Codimension computations. The main goal of this section is to compute the dimension over $\mathbb{R}$ of the solution spaces of the systems in (2.14). This is the main contribution of this work, which will be summarized in Theorem 3.3. We analyze each system independently and, in some cases, the following remark will be used.

REMARK 3.1. *If $L_1(\lambda) = A_1 + \lambda B_1$ and $L_2(\lambda) = A_2 + \lambda B_2$ are two matrix pencils, then the dimension of the solution space of the four systems* $\operatorname{syst}(L_1, L_2), \operatorname{syst}(-L_1, L_2), \operatorname{syst}(L_1, -L_2)$, *and* $\operatorname{syst}(-L_1, -L_2)$ *coincide. For instance,* $\operatorname{syst}(-L_1, L_2)$ *reads, according to (2.13), $ZA_2 + (-A_1)Y = 0 = ZB_2 + (-B_1)Y$. Replacing Y by $-Y$ (which is a change of variables), this system becomes* $\operatorname{syst}(L_1, L_2)$, *so the dimension of the solution space is the same for both systems. Using similar arguments, we can see that the dimension of the solution space of the remaining systems* $\operatorname{syst}(L_1, -L_2)$ *and* $\operatorname{syst}(-L_1, -L_2)$ *also coincides with that of* $\operatorname{syst}(L_1, L_2)$ *(note, in particular, that* $\operatorname{syst}(L_1, L_2) = \operatorname{syst}(-L_1, -L_2)$*).*

We first analyze the systems in (2.14) involving only one block of the HKCF (in parts 1–4) and then those involving two blocks (in parts 5–14).

1. $\boxed{\operatorname{syst}\left(\mathcal{J}_k^H(\infty)\right)}$ Since $\mathcal{J}_k^H(\infty) = R_k + \lambda H_k(0)$, the system $\operatorname{syst}\left(\mathcal{J}_k^H(\infty)\right)$ reads:

$$X^* R_k + R_k X = 0, \qquad X^* H_k(0) + H_k(0) X = 0. \tag{3.19}$$

Let $Y = R_k X$, so that $X^* R_k = Y^*$ by (2.1). Then, the first equation in (3.19) is equivalent to $Y^* + Y = 0$. Also, applying (2.1), (2.4), and (2.5) to the second equation in (3.19), this equation becomes $Y^* L_k(0) + J_k(0) Y = 0$. Then, by means of the previous change of variables, the system (3.19) becomes:

$$Y^* + Y = 0, \quad Y^* L_k(0) + J_k(0) Y = 0. \tag{3.20}$$

Since R_k is invertible, the solution spaces of (3.19) and (3.20) have the same dimension, so we focus on the solution space of the system (3.20).

From the first equation of (3.20), we get $Y^* = -Y$. Then, we plug it into the second equation to obtain

$$Y L_k(0) - J_k(0) Y = 0, \tag{3.21}$$

which is a Sylvester equation on Y. From (2.3), we know that $J_k(0) = R_k^{-1} L_k(0) R_k$ is the Jordan form of $L_k(0)$. Now, using (2.16) and (2.18), the solution of (3.21) is

$$Y = \begin{bmatrix} y_k & y_{k-1} & \cdots & y_2 & y_1 \\ 0 & y_k & y_{k-1} & \cdots & y_2 \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & y_k & y_{k-1} \\ 0 & \cdots & 0 & 0 & y_k \end{bmatrix} R_k = \begin{bmatrix} y_1 & y_2 & \cdots & y_{k-1} & y_k \\ y_2 & \cdots & y_{k-1} & y_k & 0 \\ \vdots & \iddots & \iddots & \iddots & \vdots \\ y_{k-1} & y_k & 0 & \cdots & 0 \\ y_k & 0 & 0 & \cdots & 0 \end{bmatrix}, \tag{3.22}$$

where $y_1, \ldots, y_k$ are free variables.

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Replacing in the first equation of (3.20), we get

$$\begin{bmatrix} \overline{y}_1 & \overline{y}_2 & \cdots & \overline{y}_{k-1} & \overline{y}_k \\ \overline{y}_2 & \cdots & \overline{y}_{k-1} & \overline{y}_k & 0 \\ \vdots & \iddots & \iddots & \iddots & \vdots \\ \overline{y}_{k-1} & \overline{y}_k & 0 & \cdots & 0 \\ \overline{y}_k & 0 & \cdots & \cdots & 0 \end{bmatrix} = \begin{bmatrix} -y_1 & -y_2 & \cdots & -y_{k-1} & -y_k \\ -y_2 & \cdots & -y_{k-1} & -y_k & 0 \\ \vdots & \iddots & \iddots & \iddots & \vdots \\ -y_{k-1} & -y_k & 0 & \cdots & 0 \\ -y_k & 0 & \cdots & \cdots & 0 \end{bmatrix},$$

that is, $\overline{y}_i = -y_i$, for $i = 1, \dots k$, which implies that y_i is an arbitrary purely imaginary number, for $i = 1, \dots k$.

Hence, the dimension over $\mathbb{R}$ of the solution space of syst $\left(\mathcal{J}_k^H(\infty)\right)$ is k.

2. $\boxed{\text{syst}\left(\mathcal{J}_k^H(a)\right), a \in \mathbb{R}}$ Since $\mathcal{J}_k^H(a) = H_k(-a) + \lambda R_k$, the system syst $\left(\mathcal{J}_k^H(a)\right)$ reads:

$$X^* H_k(-a) + H_k(-a)X = 0, \tag{3.23}$$

$$X^* R_k + R_k X = 0. \tag{3.24}$$

Since $H_k(-a) = H_k(0) - aR_k$, the equation (3.23) can be rewritten as $-a\left(X^* R_k + R_k X\right) + X^* H_k(0) + H_k(0)X = 0$, and, using (3.24), this simplifies to $X^* H_k(0) + H_k(0)X = 0$. Then, we get the same system as (3.19), namely syst $\left(\mathcal{J}_k^H(\infty)\right)$, so the dimension over $\mathbb{R}$ of the solution space of syst $\left(\mathcal{J}_k^H(a)\right)$ is the same as for syst $\left(\mathcal{J}_k^H(\infty)\right)$, namely k.

3. $\boxed{\text{syst}\left(\mathcal{J}_k^H(\mu,\bar{\mu})\right), \text{Im}(\mu) > 0}$ The pencil $\mathcal{J}_k^H(\mu,\bar{\mu})$ can be expressed as

$$\mathcal{J}_k^H(\mu,\bar{\mu}) = \begin{bmatrix} 0 & \mathcal{J}_k^H(\bar{\mu}) \\ \mathcal{J}_k^H(\mu) & 0 \end{bmatrix} = \begin{bmatrix} 0 & H_k(-\bar{\mu}) \\ H_k(-\mu) & 0 \end{bmatrix} + \lambda R_{2k}.$$

Then, syst $\left(\mathcal{J}_k^H(\mu,\bar{\mu})\right)$ reads:

$$X^* \begin{bmatrix} 0 & H_k(-\bar{\mu}) \\ H_k(-\mu) & 0 \end{bmatrix} + \begin{bmatrix} 0 & H_k(-\bar{\mu}) \\ H_k(-\mu) & 0 \end{bmatrix} X = 0, \qquad X^* R_{2k} + R_{2k} X = 0. \tag{3.25}$$

As we did before, we can perform a change of variables $Y = R_{2k}X = \left[\begin{smallmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{smallmatrix}\right]$, where Y_{11}, Y_{12}, Y_{21}, and Y_{22} are all of size $k \times k$. Then, $X = R_{2k}Y$ by (2.1), so

$$X = \begin{bmatrix} 0 & R_k \\ R_k & 0 \end{bmatrix} Y = \begin{bmatrix} R_k Y_{21} & R_k Y_{22} \\ R_k Y_{11} & R_k Y_{12} \end{bmatrix} \quad \text{and} \quad X^* = Y^* R_{2k} = Y^* \begin{bmatrix} 0 & R_k \\ R_k & 0 \end{bmatrix} = \begin{bmatrix} Y_{21}^* R_k & Y_{11}^* R_k \\ Y_{22}^* R_k & Y_{12}^* R_k \end{bmatrix}.$$

