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GENERALIZATIONS OF BRAUER'S EIGENVALUE LOCALIZATION THEOREM*

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Abstract. New eigenvalue inclusion regions are given by establishing the necessary and sufficient conditions for two classes of nonsingular matrices, named double α_1-matrices and double α_2-matrices. These results are generalizations of Brauer's eigenvalue localization theorem and improvements over the results in [L. Cvetković, V. Kostić, R. Bru, and F. Pedroche. A simple generalization of Geršgorin's theorem. *Adv. Comput. Math.*, 35:271–280, 2011.].

Key words. Matrix eigenvalue, Brauer's eigenvalue localization theorem, Double α_1-matrices, Double α_2-matrices.

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1. Introduction.

Let $\mathbb{C}^{n\times n}$ denote the collection of all $n \times n$ complex matrices and $N = \{1, 2, \ldots, n\}$. For a matrix $A = [a_{ij}] \in \mathbb{C}^{n\times n}$, we denote, for any $i, j, k \in N$,

$$r_i = \sum_{k\neq i} |a_{ik}|,\ c_i = \sum_{k\neq i} |a_{ki}|,$$

$$\Gamma_i(A) = \{z \in \mathbb{C} :\ |z - a_{ii}| \leq r_i\},$$

$$\bar{\Gamma}_i(A) = \{z \in \mathbb{C} :\ |z - a_{ii}| \leq \min\{r_i, c_i\}\},$$

$$\mathcal{H} = \{i \in N : r_i > c_i\},\ \mathcal{L} = \{i \in N : r_i < c_i\},$$

$$\tilde{\Gamma}_{i,j}(A) = \{z \in \mathbb{C} :\ |z - a_{ii}|(c_j - r_j) + |z - a_{jj}|(r_i - c_i) \leq c_j r_i - c_i r_j, i \in \mathcal{H}, j \in \mathcal{L}\},$$

$$\hat{\Gamma}_{i,j}(A) = \{z \in \mathbb{C} : \frac{|z - a_{ii}|}{c_i} \left(\frac{|z - a_{jj}|}{c_j}\right)^{\log_{\frac{c_j}{r_j}} \frac{r_i}{c_i}} \leq 1, i \in \mathcal{H}\backslash\{k : c_k = 0\},$$
$$j \in \mathcal{L}\backslash\{k : r_k = 0\}\},$$

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$$\mathcal{K}_{i,j}(A) = \{z \in \mathbb{C} : |z - a_{ii}||z - a_{jj}| \leq r_i r_j\}$$

and

$$\bar{\mathcal{K}}_{i,j}(A) = \{z \in \mathbb{C} : |z - a_{ii}||z - a_{jj}| \leq \min\{r_i r_j, c_i c_j\}\}.$$

Eigenvalue localization has been a hot topic in matrix theory and its applications. Many researchers have obtained lots of eigenvalue inclusion regions; for details, see [1]–[7], [9]–[13]. We first recall the very well known eigenvalue localization theorem of Geršgorin [6].

Theorem 1.1. [6] *Let $A = [a_{ij}] \in \mathbb{C}^{n\times n}$ and $\sigma(A)$ be the spectrum of A. Then*

$$\sigma(A) \subseteq \Gamma(A) = \bigcup_{i\in N} \Gamma_i(A).$$

Here, $\Gamma(A)$ is called the Geršgorin set of A. Recently, L. Cvetković et al. [4] gave the following two eigenvalue inclusion regions by the characterizations of two class of nonsingular H-matrices, and proved that these two regions stay within the set $\Gamma(A)\bigcap\Gamma(A^T)$, where A^T is the transpose of A.

Theorem 1.2. [4, Theorem 6] *Let $A = [a_{ij}] \in \mathbb{C}^{n\times n}$, $n \geq 2$. Then*

$$\sigma(A) \subseteq \mathcal{A}_1(A) = \bar{\Gamma}(A)\bigcup\tilde{\Gamma}(A),$$

where $\bar{\Gamma}(A) = \bigcup_{i\in N} \bar{\Gamma}_i(A)$ *and* $\tilde{\Gamma}(A) = \bigcup_{i\in\mathcal{H}, j\in\mathcal{L}} \tilde{\Gamma}_{i,j}(A)$.