Using (2.4) and (2.5), after applying the previous change of variables, the system (3.25) becomes

$$\begin{bmatrix} Y_{11}^* L_k(-\mu) + J_k(-\bar{\mu})Y_{11} & Y_{21}^* L_k(-\bar{\mu}) + J_k(-\bar{\mu})Y_{12} \\ Y_{12}^* L_k(-\mu) + J_k(-\mu)Y_{21} & Y_{22}^* L_k(-\bar{\mu}) + J_k(-\mu)Y_{22} \end{bmatrix} = 0, \quad \begin{bmatrix} Y_{11}^* + Y_{11} & Y_{21}^* + Y_{12} \\ Y_{12}^* + Y_{21} & Y_{22}^* + Y_{22} \end{bmatrix} = 0, \tag{3.26}$$

where the first equation can be written as

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$$\begin{bmatrix} Y_{11}^* L_k(-\mu) + J_k(-\bar\mu) Y_{11} & Y_{21}^* L_k(0) + J_k(0) Y_{12} - \bar\mu\,(Y_{21}^* + Y_{12}) \\ Y_{12}^* L_k(0) + J_k(0) Y_{21} - \mu\,(Y_{12}^* + Y_{21}) & Y_{22}^* L_k(-\bar\mu) + J_k(-\mu) Y_{22} \end{bmatrix} = 0. \tag{3.27}$$

From the second equation in (3.26), we get $Y_{11}^* = -Y_{11},\ Y_{21}^* = -Y_{12}$, and $Y_{22}^* = -Y_{22}$, and plugging these identities into (3.27) we obtain

$$\begin{bmatrix} J_k(-\bar\mu) Y_{11} & J_k(0) Y_{12} \\ J_k(0) Y_{21} & J_k(-\mu) Y_{22} \end{bmatrix} = \begin{bmatrix} Y_{11} L_k(-\mu) & Y_{12} L_k(0) \\ Y_{21} L_k(0) & Y_{22} L_k(-\bar\mu) \end{bmatrix}.$$

To find the dimension of the solution space of this equation, we need to solve for the four blocks separately, which is possible because the corresponding equations are decoupled. Moreover, the (2,1)-block can be obtained from the (1,2)-block by conjugate transpose, so we only need to solve the equations corresponding to the blocks (1,1), (1,2), and (2,2). Now we are going to discuss the dimension of the solution space for these three equations.

- (1,1)-block and (2,2)-block.
  The equation corresponding to the (1,1)-block is

  $$J_k(-\bar\mu) Y_{11} = Y_{11} L_k(-\mu). \tag{3.28}$$

  This is a Sylvester equation. Since μ has positive imaginary part, then $\bar\mu \neq \mu$. This implies that $J_k(\bar\mu)$ and $L_k(-\mu)$ have no common eigenvalues, so the only solution of this equation is $Y_{11} = 0$.
  As for the (2,2)-block, the corresponding equation is $J_k(-\mu) Y_{22} = Y_{22} L_k(-\bar\mu)$, which is similar to (3.28), so we also get $Y_{22} = 0$.
- (1,2)-block.
  The equation corresponding to the (1,2)-block is $J_k(0) Y_{12} = Y_{12} L_k(0)$. This equation is the same as (3.21), so Y_{12} is as in the right-hand side of (3.22), which depends on k free complex parameters (namely, $2k$ real parameters).

Summarizing, we have $Y_{11} = Y_{22} = 0$ and $Y_{21}^* = -Y_{12}$, where Y_{12} depends on $2k$ real parameters. Hence, the dimension over $\mathbb{R}$ of the solution space of syst $\left(\mathcal{J}_k^H(\mu, \bar\mu)\right)$ is $2k$.

4. $\boxed{\text{syst}\,(\mathcal{M}_d)}$ The dimension of the solution space of this system has been computed in [5, Section 6.1], and it is equal to $2d + 2$.
5. $\boxed{\text{syst}\left(\sigma_1 \mathcal{J}_{k_1}^H(\infty), \sigma_2 \mathcal{J}_{k_2}^H(\infty)\right)}$ By Remark 3.1, we can get rid of σ_1 and σ_2 and consider only syst $\left(\mathcal{J}_{k_1}^H(\infty), \mathcal{J}_{k_2}^H(\infty)\right)$.
   By (2.8), the system syst $\left(\mathcal{J}_{k_1}^H(\infty), \mathcal{J}_{k_2}^H(\infty)\right)$ reads

   $$Z R_{k_2} + R_{k_1} Y = 0, \qquad Z H_{k_2}(0) + H_{k_1}(0) Y = 0. \tag{3.29}$$

   From the first equation of (3.29), and using (2.1), we get $Z = -R_{k_1} Y R_{k_2}$. Then, we plug it into the second equation and use (2.4) to obtain $-R_{k_1} Y R_{k_2} H_{k_2}(0) + H_{k_1}(0) Y = -R_{k_1} Y L_{k_2}(0) + H_{k_1}(0) Y = 0$. Left-multiplying this equation by $-R_{k_1}$ and using (2.4), we get $Y L_{k_2}(0) - L_{k_1}(0) Y = 0$, which is a Sylvester equation.
   The elementary divisors of $L_{k_1}(0)$ and $L_{k_2}(0)$ are λ^{k_1} and λ^{k_2}, respectively, whose greatest common divisor is λ^k, where $k = \min\{k_1, k_2\}$. Hence, the dimension over $\mathbb{R}$ of the solution space of syst $\left(\mathcal{J}_{k_1}^H(\infty), \mathcal{J}_{k_2}^H(\infty)\right)$ is $2\min\{k_1, k_2\}$ (see Section 2.3).
6. $\boxed{\text{syst}\left(\sigma_1 \mathcal{J}_{k_1}^H(a), \sigma_2 \mathcal{J}_{k_2}^H(b)\right), a, b \in \mathbb{R}}$ By Remark 3.1, we can get rid of σ_1 and σ_2 and consider only syst $\left(\mathcal{J}_{k_1}^H(a), \mathcal{J}_{k_2}^H(b)\right)$, as we did with syst $\left(\sigma_1 \mathcal{J}_{k_1}^H(\infty), \sigma_2 \mathcal{J}_{k_2}^H(\infty)\right)$.

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By (2.7), the system syst $\left(\mathcal{J}_{k_1}^H(a), \mathcal{J}_{k_2}^H(b)\right)$ reads

$$ZH_{k_2}(-b) + H_{k_1}(-a)Y = 0, \qquad ZR_{k_2} + R_{k_1}Y = 0. \tag{3.30}$$

Since the only difference between syst $\left(\mathcal{J}_{k_1}^H(a), \mathcal{J}_{k_2}^H(b)\right)$ and syst $\left(\mathcal{J}_{k_1}^H(\infty), \mathcal{J}_{k_2}^H(\infty)\right)$ is that $H_{k_1}(0)$ and $H_{k_2}(0)$ in (3.29) are replaced by $H_{k_1}(-a)$ and $H_{k_2}(-b)$ in (3.30), applying the same method used for syst $\left(\mathcal{J}_{k_1}^H(\infty), \mathcal{J}_{k_2}^H(\infty)\right)$ to syst $\left(\mathcal{J}_{k_1}^H(a), \mathcal{J}_{k_2}^H(b)\right)$, we end up with the Sylvester equation $YL_{k_2}(-b) - L_{k_1}(-a)Y = 0$.
The elementary divisors of the matrices $L_{k_1}(-a)$ and $L_{k_2}(-b)$ are $(\lambda+a)^{k_1}$ and $(\lambda+b)^{k_2}$, respectively. If $a = b$, then their greatest common divisor is $(\lambda + a)^k$, where $k = \min\{k_1, k_2\}$; and if $a \neq b$, then, they are coprime. Hence, the dimension over $\mathbb{R}$ of the solution space of syst $\left(\mathcal{J}_{k_1}^H(a), \mathcal{J}_{k_2}^H(b)\right)$ is

$$\begin{cases} 2\min\{k_1, k_2\}, & \text{if } a = b, \\ 0, & \text{if } a \neq b. \end{cases} \tag{3.31}$$