Theorem 1.3. [4, Theorem 7] *Let $A = [a_{ij}] \in \mathbb{C}^{n\times n}$, $n \geq 2$. Then*

$$\sigma(A) \subseteq \mathcal{A}_2(A) = \bar{\Gamma}(A)\bigcup\hat{\Gamma}(A),$$

where $\bar{\Gamma}(A) = \bigcup_{i\in N} \bar{\Gamma}_i(A)$ *and* $\hat{\Gamma}(A) = \bigcup_{i\in\mathcal{H}, j\in\mathcal{L}} \hat{\Gamma}_{i,j}(A)$.

In [1], Brauer obtained the following eigenvalue localization theorem.

Theorem 1.4. [1] *Let $A = [a_{ij}] \in \mathbb{C}^{n\times n}$, $n \geq 2$. Then*

$$\sigma(A) \subseteq \mathcal{K}(A) = \bigcup_{i,j\in N,\ i\neq j} \mathcal{K}_{i,j}(A).$$

The set $\mathcal{K}(A)$ is called the Brauer set of A, and $\mathcal{K}_{i,j}(A)$ is called the (i,j)-th Brauer Cassini oval. It is well known that $\mathcal{K}(A) \subseteq \Gamma(A)$ (see [12, 13]). Since A and

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its transpose A^T have the same spectrum, we have that $\sigma(A) = \sigma(A^T) \subseteq \mathcal{K}(A^T) \subseteq \Gamma(A^T)$, and thus, $\sigma(A) \subseteq \left(\mathcal{K}(A) \bigcap \mathcal{K}(A^T)\right) \subseteq \left(\Gamma(A) \bigcap \Gamma(A^T)\right)$.

We now let

$$\bar{\mathcal{K}}(A) = \bigcup_{i,j \in N, i \neq j} \bar{\mathcal{K}}_{i,j}(A). \tag{1.1}$$

Note that $\mathcal{K}_{i,j}(A) = \mathcal{K}_{j,i}(A)$, $\bar{\mathcal{K}}_{i,j}(A) = \bar{\mathcal{K}}_{j,i}(A)$, $\bar{\mathcal{K}}_{i,j}(A) \subseteq \mathcal{K}_{i,j}(A)$ and $\bar{\mathcal{K}}_{i,j}(A) \subseteq \mathcal{K}_{i,j}(A^T)$ for $i, j \in N,\ i \neq j$. These show that $\bar{\mathcal{K}}(A) \subseteq \mathcal{K}(A)$ and $\bar{\mathcal{K}}(A) \subseteq \mathcal{K}(A^T)$, and thus,

$$\bar{\mathcal{K}}(A) \subseteq \left(\mathcal{K}(A) \bigcap \mathcal{K}(A^T)\right).$$

An interesting problem arises: whether $\bar{\mathcal{K}}(A)$ includes all eigenvalues of A or not? The following example provides a negative answer.

Example 1.5. Let

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 2 & 0 & 3 \end{bmatrix}.$$

By calculation, we get

$$\sigma(A) = \{-0.1149, 2.2541, 3.8608\},$$

$$\bar{\mathcal{K}}_{1,2}(A) = \{z \in \mathbb{C} : |z-1||z-2| \leq 2\},$$

$$\bar{\mathcal{K}}_{1,3}(A) = \{z \in \mathbb{C} : |z-1||z-3| \leq 3\}$$

and

$$\bar{\mathcal{K}}_{2,3}(A) = \{z \in \mathbb{C} : |z-2||z-3| \leq 1\}.$$

Obviously, $-0.1149 \notin \bar{\mathcal{K}}(A) = \left(\bar{\mathcal{K}}_{1,2}(A) \bigcup \bar{\mathcal{K}}_{1,3}(A) \bigcup \bar{\mathcal{K}}_{2,3}(A)\right)$.

In this paper, we also focus on the subject of eigenvalue localization. In Section 2, we establish necessary and sufficient conditions for two classes of nonsingular matrices, named double α_1-matrices and double α_2-matrices. In Section 3, new regions $\mathcal{K}_1(A)$ and $\mathcal{K}_2(A)$ including all the eigenvalues of A are obtained, which include $\bar{\mathcal{K}}(A)$ and stay within the set $\mathcal{K}(A) \bigcap \mathcal{K}(A^T)$. Specially, we compare the new eigenvalue inclusion region $\mathcal{K}_2(A)$ with $\mathcal{A}_1(A)$ in Theorem 1.2 (Theorem 6 of [4]) and $\mathcal{A}_2(A)$ in Theorem 1.3 (Theorem 7 of [4]), and prove $\mathcal{K}_2(A) \subseteq \mathcal{A}_1(A)$ and $\mathcal{K}_2(A) \subseteq \mathcal{A}_2(A)$.

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2. Necessary and sufficient conditions of double α_1-matrices and double α_2-matrices. In this section, double α_1-matrices and double α_2-matrices are presented. And their characterizations are given.

DEFINITION 2.1. A matrix $A = [a_{ij}] \in \mathbb{C}^{n\times n}$ is said to be a *double α_1-matrix*, if there is $\alpha \in [0,\ 1]$ such that for all $i,\ j \in N,\ i \neq j$,

$$|a_{ii}||a_{jj}| > \alpha r_i r_j + (1-\alpha)c_i c_j.$$

DEFINITION 2.2. A matrix $A = [a_{ij}] \in \mathbb{C}^{n\times n}$ is said to be a *double α_2-matrix*, if there is $\alpha \in [0,\ 1]$ such that for all $i,\ j \in N,\ i \neq j$,

$$|a_{ii}||a_{jj}| > (r_i r_j)^{\alpha}(c_i c_j)^{1-\alpha}. \tag{2.1}$$

As shown in [8], double α_2-matrices are nonsingular. And moreover, from the generalized arithmetic-geometric mean inequality:

$$\alpha a + (1-\alpha)b \geq a^{\alpha} b^{1-\alpha}$$

where $a, b \geq 0$ and $0 \leq \alpha \leq 1$, we easily get that double α_1-matrices are also nonsingular.

Now we establish necessary and sufficient conditions for double α_1-matrices and double α_2-matrices, respectively. First, some notations are given. For a matrix $A = [a_{ij}] \in \mathbb{C}^{n\times n},\ n \geq 2$, we denote

$$\mathcal{R} = \{(i,j) : r_i r_j > c_i c_j,\ i \neq j,\ i,\ j \in N\},$$

$$\mathcal{C} = \{(i,j) : c_i c_j > r_i r_j,\ i \neq j,\ i,\ j \in N\},$$

$$\mathcal{E} = \{(i,j) : r_i r_j = c_i c_j,\ i \neq j,\ i,\ j \in N\}.$$

Note here that $(i,j) \in \mathcal{R}$ ($\mathcal{C}$ or $\mathcal{E}$) implies $(j,i) \in \mathcal{R}$ ($\mathcal{C}$ or $\mathcal{E}$, respectively).

THEOREM 2.3. *A matrix $A = [a_{ij}] \in \mathbb{C}^{n\times n}, n \geq 2$, is a double α_2-matrix if and only if the following two conditions hold:*

(i) $|a_{ii}||a_{jj}| > \min\{r_i r_j,\ c_i c_j\}$ *for all* $i, j \in N$, $i \neq j$.