7. $\boxed{\text{syst}\left(\mathcal{J}_{k_1}^H(\mu_1, \bar{\mu}_1), \mathcal{J}_{k_2}^H(\mu_2, \bar{\mu}_2)\right)}$ By (2.7) and (2.9), the system syst $\left(\mathcal{J}_{k_1}^H(\mu_1, \bar{\mu}_1), \mathcal{J}_{k_2}^H(\mu_2, \bar{\mu}_2)\right)$ reads

$$Z\begin{bmatrix} 0 & H_{k_2}(-\bar{\mu}_2) \\ H_{k_2}(-\mu_2) & 0 \end{bmatrix} + \begin{bmatrix} 0 & H_{k_1}(-\bar{\mu}_1) \\ H_{k_1}(-\mu_1) & 0 \end{bmatrix} Y = 0, \qquad ZR_{2k_2} + R_{2k_1}Y = 0. \tag{3.32}$$

From the second equation of the system (3.32) and using (2.1), we get $Z = -R_{2k_1}YR_{2k_2}$. Then, we plug it into the first equation and use (2.4) to obtain

$$\begin{aligned} &- R_{2k_1}YR_{2k_2}\begin{bmatrix} 0 & H_{k_2}(-\bar{\mu}_2) \\ H_{k_2}(-\mu_2) & 0 \end{bmatrix} + \begin{bmatrix} 0 & H_{k_1}(-\bar{\mu}_1) \\ H_{k_1}(-\mu_1) & 0 \end{bmatrix} Y \\ =& - R_{2k_1}Y\begin{bmatrix} L_{k_2}(-\mu_2) & 0 \\ 0 & L_{k_2}(-\bar{\mu}_2) \end{bmatrix} + \begin{bmatrix} 0 & H_{k_1}(-\bar{\mu}_1) \\ H_{k_1}(-\mu_1) & 0 \end{bmatrix} Y = 0. \end{aligned}$$

Left-multiplying this equation by $-R_{2k_1}$ and using (2.4) again, we end up with

$$Y\begin{bmatrix} L_{k_2}(-\mu_2) & 0 \\ 0 & L_{k_2}(-\bar{\mu}_2) \end{bmatrix} - \begin{bmatrix} L_{k_1}(-\mu_1) & 0 \\ 0 & L_{k_1}(-\bar{\mu}_1) \end{bmatrix} Y = 0,$$

which is a Sylvester equation.
The elementary divisors of the matrix $\begin{bmatrix} L_{k_1}(-\mu_1) & 0 \\ 0 & L_{k_1}(-\bar{\mu}_1) \end{bmatrix}$ are $(\lambda + \mu_1)^{k_1}$ and $(\lambda + \bar{\mu}_1)^{k_1}$, and the elementary divisors of $\begin{bmatrix} L_{k_2}(-\mu_2) & 0 \\ 0 & L_{k_2}(-\bar{\mu}_2) \end{bmatrix}$ are $(\lambda + \mu_2)^{k_2}$ and $(\lambda + \bar{\mu}_2)^{k_2}$. Hence, if $\mu_1 = \mu_2$, the greatest common divisor of $(\lambda + \mu_1)^{k_1}$ and $(\lambda + \mu_2)^{k_2}$ is $(\lambda + \mu_1)^k$, and the greatest common divisor of $(\lambda + \bar{\mu}_1)^{k_1}$ and $(\lambda + \bar{\mu}_2)^{k_2}$ is $(\lambda + \bar{\mu}_1)^k$, where $k = \min\{k_1, k_2\}$ in both cases; however, if $\mu_1 \neq \mu_2$, these elementary divisors are coprime. Hence, by (2.17), the dimension over $\mathbb{R}$ of the solution space of syst $\left(\mathcal{J}_{k_1}^H(\mu_1, \bar{\mu}_1), \mathcal{J}_{k_2}^H(\mu_2, \bar{\mu}_2)\right)$ is

$$\begin{cases} 4\min\{k_1, k_2\}, & \text{if } \mu_1 = \mu_2, \\ 0, & \text{if } \mu_1 \neq \mu_2. \end{cases} \tag{3.33}$$

8. $\boxed{\text{syst}\left(\sigma_1\mathcal{J}_{k_1}^H(a), \sigma_2\mathcal{J}_{k_2}^H(\infty)\right)}$ As we did with syst $\left(\sigma_1\mathcal{J}_{k_1}^H(\infty), \sigma_2\mathcal{J}_{k_2}^H(\infty)\right)$, we can get rid of σ_1 and σ_2 and consider only syst $\left(\mathcal{J}_{k_1}^H(a), \mathcal{J}_{k_2}^H(\infty)\right)$ by Remark 3.1.

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By (2.7) and (2.8), the system syst $\left(\mathcal{J}_{k_1}^H(a), \mathcal{J}_{k_2}^H(\infty)\right)$ is

$$ZR_{k_2} + H_{k_1}(-a)Y = 0, \qquad ZH_{k_2}(0) + R_{k_1}Y = 0.$$

Multiplying both sides of the second equation on the left by R_{k_1}, we get $R_{k_1}ZH_{k_2}(0) + Y = 0$, which implies $Y = -R_{k_1}ZH_{k_2}(0)$. Replacing this identity into the first equation using (2.5), we obtain

$$ZR_{k_2} - H_{k_1}(-a)R_{k_1}ZH_{k_2}(0) = ZR_{k_2} - J_{k_1}(-a)ZH_{k_2}(0) = 0.$$

Multiplying both sides on the right by R_{k_2} and using (2.5) again, we end up with

$$Z - J_{k_1}(-a)ZJ_{k_2}(0) = 0. \tag{3.34}$$

If $a = 0$, then this becomes $Z = J_{k_1}(0)ZJ_{k_2}(0) = J_{k_1}^2(0)ZJ_{k_2}^2(0) = \cdots = J_{k_1}^k(0)ZJ_{k_2}^k(0) = 0$, for $k = \min\{k_1, k_2\}$. Hence, $Z = 0$, which implies $Y = 0$ as well.
If $a \neq 0$, then $J_{k_1}(-a)$ is invertible. Let us denote it by S. Then, we multiply both sides of (3.34) on the left by S^{-1} to obtain $S^{-1}Z - ZJ_{k_2}(0) = 0$, which is a Sylvester equation. Note that the only eigenvalue of S^{-1} is $-1/a$ and the only eigenvalue of $J_{k_2}(0)$ is 0. Since they are different, S^{-1} and $J_{k_2}(0)$ have no common eigenvalues. Therefore, the only solution of this equation is $Z = 0$, from which we obtain $Y = 0$ as well. Hence, the dimension over $\mathbb{R}$ of the solution space of syst $\left(\mathcal{J}_{k_1}^H(a), \mathcal{J}_{k_2}^H(\infty)\right)$ is 0.