(ii) $\log_{\frac{r_i r_j}{c_i c_j}} \frac{|a_{ii}||a_{jj}|}{c_i c_j} > \log_{\frac{c_m c_n}{r_m r_n}} \frac{c_m c_n}{|a_{mm}||a_{nn}|}$ *for* $(i,j) \in \mathcal{R}\backslash\{(l,k) : c_l c_k = 0\}$, *and* $(m,n) \in \mathcal{C}\backslash\{(l,k) : r_l r_k = 0\}$.

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Proof. Firstly, suppose that A is a double α_2-matrix. Then there is $\alpha \in [0,1]$ such that

$$|a_{ii}||a_{jj}| > (r_i r_j)^{\alpha}(c_i c_j)^{1-\alpha}$$

for all $i,\ j \in N$, $i \neq j$. Condition **(i)** follows from the fact

$$(r_i r_j)^{\alpha}(c_i c_j)^{1-\alpha} \geq \min\{r_i r_j,\ c_i c_j\}.$$

Now, for $(i,j) \in \mathcal{R}\backslash\{(l,k) : c_l c_k = 0\}$, we have

$$\frac{|a_{ii}||a_{jj}|}{c_i c_j} > \left(\frac{r_i r_j}{c_i c_j}\right)^{\alpha}.$$

Note that $r_i r_j > c_i c_j$, taking the logarithm of the above inequality for the base $\frac{r_i r_j}{c_i c_j} > 1$, and using the monotonicity, we obtain that

$$\log_{\frac{r_i r_j}{c_i c_j}} \frac{|a_{ii}||a_{jj}|}{c_i c_j} > \alpha.$$

Similarly, for $(m,n) \in \mathcal{C}\backslash\{(l,k) : r_l r_k = 0\}$, we obtain that

$$\log_{\frac{c_m c_n}{r_m r_n}} \frac{c_m c_n}{|a_{mm}||a_{nn}|} < \alpha.$$

Thus, condition **(ii)** holds.

Conversely, suppose that the conditions **(i)** and **(ii)** hold. For each $(i,j) \in \mathcal{E}$, condition **(i)** directly implies inequality (2.1). And for $(i,j) \in \mathcal{R}$ such that $c_i c_j = 0$, or $(m,n) \in \mathcal{C}$ such that $r_m r_n = 0$, inequality (2.1) follows immediately. Thus, it remains to prove that inequality (2.1) holds for all $(i,j) \in (\mathcal{R}\backslash\{(l,k) : c_l c_k = 0\})\bigcup(\mathcal{C}\backslash\{(l,k) : r_l r_k = 0\})$.

For each $(i,j) \in \mathcal{R}\backslash\{(l,k) : c_l c_k = 0\}$, we have $r_i r_j > c_i c_j$, which, from condition **(i)**, leads to $|a_{ii}||a_{jj}| > c_i c_j$. Using the properties of the log function for the base greater than one, we obtain

$$\log_{\frac{r_i r_j}{c_i c_j}} \frac{|a_{ii}||a_{jj}|}{c_i c_j} > 0. \tag{2.2}$$

Similarly, for each $(m,n) \in \mathcal{C}\backslash\{(l,k) : r_l r_k = 0\}$, we have

$$\log_{\frac{c_m c_n}{r_m r_n}} \frac{c_m c_n}{|a_{mm}||a_{nn}|} < 1. \tag{2.3}$$

From inequalities (2.2), (2.3) and condition **(ii)**, we have that there is α such that, for each $(i,j) \in \mathcal{R}\backslash\{(l,k) : c_l c_k = 0\}$ and each $(m,n) \in \mathcal{C}\backslash\{(l,k) : r_l r_k = 0\}$,

$$\max\left\{0, \log_{\frac{c_m c_n}{r_m r_n}} \frac{c_m c_n}{|a_{mm} a_{nn}|}\right\} < \alpha < \min\left\{\log_{\frac{r_i r_j}{c_i c_j}} \frac{|a_{ii} a_{jj}|}{c_i c_j}, 1\right\}. \tag{2.4}$$