9. syst $\left(\sigma\mathcal{J}_{k_1}^H(\infty), \mathcal{J}_{k_2}^H(\mu, \bar{\mu})\right)$ Again by Remark 3.1, we can ignore the parameter σ.
By (2.8) and (2.9), the system syst $\left(\mathcal{J}_{k_1}^H(\infty), \mathcal{J}_{k_2}^H(\mu, \bar{\mu})\right)$ reads:

$$Z\begin{bmatrix} 0 & H_{k_2}(-\bar{\mu}) \\ H_{k_2}(-\mu) & 0 \end{bmatrix} + R_{k_1}Y = 0, \qquad ZR_{2k_2} + H_{k_1}(0)Y = 0.$$

Multiplying both sides of the second equation on the right by R_{2k_2} and using (2.1), we obtain $Z + H_{k_1}(0)YR_{2k_2} = 0$, which implies $Z = -H_{k_1}(0)YR_{2k_2}$. Then, we replace this expression into the first equation and use (2.4) to get

$$-H_{k_1}(0)YR_{2k_2}\begin{bmatrix} 0 & H_{k_2}(-\bar{\mu}) \\ H_{k_2}(-\mu) & 0 \end{bmatrix} + R_{k_1}Y = -H_{k_1}(0)Y\begin{bmatrix} L_{k_2}(-\mu) & 0 \\ 0 & L_{k_2}(-\bar{\mu}) \end{bmatrix} + R_{k_1}Y = 0.$$

Multiplying both sides on the left by R_{k_1} and using (2.4) again, this is equivalent to

$$-L_{k_1}(0)Y\begin{bmatrix} L_{k_2}(-\mu) & 0 \\ 0 & L_{k_2}(-\bar{\mu}) \end{bmatrix} + Y = 0. \tag{3.35}$$

Since μ has positive imaginary part, the matrix $\left[\begin{smallmatrix} L_{k_2}(-\mu) & 0 \\ 0 & L_{k_2}(-\bar{\mu}) \end{smallmatrix}\right]$ is invertible. We denote it by S. Then, we multiply both sides of (3.35) on the right by S^{-1} to obtain $-L_{k_1}(0)Y + YS^{-1} = 0$. This is, again, a Sylvester equation. Note that the eigenvalues of S^{-1} are $-1/\mu$ and $-1/\bar{\mu}$, and the only eigenvalue of $L_{k_1}(0)$ is 0, so these two matrices have no common eigenvalues. Hence, the solution of this equation is $Y = 0$, which implies $Z = 0$ as well. Therefore, the dimension over $\mathbb{R}$ of the solution space of syst $\left(\mathcal{J}_{k_1}^H(\infty), \mathcal{J}_{k_2}^H(\mu, \bar{\mu})\right)$ is 0.

10. syst $\left(\sigma\mathcal{J}_{k_1}^H(a), \mathcal{J}_{k_2}^H(\mu, \bar{\mu})\right)$ We disregard again the sign σ by Remark 3.1.

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By (2.7) and (2.9), the system syst $\left(\mathcal{J}^H_{k_1}(a), \mathcal{J}^H_{k_2}(\mu,\bar\mu)\right)$ is

$$Z\begin{bmatrix} 0 & H_{k_2}(-\bar\mu) \\ H_{k_2}(-\mu) & 0\end{bmatrix} + H_{k_1}(-a)Y = 0, \qquad ZR_{2k_2} + R_{k_1}Y = 0.$$

Multiplying both sides of the second equation on the right by R_{2k_2}, we have $Z + R_{k_1}YR_{2k_2} = 0$, which implies $Z = -R_{k_1}YR_{2k_2}$. Then, we replace it into the first equation to obtain

$$-R_{k_1}YR_{2k_2}\begin{bmatrix} 0 & H_{k_2}(-\bar\mu) \\ H_{k_2}(-\mu) & 0\end{bmatrix} + H_{k_1}(-a)Y = -R_{k_1}Y\begin{bmatrix} L_{k_2}(-\mu) & 0 \\ 0 & L_{k_2}(-\bar\mu)\end{bmatrix} + H_{k_1}(-a)Y = 0,$$

where to get the first identity we have used (2.4).
Multiplying both sides on the left by R_{k_1} and using (2.4) again, we end up with

$$-Y\begin{bmatrix} L_{k_2}(-\mu) & 0 \\ 0 & L_{k_2}(-\bar\mu)\end{bmatrix} + L_{k_1}(-a)Y = 0,$$

which is equivalent to

$$L_{k_1}(-a)Y = Y\begin{bmatrix} L_{k_2}(-\mu) & 0 \\ 0 & L_{k_2}(-\bar\mu)\end{bmatrix},$$

and this is, again, a Sylvester equation. Note that μ has positive imaginary part and a is a real number, so $\mu \neq a$ and the matrices $L_{k_1}(-a)$ and $\left[\begin{smallmatrix} L_{k_2}(-\mu) & 0 \\ 0 & L_{k_2}(-\bar\mu)\end{smallmatrix}\right]$ have no common eigenvalues, which implies $Y = 0$ and this in turn implies $Z = 0$. Then, the dimension over $\mathbb{R}$ of the solution space of syst $\left(\mathcal{J}^H_{k_1}(a), \mathcal{J}^H_{k_2}(\mu,\bar\mu)\right)$ is 0.

11. $\boxed{\text{syst}\left(\sigma\mathcal{J}^H_k(\infty), \mathcal{M}_d\right)}$ We still ignore σ and consider only syst $\left(\mathcal{J}^H_k(\infty), \mathcal{M}_d\right)$.
By (2.6) and (2.8), the system syst $\left(\mathcal{J}^H_k(\infty), \mathcal{M}_d\right)$ reads

$$Z\begin{bmatrix} 0 & F_d^\top \\ F_d & 0\end{bmatrix} + R_kY = 0, \qquad Z\begin{bmatrix} 0 & G_d^\top \\ G_d & 0\end{bmatrix} + H_k(0)Y = 0.$$

From the first equation and (2.1), we get $Y = -R_kZ\left[\begin{smallmatrix} 0 & F_d^\top \\ F_d & 0\end{smallmatrix}\right]$. Then, we replace it into the second equation and use (2.5) to obtain

$$Z\begin{bmatrix} 0 & G_d^\top \\ G_d & 0\end{bmatrix} - J_k(0)Z\begin{bmatrix} 0 & F_d^\top \\ F_d & 0\end{bmatrix} = 0. \tag{3.36}$$

Let us write $Z = \begin{bmatrix} Z_1 & Z_2\end{bmatrix}$ with $Z_1 \in \mathbb{C}^{k\times(d+1)}$ and $Z_2 \in \mathbb{C}^{k\times d}$. Then, (3.36) becomes

$$\begin{aligned}&\begin{bmatrix} Z_1 & Z_2\end{bmatrix}\begin{bmatrix} 0 & G_d^\top \\ G_d & 0\end{bmatrix} - J_k(0)\begin{bmatrix} Z_1 & Z_2\end{bmatrix}\begin{bmatrix} 0 & F_d^\top \\ F_d & 0\end{bmatrix} \\ &= \begin{bmatrix} Z_2G_d & Z_1G_d^\top\end{bmatrix} - J_k(0)\begin{bmatrix} Z_2F_d & Z_1F_d^\top\end{bmatrix} = 0,\end{aligned}$$

which is equivalent to the following system of two equations

$$Z_2G_d - J_k(0)Z_2F_d = 0, \qquad Z_1G_d^\top - J_k(0)Z_1F_d^\top = 0. \tag{3.37}$$

Note that this system is decoupled, so the equations can be solved independently.

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From [9, Lemma 18], we know that the solution of the system $F_d^\top X + Y^\top = 0 = J_k(0)Y + X^\top G_d$ is $X = Y = 0$. Solving for Y in the first equation of this system (namely, $Y = -X^\top F_d$) and replacing in the second one, we get $X^\top G_d - J_k(0)X^\top F_d = 0$, which is the first equation of (3.37) with $Z_2 = X^\top$. Therefore, $Z_2 = 0$.

Also, from [9, Lemma 16], we know that the (complex) dimension of the solution space of the system $F_d X + Y^\top = 0 = J_k(a)Y + X^\top G_d^\top$ is k. Solving for Y in the first equation (namely, $Y = -X^\top F_d^\top$) and replacing it in the second equation, we arrive at $X^\top G_d^\top - J_k(a)X^\top F_d^\top = 0$, which is the second equation in (3.37) with $Z_1 = X^\top$ and $a = 0$.

Hence, the dimension over $\mathbb{R}$ of the solution space of syst $\left(\mathcal{J}_k^H(\infty), \mathcal{M}_d\right)$ is $2k$.

12. $\boxed{\text{syst}\left(\sigma\mathcal{J}_k^H(a), \mathcal{M}_d\right)}$ Getting rid of σ again, we consider only syst $\left(\mathcal{J}_k^H(a), \mathcal{M}_d\right)$, which, by (2.6) and (2.7), reads

$$Z\begin{bmatrix} 0 & F_d^\top \\ F_d & 0 \end{bmatrix} + H_k(-a)Y = 0, \qquad Z\begin{bmatrix} 0 & G_d^\top \\ G_d & 0 \end{bmatrix} + R_k Y = 0.$$

From the second equation and (2.1), we get $Y = -R_k Z \begin{bmatrix} 0 & G_d^\top \\ G_d & 0 \end{bmatrix}$. Then, we replace it into the first equation and use (2.5) to obtain

$$Z\begin{bmatrix} 0 & F_d^\top \\ F_d & 0 \end{bmatrix} - J_k(-a)Z\begin{bmatrix} 0 & G_d^\top \\ G_d & 0 \end{bmatrix} = 0. \tag{3.38}$$

We write $Z = \begin{bmatrix} Z_1 & Z_2 \end{bmatrix}$ again, with $Z_1 \in \mathbb{C}^{k\times(d+1)}$ and $Z_2 \in \mathbb{C}^{k\times d}$. Then, (3.38) becomes

$$\begin{aligned} &\begin{bmatrix} Z_1 & Z_2 \end{bmatrix}\begin{bmatrix} 0 & F_d^\top \\ F_d & 0 \end{bmatrix} - J_k(-a)\begin{bmatrix} Z_1 & Z_2 \end{bmatrix}\begin{bmatrix} 0 & G_d^\top \\ G_d & 0 \end{bmatrix} \\ &= \begin{bmatrix} Z_2 F_d & Z_1 F_d^\top \end{bmatrix} - J_k(-a)\begin{bmatrix} Z_2 G_d & Z_1 G_d^\top \end{bmatrix} = 0, \end{aligned}$$

which is equivalent to the following system of two equations

$$Z_2 F_d - J_k(-a)Z_2 G_d = 0, \qquad Z_1 F_d^\top - J_k(-a)Z_1 G_d^\top = 0. \tag{3.39}$$

The previous system resembles very much (3.37) for syst $\left(\mathcal{J}_k^H(\infty), \mathcal{M}_d\right)$. However, the presence of the matrix $J_k(-a)$ instead of $J_k(0)$ requires a slightly different treatment. Let us address the two matrix equations independently, since they are decoupled.

Multiplying on the right the first equation of (3.39) by $G_d^\top$, and using $G_d G_d^\top = I_d$ and $F_d G_d^\top = 0$, we end up with $-J_k(-a)Z_2 = 0$, which implies $Z_2 = 0$, since $J_k(-a)$ is invertible, as $a \neq 0$.

As for the second equation, we use again [9, Lemma 16], namely the (complex) dimension of the solution space of the system $F_d X + Y^\top = 0 = J_k(-a)Y + X^\top G_d^\top$ is k. Proceeding as with the right-hand side equation in (3.37), we first solve for Y ($Y = -X^\top F_d^\top$) and replace it in the second equation, to get $X^\top G_d^\top - J_k(-a)X^\top F_d^\top = 0$. Using (2.1), and multiplying on the right by R_d (which is invertible), this equation is equivalent to $(R_{d+1}X)^\top(R_{d+1}G_d^\top R_d) - J_k(-a)(R_{d+1}X)^\top(R_{d+1}F_d^\top R_d) = 0$ which, using (2.2) and performing the change of variables $Z_1 = R_{d+1}X$, reads $Z_1 F_d^\top - J_k(-a)Z_1 G_d^\top = 0$. This is, precisely, the second equation in (3.39).

Therefore, the dimension over $\mathbb{R}$ of the solution space of syst $\left(\sigma\mathcal{J}_k^H(a), \mathcal{M}_d\right)$ is $2k$.

13. $\boxed{\text{syst}\left(\mathcal{J}_k^H(\mu, \bar{\mu}), \mathcal{M}_d\right)}$ By (2.9) and (2.6), the system syst $\left(\mathcal{J}_k^H(\mu, \bar{\mu}), \mathcal{M}_d\right)$ is

$$Z\begin{bmatrix} 0 & F_d^\top \\ F_d & 0 \end{bmatrix} + \begin{bmatrix} 0 & H_k(-\bar{\mu}) \\ H_k(-\mu) & 0 \end{bmatrix} Y = 0, \qquad Z\begin{bmatrix} 0 & G_d^\top \\ G_d & 0 \end{bmatrix} + R_{2k} Y = 0.$$

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From the second equation and (2.1), we get $Y = -R_{2k} Z \begin{bmatrix} 0 & G_d^\top \\ G_d & 0 \end{bmatrix}$, and replacing it into the first equation, we obtain, using (2.5) as well,

$$Z \begin{bmatrix} 0 & F_d^\top \\ F_d & 0 \end{bmatrix} - \begin{bmatrix} J_k(-\bar{\mu}) & 0 \\ 0 & J_k(-\mu) \end{bmatrix} Z \begin{bmatrix} 0 & G_d^\top \\ G_d & 0 \end{bmatrix} = 0. \tag{3.40}$$

Let us write $Z = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}$ with $Z_{11}, Z_{21} \in \mathbb{C}^{k\times(d+1)}$ and $Z_{12}, Z_{22} \in \mathbb{C}^{k\times d}$. Then, (3.40) becomes

$$\begin{bmatrix} Z_{12}F_d - J_k(-\bar{\mu})Z_{12}G_d & Z_{11}F_d^\top - J_k(-\bar{\mu})Z_{11}G_d^\top \\ Z_{22}F_d - J_k(-\mu)Z_{22}G_d & Z_{21}F_d^\top - J_k(-\mu)Z_{21}G_d^\top \end{bmatrix} = 0.$$

The equations corresponding to the (1,1)-block and (2,1)-block are similar to the first equation of (3.39), so $Z_{12} = Z_{22} = 0$. Similarly, the equations corresponding to the (1,2)-block and (2,2)-block are like the second equation of (3.39). Hence, both Z_{11} and Z_{21} have k independent complex parameters. Therefore, the dimension over $\mathbb{R}$ of the solution space of syst $\left(\mathcal{J}_k^H(\mu,\bar{\mu}), \mathcal{M}_d\right)$ is $4k$.

14. $\boxed{\text{syst}\,(\mathcal{M}_{d_1}, \mathcal{M}_{d_2})}$ The dimension of the solution space of syst $(\mathcal{M}_{d_1}, \mathcal{M}_{d_2})$ (over $\mathbb{C}$) has been shown in [14, Corollary 2.2] to be equal to

$$\begin{cases} 2d_1 + 2, & \text{if } d_1 = d_2, \\ 2\max(d_1, d_2) + 1, & \text{if } d_1 \neq d_2. \end{cases}$$

Since we are considering the solution space over $\mathbb{R}$ instead, this dimension is

$$\begin{cases} 2\cdot(2d_1 + 2), & \text{if } d_1 = d_2, \\ 2\cdot(2\max\{d_1, d_2\} + 1), & \text{if } d_1 \neq d_2. \end{cases}$$

We summarize the above results in Table 1.

Remark 3.2. *If, in Table 1, we compare the contribution to the codimension coming from finite and infinite Jordan blocks, namely $\sigma\mathcal{J}_k^H(a)$ and $\sigma\mathcal{J}_k^H(\infty)$, we see that there is no difference between them. Therefore, we can join both blocks in a single block $\sigma\mathcal{J}_k^H(a)$, with $a \in \mathbb{R}\cup\{\infty\}$.*

Based on Table 1 and Theorem 2.1, we present in Theorem 3.3 the breakdown of the codimension count for the Hermitian orbit of an arbitrary Hermitian pencil given in HKCF, which is the main result of this work. We want to emphasize that, according to Remark 3.2, the Jordan blocks associated with finite and infinite eigenvalues are not distinguished.