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From the left inequality and right inequality of inequality (2.4), we get, respectively, that for each $(i,j) \in \mathcal{R}\backslash\{(l,k) : c_l c_k = 0\}$,

$$\frac{|a_{ii}a_{jj}|}{c_i c_j} > \left(\frac{r_i r_j}{c_i c_j}\right)^{\alpha}$$

and for each $(m,n) \in \mathcal{C}\backslash\{(l,k) : r_l r_k = 0\}$,

$$\frac{c_m c_n}{|a_{mm}a_{nn}|} > \left(\frac{c_m c_n}{r_m r_m}\right)^{\alpha}.$$

Thus, the proof is completed. ◻

Similar to the proof of Theorem 2.3, we can obtain the following necessary and sufficient conditions for double α_1-matrices, and its proof is omitted.

Theorem 2.4. *A matrix $A = [a_{ij}] \in \mathbb{C}^{n\times n}, n \geq 2$, is a double α_1-matrix if and only if the following two conditions hold:*

(i) $|a_{ii}||a_{jj}| > \min\{r_i r_j,\ c_i c_j\}$ *for all $i, j \in N$, $i \neq j$.*

(ii) $\frac{|a_{ii}||a_{jj}|-c_i c_j}{r_i r_j - c_i c_j} > \frac{c_m c_n - |a_{mm}||a_{nn}|}{c_m c_n - r_m r_n}$ *for all $(i,j) \in \mathcal{R}$, and all $(m,n) \in \mathcal{C}$.*

3. Eigenvalue localizations. By the necessary and sufficient conditions of double α_1-matrices and double α_2-matrices in Section 2, we give two new eigenvalue inclusion regions.

Theorem 3.1. *Let $A = [a_{ij}] \in \mathbb{C}^{n\times n}$, $n \geq 2$, and $\sigma(A)$ be the spectrum of A. Then*

$$\sigma(A) \subseteq \mathcal{K}_2(A) = \bar{\mathcal{K}}(A) \bigcup \hat{\mathcal{K}}(A), \tag{3.1}$$

where $\bar{\mathcal{K}}(A)$ is given by (1.1), $\hat{\mathcal{K}}(A) = \bigcup\limits_{(i,j)\in\mathcal{R},(m,n)\in\mathcal{C}} \hat{\mathcal{K}}_{i,j,m,n}(A)$ *and*

$$\hat{\mathcal{K}}_{i,j,m,n}(A) = \{z \in \mathbb{C} : \frac{|\lambda - a_{ii}||\lambda - a_{jj}|}{c_i c_j}\left(\frac{|\lambda - a_{mm}||\lambda - a_{nn}|}{c_m c_n}\right)^{\log_{\frac{c_m c_n}{r_m r_n}} \frac{r_i r_j}{c_i c_j}} \leq 1,$$
$$(i,j) \in \mathcal{R}\backslash\{(l,k) : c_l c_k = 0\},\ (m,n) \in \mathcal{C}\backslash\{(l,k) : r_l r_k = 0\}\}.$$

Proof. For any $\lambda \in \sigma(A)$, $\lambda I - A$ is singular. Note that the moduli of every off-diagonal entry of $\lambda I - A$ is the same as A. Hence, the sets $\mathcal{R}$ and $\mathcal{C}$ for the matrix $\lambda I - A$ remain the same. If $\lambda \notin \mathcal{K}_2(A)$, then $\lambda I - A$ satisfies conditions **(i)** and **(ii)** of Theorem 2.3, hence $\lambda I - A$ is a double α_2-matrix, which implies that $\lambda I - A$ is nonsingular. This is a contradiction. Hence, $\lambda \in \mathcal{K}_2(A)$, that is, $\sigma(A) \subseteq \mathcal{K}_2(A)$. ◻

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REMARK 3.2. (**i**) From the original definition of double α_2-matrices, we can derive directly the following eigenvalue inclusion region (see [9]):

$$\mathcal{K}_2(A) = \bigcap_{0\le\alpha\le 1} \bigcup_{i,j\in N, i\neq j} \left\{ z\in\mathbb{C} : |z-a_{ii}||z-a_{jj}| \le (r_ir_j)^{\alpha}(c_ic_j)^{1-\alpha} \right\}. \tag{3.2}$$

Obviously, the form of $\mathcal{K}_2(A)$ obtained in (3.1) is much more convenient than that in (3.2).