The statement of Theorem 3.3 follows the ones of [11, Theorem 2.2] and [13, Corollary 2] for, respectively, orbits under strict equivalence of unstructured pencils and orbits under congruence of skew-symmetric pencils, as well as the one in [9, Th. 3, Th. 4] for the solution of the equation $AX + X^\star B = 0$ and [14, Corollary 2.2], [13, Corollary 2] for the solution of the systems $X^\top A + AX = 0 = X^\top B + BX$, with A, B both being symmetric and skew-symmetric, respectively.

Theorem 3.3. *Let $\mathcal{H}$ be an $n\times n$ Hermitian pencil and assume that*

$$\mathrm{HKCF}(\mathcal{H}) = \left(\bigoplus_{i=1}^{p} \sigma_i \mathcal{J}_{k_i}^H(a_i)\right) \oplus \left(\bigoplus_{u=1}^{s} \mathcal{J}_{h_u}^H(\lambda_u, \overline{\lambda}_u)\right) \oplus \left(\bigoplus_{v=1}^{t} \mathcal{M}_{d_v}\right), \tag{3.41}$$

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TABLE 1
The dimension over $\mathbb{R}$ of the solution space of the systems in (2.14).

System	Dimension
$\operatorname{syst}\left(\mathcal{J}_k^H(\infty)\right)$	k
$\operatorname{syst}\left(\mathcal{J}_k^H(a)\right)$	k
$\operatorname{syst}\left(\mathcal{J}_k^H(\mu,\bar\mu)\right)$	$2k$
$\operatorname{syst}\left(\mathcal{M}_d\right)$	$2d+2$
$\operatorname{syst}\left(\sigma_1\mathcal{J}_{k_1}^H(\infty),\sigma_2\mathcal{J}_{k_2}^H(\infty)\right)$	$2\min\{k_1,k_2\}$
$\operatorname{syst}\left(\sigma_1\mathcal{J}_{k_1}^H(a),\sigma_2\mathcal{J}_{k_2}^H(b)\right)$	$\begin{cases}2\min\{k_1,k_2\}, & \text{if } a=b,\\ 0, & \text{if } a\neq b.\end{cases}$
$\operatorname{syst}\left(\mathcal{J}_{k_1}^H(\mu_1,\bar\mu_1),\mathcal{J}_{k_2}^H(\mu_2,\bar\mu_2)\right)$	$\begin{cases}4\min\{k_1,k_2\}, & \text{if } \mu_1=\mu_2,\\ 0, & \text{if } \mu_1\neq \mu_2.\end{cases}$
$\operatorname{syst}\left(\sigma_1\mathcal{J}_{k_1}^H(a),\sigma_2\mathcal{J}_{k_2}^H(\infty)\right)$	0
$\operatorname{syst}\left(\sigma\mathcal{J}_{k_1}^H(\infty),\mathcal{J}_{k_2}^H(\mu,\bar\mu)\right)$	0
$\operatorname{syst}\left(\sigma\mathcal{J}_{k_1}^H(a),\mathcal{J}_{k_2}^H(\mu,\bar\mu)\right)$	0
$\operatorname{syst}\left(\sigma\mathcal{J}_k^H(\infty),\mathcal{M}_d\right)$	$2k$
$\operatorname{syst}\left(\sigma\mathcal{J}_k^H(a),\mathcal{M}_d\right)$	$2k$
$\operatorname{syst}\left(\mathcal{J}_k^H(\mu,\bar\mu),\mathcal{M}_d\right)$	$4k$
$\operatorname{syst}\left(\mathcal{M}_{d_1},\mathcal{M}_{d_2}\right)$	$\begin{cases}2\cdot(2d_1+2), & \text{if } d_1=d_2,\\ 2\cdot(2\max\{d_1,d_2\}+1), & \text{if } d_1\neq d_2.\end{cases}$

with $a_i \in \mathbb{R}\cup\{\infty\}$, for $i=1,\dots,p$, and where any of p,s, or t being 0 is allowed, and it means that the corresponding summand does not appear in (3.41).

Then, the real codimension of the Hermitian orbit of $\mathcal{H}$ is equal to the sum

$$\operatorname{codim}_{\mathbb{R}}\mathcal{O}^{\mathrm{H}}(\mathcal{H}) = \mathrm{c_H}+\mathrm{c_L}+\mathrm{c_M}+\mathrm{c_{HH}}+\mathrm{c_{LL}}+\mathrm{c_{MM}}+\mathrm{c_{HM}}+\mathrm{c_{LM}}, \tag{3.42}$$

whose summands correspond to

- *the dimension of the solution space of (2.12) corresponding to the blocks in (3.41):*

$$\begin{aligned}
&\operatorname{syst}\left(\mathcal{J}_{k_i}^H(a_i)\right):\ c_H := \sum_{i=1}^{p} k_i,\\
&\operatorname{syst}\left(\mathcal{J}_{h_u}^H(\lambda_u,\overline{\lambda}_u)\right):\ c_L := 2\sum_{u=1}^{s} h_u,\\
&\operatorname{syst}\left(\mathcal{M}_{d_v}\right):\ c_M := 2\sum_{v=1}^{t}(d_v+1),
\end{aligned}$$

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- *the dimension of the solution space of* (2.13) *corresponding to a couple of blocks of the same type in* (3.41)*:*

$$\text{syst}\left(\sigma_i \mathcal{J}_{k_i}^H(a_i), \sigma_{i'}\mathcal{J}_{k_{i'}}^H(a_{i'})\right): \quad c_{HH} := 2\sum_{\substack{i<i' \\ a_i=a_{i'}}} \min\{k_i, k_{i'}\},$$

$$\text{syst}\left(\mathcal{J}_{h_u}^H(\lambda_u, \overline{\lambda}_u), \mathcal{J}_{h_{u'}}^H(\lambda_{u'}, \overline{\lambda}_{u'})\right): \quad c_{LL} := 4\sum_{\substack{u<u' \\ \lambda_u=\lambda_{u'}}} \min\{h_u, h_{u'}\},$$

$$\text{syst}\left(\mathcal{M}_{d_v}, \mathcal{M}_{d_{v'}}\right): \quad c_{MM} := 2\sum_{\substack{v<v' \\ d_v\neq d_{v'}}} (2\max\{d_v, d_{v'}\}+1) + 4\sum_{\substack{v<v' \\ d_v=d_{v'}}} (d_v+1),$$

- *the dimension of the solution space of* (2.13) *corresponding to a couple of blocks of different type in* (3.41)*:*

$$\text{syst}\left(\sigma_i \mathcal{J}_{k_i}^H(a_i), \mathcal{M}_{d_v}\right): \quad c_{HM} := 2t\sum_{i=1}^{p} k_i,$$

$$\text{syst}\left(\mathcal{J}_{h_u}^H(\lambda_u, \overline{\lambda}_u), \mathcal{M}_{d_v}\right): \quad c_{LM} := 4t\sum_{u=1}^{s} h_u.$$

4. An example: The case of 3×3 Hermitian pencils. In this section, we compute the codimension of the orbits of all 3×3 Hermitian pencils according to their HKCFs. Table 2 summarizes the results. More specifically, we display all possible HKCFs of 3×3 Hermitian pencils, classified according to their rank, together with their codimension and dimension.

The codimensions are calculated by means of Theorem 3.3, and the dimensions are then obtained using (2.10), together with the fact that $\dim_{\mathbb{R}}(\text{PENCIL}_{3\times3}^H) = 18$.