(**ii**) Since $\mathcal{K}_2(A) = \mathcal{K}_2(A^T)$, we have that $\mathcal{K}_2(A) \subseteq \left(\mathcal{K}(A)\bigcap\mathcal{K}(A^T)\right)$.

Similar to the proof of Theorem 3.1, we can obtain easily the following eigenvalue localization theorem.

THEOREM 3.3. *Let $A=[a_{ij}]\in\mathbb{C}^{n\times n}$, $n\ge 2$, and $\sigma(A)$ be the spectrum of A. Then*

$$\sigma(A) \subseteq \mathcal{K}_1(A) = \bar{\mathcal{K}}(A)\bigcup\tilde{\mathcal{K}}(A),$$

where $\bar{\mathcal{K}}(A)$ is given by (1.1), $\tilde{\mathcal{K}}(A) = \bigcup_{(i,j)\in\mathcal{R},(m,n)\in\mathcal{C}} \tilde{\mathcal{K}}_{i,j,m,n}(A)$ *and*

$$\begin{aligned}\tilde{\mathcal{K}}_{i,j,m,n}(A) = \{z\in\mathbb{C} :&|\lambda-a_{ii}||\lambda-a_{jj}|(c_mc_n-r_mr_n)+|\lambda-a_{mm}||\lambda-a_{nn}|(r_ir_j\\ &-c_ic_j)\le c_mc_nr_ir_j-c_ic_jr_mr_n, (i,j)\in\mathcal{R}, (m,n)\in\mathcal{C}\}.\end{aligned}$$

Similar to Remark 3.2, we also obtain that $\mathcal{K}_1(A)\subseteq\left(\mathcal{K}(A)\bigcap\mathcal{K}(A^T)\right)$. Next, we compare $\mathcal{K}_2(A)$ in Theorem 3.1 with $\mathcal{A}_1(A)$ in Theorem 1.2 (Theorem 6 of [4]) and $\mathcal{A}_2(A)$ in Theorem 1.3 (Theorem 7 of [4]).

THEOREM 3.4. *Let $A=[a_{ij}]\in\mathbb{C}^{n\times n}$, $n\ge 2$. And $\mathcal{A}_2(A)$ and $\mathcal{K}_2(A)$ are defined in* Theorems 1.3*, and* 3.1*, respectively. Then*

$$\mathcal{K}_2(A)\subseteq\mathcal{A}_2(A).$$

Proof. We prove $\mathcal{K}_2(A)\subseteq\mathcal{A}_2(A)$. Equivalently, we prove that if $z\notin\mathcal{A}_2(A)$, then $z\notin\mathcal{K}_2(A)$. In fact, if $z\notin\mathcal{A}_2(A)$, from Theorem 1.3, we have that for any $i\in N$,

$$|z-a_{ii}| > \min\{r_i, c_i\}, \tag{3.3}$$

and for $i\in\mathcal{H}\backslash\{k : c_k=0\}$ and $j\in\mathcal{L}\backslash\{k : r_k=0\}$,

$$\frac{|z-a_{ii}|}{c_i}\left(\frac{|z-a_{jj}|}{c_j}\right)^{\log_{\frac{c_j}{r_j}}\frac{r_i}{c_i}} > 1. \tag{3.4}$$

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From Theorems 5 and 7 of [4], inequalities (3.3) and (3.4) imply that for any $i \in N$,

$$|z - a_{ii}| > (r_i)^\alpha (c_i)^{1-\alpha}$$

for some $\alpha \in [0,\ 1]$. Hence, for any $i, j \in N$ and $i \neq j$, we have

$$|z - a_{ii}||z - a_{jj}| > (r_i)^\alpha (c_i)^{1-\alpha} (r_j)^\alpha (c_j)^{1-\alpha} = (r_i r_j)^\alpha (c_i c_j)^{1-\alpha}$$

for some $\alpha \in [0,\ 1]$. This implies that $zI - A$ is a double α_2-matrix. From Theorem 2.3, the following two inequalities hold:

$$|z - a_{ii}||z - a_{jj}| > \min\{r_i r_j,\ c_i c_j\} \tag{3.5}$$

for all $i, j \in N$, $i \neq j$, and

$$\log_{\frac{r_i r_j}{c_i c_j}} \frac{|z - a_{ii}||z - a_{jj}|}{c_i c_j} > \log_{\frac{c_m c_n}{r_m r_n}} \frac{c_m c_n}{|z - a_{mm}||z - a_{nn}|} \tag{3.6}$$

for $(i, j) \in \mathcal{R} \backslash \{(l, k) : c_l c_k = 0\}$, and $(m, n) \in \mathcal{C} \backslash \{(l, k) : r_l r_k = 0\}$. Moveover, inequality (3.6) is written equivalently as

$$\frac{|z - a_{ii}||z - a_{jj}|}{c_i c_j} \left(\frac{|z - a_{mm}||z - a_{nn}|}{c_m c_n} \right)^{\log_{\frac{c_m c_n}{r_m r_n}} \frac{r_i r_j}{c_i c_j}} > 1. \tag{3.7}$$

Hence, from inequalities (3.5) and (3.7), $z \notin \bar{\mathcal{K}}(A)$ and $z \notin \hat{\mathcal{K}}(A)$, that is, $z \notin \mathcal{K}_2(A)$. The proof is completed. ▯

LEMMA 3.5. *Let $A = [a_{ij}] \in \mathbb{C}^{n \times n}$, $n \geq 2$. And $\mathcal{A}_1(A)$ and $\mathcal{A}_2(A)$ are defined in* Theorems 1.2*, and* 1.3*, respectively. Then*

$$\mathcal{A}_2(A) \subseteq \mathcal{A}_1(A).$$

Proof. Similar to the proof of Theorem 3.4 and from the fact that if

$$|z - a_{ii}| > \alpha r_i + (1 - \alpha) c_i,\ i \in N$$

for some $\alpha \in [0, 1]$, then

$$|z - a_{ii}| > r_i^\alpha c_i^{1-\alpha},$$

we can easily get that if $z \notin \mathcal{A}_1(A)$, then $z \notin \mathcal{A}_2(A)$, that is, $\mathcal{A}_2(A) \subseteq \mathcal{A}_1(A)$. ▯

From Theorem 3.4 and Lemma 3.5, we have easily the following result.

COROLLARY 3.6. *Let $A = [a_{ij}] \in \mathbb{C}^{n \times n}$, $n \geq 2$. And $\mathcal{A}_1(A)$ and $\mathcal{K}_2(A)$ are defined in* Theorems 1.2 *and* 3.1*, respectively. Then*

$$\mathcal{K}_2(A) \subseteq \mathcal{A}_1(A).$$

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Similar to the proof of Lemma 3.5, we can establish easily the following comparison result.

THEOREM 3.7. *Let* $A = [a_{ij}] \in \mathbb{C}^{n\times n}$, $n \geq 2$. *And* $\mathcal{K}_1(A)$ *and* $\mathcal{K}_2(A)$ *are defined in* Theorems 3.3, *and* 3.1, *respectively. Then*

$$\mathcal{K}_2(A) \subseteq \mathcal{K}_1(A).$$

REMARK 3.8. In Theorem 3.7, it is proved that $\mathcal{K}_2(A) \subseteq \mathcal{K}_1(A)$. However, $\mathcal{K}_2(A)$ is determined with more difficultly than $\mathcal{K}_1(A)$ because it is difficult to compute exactly $\log_{\frac{c_m c_n}{r_m r_n}} \frac{r_i r_j}{c_i c_j}$ in some cases.