TABLE 2
Dimension and codimension of all HKCFs *of* 3×3 *Hermitian pencils. Here,* $\sigma, \sigma_i \in \{1,-1\}$ *and* $a, a_i \in \mathbb{R}\cup\{\infty\}$*, for* $i = 1, 2, 3$*, and* $\text{Im}(\mu) > 0$.

Rank	HKCF	codim	dim
3	$\sigma_1\mathcal{J}_1^H(a_1)\oplus\sigma_2\mathcal{J}_1^H(a_2)\oplus\sigma_3\mathcal{J}_1^H(a_3)$, $a_1=a_2=a_3$	9	9
	$\sigma_1\mathcal{J}_1^H(a_1)\oplus\sigma_2\mathcal{J}_1^H(a_2)\oplus\sigma_3\mathcal{J}_1^H(a_3)$, $a_1=a_2\neq a_3$	5	13
	$\sigma_1\mathcal{J}_1^H(a_1)\oplus\sigma_2\mathcal{J}_1^H(a_2)\oplus\sigma_3\mathcal{J}_1^H(a_3)$, $a_1\neq a_2\neq a_3\neq a_1$	3	15
	$\mathcal{J}_1^H(\mu,\bar{\mu})\oplus\sigma\mathcal{J}_1^H(a)$	3	15
	$\sigma_1\mathcal{J}_1^H(a_1)\oplus\sigma_2\mathcal{J}_2^H(a_2)$, $a_1=a_2$	5	13
	$\sigma_1\mathcal{J}_1^H(a_1)\oplus\sigma_2\mathcal{J}_2^H(a_2)$, $a_1\neq a_2$	3	15
	$\sigma\mathcal{J}_3^H(a)$	3	15
2	$\sigma_1\mathcal{J}_1^H(a_1)\oplus\sigma_2\mathcal{J}_1^H(a_2)\oplus\mathcal{M}_0$, $a_1=a_2$	10	8
	$\sigma_1\mathcal{J}_1^H(a_1)\oplus\sigma_2\mathcal{J}_1^H(a_2)\oplus\mathcal{M}_0$, $a_1\neq a_2$	8	10
	$\mathcal{J}_1^H(\mu,\bar{\mu})\oplus\mathcal{M}_0$	8	10
	$\mathcal{M}_1$	4	14
	$\sigma\mathcal{J}_2^H(a)\oplus\mathcal{M}_0$	8	10
1	$\sigma\mathcal{J}_1^H(a)\oplus\mathcal{M}_0\oplus\mathcal{M}_0$	13	5
0	$\mathcal{M}_0\oplus\mathcal{M}_0\oplus\mathcal{M}_0$	18	0

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In the following, we calculate the codimension of each orbit, following the order in Table 2.

1. If $\mathrm{HKCF}(\mathcal{H}) = \sigma_1\mathcal{J}_1^H(a_1) \oplus \sigma_2\mathcal{J}_1^H(a_2) \oplus \sigma_3\mathcal{J}_1^H(a_3)$, then $p = 3$ in (3.41) and the only nonzero summands in (3.42) are c_H and c_{HH}. In this case, $c_H = p = 3$. The remaining term c_{HH} depends on the values a_1, a_2, and a_3, according to the following three cases:
   1.1. When $a_1 = a_2 = a_3$, $c_{HH} = \binom{3}{2}\cdot 2 = 6$, and this implies $\mathrm{codim}_{\mathbb{R}}\mathcal{O}^{\mathrm{H}}(\mathcal{H}) = \mathrm{c_H} + \mathrm{c_{HH}} = 3+6 = 9$.
   1.2. If $a_1 = a_2$, $a_1 \neq a_3$, then $c_{HH} = 1\cdot 2 = 2$, so $\mathrm{codim}_{\mathbb{R}}\mathcal{O}^{\mathrm{H}}(\mathcal{H}) = \mathrm{c_H} + \mathrm{c_{HH}} = 3 + 2 = 5$.
   1.3. When a_1, a_2, and a_3 are pairwise distinct, $c_{HH} = 0\cdot 2 = 0$, so $\mathrm{codim}_{\mathbb{R}}\mathcal{O}^{\mathrm{H}}(\mathcal{H}) = \mathrm{c_H} + \mathrm{c_{HH}} = 3 + 0 = 3$.
2. If $\mathrm{HKCF}(\mathcal{H}) = \mathcal{J}_1^H(\mu,\bar{\mu}) \oplus \sigma\mathcal{J}_1^H(a)$, the (nonzero) summands that remain in (3.42) are c_L and c_H. In this case, $c_L = 2$ and $c_H = 1$. Hence, $\mathrm{codim}_{\mathbb{R}}\mathcal{O}^{\mathrm{H}}(\mathcal{H}) = 2 + 1 = 3$.
3. If $\mathrm{HKCF}(\mathcal{H}) = \sigma_1\mathcal{J}_1^H(a_1) \oplus \sigma_2\mathcal{J}_2^H(a_2)$, then the only nonzero summands in (3.42) are c_H and c_{HH}. In this case, $c_H = 3$. The remaining term c_{HH} depends on the relations between a_1 and a_2, according to the following two cases:
   3.1. If $a_1 = a_2$, then $c_{HH} = 2\min\{1,2\} = 2$, so $\mathrm{codim}_{\mathbb{R}}\mathcal{O}^{\mathrm{H}}(\mathcal{H}) = \mathrm{c_H} + \mathrm{c_{HH}} = 3 + 2 = 5$.
   3.2. If $a_1 \neq a_2$, then $c_{HH} = 0$, so $\mathrm{codim}_{\mathbb{R}}\mathcal{O}^{\mathrm{H}}(\mathcal{H}) = \mathrm{c_H} + \mathrm{c_{HH}} = 3 + 0 = 3$.
4. If $\mathrm{HKCF}(\mathcal{H}) = \sigma\mathcal{J}_3^H(a)$, we immediately get from (3.42) that $\mathrm{codim}_{\mathbb{R}}\mathcal{O}^{\mathrm{H}}(\mathcal{H}) = 3$.
5. If $\mathrm{HKCF}(\mathcal{H}) = \sigma_1\mathcal{J}_1^H(a_1) \oplus \sigma_2\mathcal{J}_1^H(a_2) \oplus \mathcal{M}_0$, then $p = 2$ in (3.41), and the nonzero summands in (3.42) are c_H, c_M, c_{HH}, and c_{HM}. It is straightforward to get $c_M = 2$. From $p = 2$, we get $c_H = p = 2$ and $c_{HM} = 2p = 4$. The remaining term c_{HH} depends on the relations between a_1 and a_2, according to the following two cases:
   5.1. If $a_1 = a_2$, then $c_{HH} = 2$, which implies that $\mathrm{codim}_{\mathbb{R}}\mathcal{O}^{\mathrm{H}}(\mathcal{H}) = \mathrm{c_H} + \mathrm{c_M} + \mathrm{c_{HH}} + \mathrm{c_{HM}} = 2 + 2 + 2 + 4 = 10$.
   5.2. If $a_1 \neq a_2$, then $c_{HH} = 0$, and this implies that $\mathrm{codim}_{\mathbb{R}}\mathcal{O}^{\mathrm{H}}(\mathcal{H}) = \mathrm{c_H} + \mathrm{c_M} + \mathrm{c_{HH}} + \mathrm{c_{HM}} = 2 + 2 + 0 + 4 = 8$.
6. If $\mathrm{HKCF}(\mathcal{H}) = \mathcal{J}_1^H(\mu,\bar{\mu}) \oplus \mathcal{M}_0$, the nonzero summands in (3.42) are c_L, c_M, and c_{LM}. Moreover, $c_L = 2$, $c_M = 2$, and $c_{LM} = 4$, so $\mathrm{codim}_{\mathbb{R}}\mathcal{O}^{\mathrm{H}}(\mathcal{H}) = \mathrm{c_L} + \mathrm{c_M} + \mathrm{c_{LM}} = 2 + 2 + 4 = 8$.
7. If $\mathrm{HKCF}(\mathcal{H}) = \mathcal{M}_1$, we directly get from (3.42) that $\mathrm{codim}_{\mathbb{R}}\mathcal{O}^{\mathrm{H}}(\mathcal{H}) = \mathrm{c_M} = 2\cdot(1+1) = 4$.
8. If $\mathrm{HKCF}(\mathcal{H}) = \sigma\mathcal{J}_2^H(a) \oplus \mathcal{M}_0$, the nonzero summands in (3.42) are c_H, c_M, and c_{HM}, with $c_H = 2$, $c_{HM} = 4$, and $c_M = 2\cdot(0+1) = 2$. Hence, $\mathrm{codim}_{\mathbb{R}}\mathcal{O}^{\mathrm{H}}(\mathcal{H}) = \mathrm{c_H} + \mathrm{c_M} + \mathrm{c_{HM}} = 2 + 2 + 4 = 8$.
9. The case $\mathrm{HKCF}(\mathcal{H}) = \sigma\mathcal{J}_1^H(a) \oplus \mathcal{M}_0 \oplus \mathcal{M}_0$ is similar to the case $\mathrm{HKCF}(\mathcal{H}) = \sigma\mathcal{J}_2^H(a) \oplus \mathcal{M}_0$, but now $c_H = 1$ and $c_M = 2\cdot 2 = 4$, and there is one more nonzero summand c_{MM} in (3.42) which equals $2\cdot 2\cdot 1 = 4$. Hence, $\mathrm{codim}_{\mathbb{R}}\mathcal{O}^{\mathrm{H}}(\mathcal{H}) = \mathrm{c_H} + \mathrm{c_M} + \mathrm{c_{HM}} + \mathrm{c_{MM}} = 1 + 4 + 4 + 4 = 13$.
10. If $\mathrm{HKCF}(\mathcal{H}) = \mathcal{M}_0 \oplus \mathcal{M}_0 \oplus \mathcal{M}_0 = 0_{3\times 3}$, then the only nonzero summands in (3.42) are c_M and c_{MM}. In this case, $c_M = 3\cdot 2 = 6$ and $c_{MM} = \binom{3}{2}\cdot 2\cdot(2\cdot 0+2) = 12$, so $\mathrm{codim}_{\mathbb{R}}\mathcal{O}^{\mathrm{H}}(\mathcal{H}) = \mathrm{c_M} + \mathrm{c_{MM}} = 6 + 12 = 18$.

5. Bundles and generic eigenstructure. The following question has been considered in [5], which is the most likely (termed as "generic") HKCF of $n\times n$ Hermitian pencils? In order to answer this question, the notion of "bundle" is key. The *Hermitian bundle* of $\mathcal{H}$, denoted by $\mathcal{B}^H(\mathcal{H})$, is the set of all Hermitian pencils having the same HKCF as $\mathcal{H}$ except that the values of the distinct finite eigenvalues of each pencil may be different (see, for instance, [5, page 264]). The *codimension* of $\mathcal{B}^H(\mathcal{H})$ over $\mathbb{R}$, denoted by $\mathrm{codim}_{\mathbb{R}}\mathcal{B}^{\mathrm{H}}(\mathcal{H})$, is the codimension of $\mathcal{O}^H(\mathcal{H})$ minus the number of different eigenvalues of $\mathcal{H}$ (see [5, Section 6]). Then, the most likely HKCFs will be those corresponding to bundles with the largest dimension or, equivalently, with the smallest codimension.

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The following result, which is an immediate consequence of Theorem 3.3 and the previous definition of codimension of a bundle, provides a formula for the codimension of the Hermitian bundles.

COROLLARY 5.1. *Let $\mathcal{H}(\lambda)$ be an $n \times n$ Hermitian pencil with* HKCF$(\mathcal{H})$ *as in* (3.41)*. Then,*

$$\mathrm{codim}_{\mathbb{R}}\mathcal{B}^{\mathrm{H}}(\mathcal{H}) = \mathrm{codim}_{\mathbb{R}}\mathcal{O}^{\mathrm{H}}(\mathcal{H}) - (\kappa_{\mathrm{r}} + \kappa_{\infty} + 2\kappa_{\mathrm{c}}),$$

with $\mathrm{codim}_{\mathbb{R}}\mathcal{O}^{\mathrm{H}}(\mathcal{H})$ *as in* (3.42) *and*

- κ_r *is the number of different values* a_i *in* (3.41)*;*
- $\kappa_\infty = 1$*, if there is some* $1 \le i \le p$ *such that* $a_i = \infty$ *in* (3.41)*, or* 0 *otherwise; and*
- κ_c *is the number of different values* λ_u *in* (3.41)*.*

If the different blocks of finite eigenvalues in HKCF$(\mathcal{H})$ have different eigenvalues, then the summands c_{HH} and c_{LL} in (3.42) will vanish. Moreover, if $\mathcal{H}$ is of full rank (namely, n), then $c_{MM} = c_{HM} = c_{LM} = 0$, and the summands in (3.42) reduce to just c_H and c_L, namely $\mathrm{codim}_{\mathbb{R}}\mathcal{O}^{\mathrm{H}}(\mathcal{H}) = \mathrm{c_H} + \mathrm{c_L}$. To make the codimension of $\mathcal{B}^H(\mathcal{H})$ as small as possible, the number of distinct finite eigenvalues should be as large as possible. This means that each block associated with an eigenvalue a_i, for $i = 1, \ldots, p$, should be of minimal size, namely 1×1. In this case, the HKCF of $\mathcal{H}$ becomes

$$\mathrm{HKCF}(\mathcal{H}) = \left(\bigoplus_{i=1}^{p} \sigma_i \mathcal{J}_1^H(a_i)\right) \oplus \left(\bigoplus_{u=1}^{s} \mathcal{J}_1^H(\lambda_u, \overline{\lambda}_u)\right), \tag{5.43}$$

where $p + 2s = n$, $a_i \neq a_j$, and $\lambda_i \neq \lambda_j$ for $i \neq j$, and the codimension of $\mathcal{B}^H(\mathcal{H})$ is equal to

$$\mathrm{codim}_{\mathbb{R}}\mathcal{B}^{\mathrm{H}}(\mathcal{H}) = \mathrm{c_H} + \mathrm{c_L} - \mathrm{n} = \mathrm{p} + 2\mathrm{s} - \mathrm{n} = 0.$$

In fact, (5.43) is equivalent to $\mathcal{R}_{c,d}(\lambda)$ in [5, Theorem 4.1], and $\mathrm{codim}_{\mathbb{R}}\mathcal{B}^{\mathrm{H}}(\mathcal{H}) = 0$ recovers the result of [5, Theorem 6.2].

We emphasize that these are the only bundles with codimension 0, since there is no other way to minimize the codimension than the one described above. For this, note that having full rank is a generic condition on $\mathrm{PENCIL}^H_{n\times n}$, since those pencils, $\mathcal{H}(\lambda)$, not having full rank satisfy the equation $\det \mathcal{H}(\lambda) \equiv 0$ (namely, the determinant is the zero polynomial in λ). This determines a proper algebraic set on $\mathrm{PENCIL}^H_{n\times n}$, defined by the polynomial equations on the entries of $\mathcal{H}(\lambda)$, which result by equating to 0 all the coefficients of $\det \mathcal{H}(\lambda)$.

6. Conclusions and future work. We have obtained an explicit expression for the codimension of all orbits and bundles of complex $n \times n$ Hermitian matrix pencils under $*$-congruence in terms of the canonical blocks appearing in the Hermitian Kronecker canonical form. This allows us to classify these orbits according to their relevance. Since all pencils in the same orbit have the same eigenstructure, our results provide a classification of all possible eigenstructures according to their relevance.

The results in this work can be considered as the first step toward the complete description of the geometry of the set of (complex) $n \times n$ Hermitian matrix pencils that would come from the stratification of the set of orbits and bundles under $*$-congruence.

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