EXAMPLE 3.9. Let

$$A = \begin{bmatrix} 1 & 0.5 & 0.5 & 0 \\ 1 & -1 & 0.5 & 0 \\ 0.5 & 0 & i & 0.05 \\ 0.1 & 0 & 0.1i & i \end{bmatrix}.$$

The eigenvalue inclusion regions of Theorems 1.2, 1.3, 1.4, 3.3 and 3.1 are given, respectively, by Figs. 3.1, 3.2, 3.3, 3.8 and 3.9. And $\tilde{\mathcal{K}}(A)$, $\bar{\mathcal{K}}(A)$ and $\hat{\mathcal{K}}(A)$ are shown in Figs. 3.5, 3.6 and 3.7, respectively. Note that the exact eigenvalues are plotted with asterisks. As we can see, $\bar{\mathcal{K}}(A)$ fails to capture all the eigenvalues of A, so, the necessity of $\tilde{\mathcal{K}}(A)$ or $\hat{\mathcal{K}}(A)$ is evident. Also, it is easy to see that $\mathcal{K}_1(A) \subset \mathcal{A}_1(A)$, $\mathcal{K}_2(A) \subset \mathcal{A}_2(A) \subset \mathcal{A}_1(A)$ and $\mathcal{K}_2(A) \subset \mathcal{K}_1(A) \subset \left(\mathcal{K}(A) \bigcap \mathcal{K}(A^T)\right)$. This example shows that the two new eigenvalue inclusion regions are smaller than the intersection of the Brauer sets of a matrix and its transpose, and the region of Theorem 3.1 is smaller than those of Theorem 6 and Theorem 7 in [4].

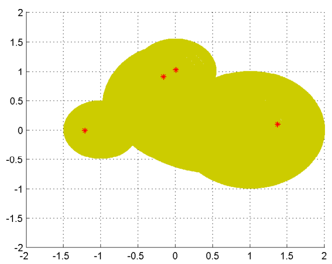

FIG. 3.1. $\mathcal{A}_1(A)$

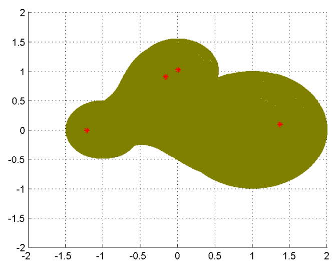

FIG. 3.2. $\mathcal{A}_2(A)$

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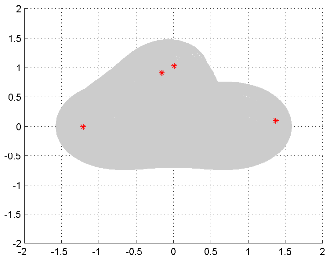

FIG. 3.3. $\mathcal{K}(A)$

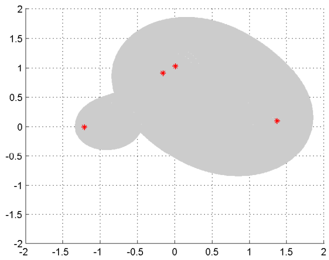

FIG. 3.4. $\mathcal{K}(A^T)$

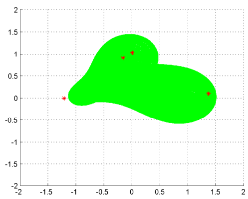

FIG. 3.5. $\tilde{\mathcal{K}}(A)$

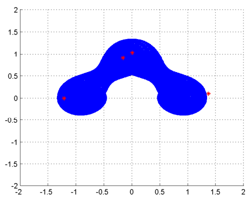

FIG. 3.6. $\bar{\mathcal{K}}(A)$

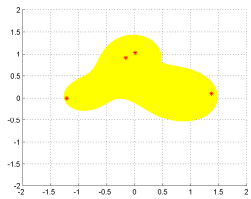

FIG. 3.7. $\hat{\mathcal{K}}(A)$

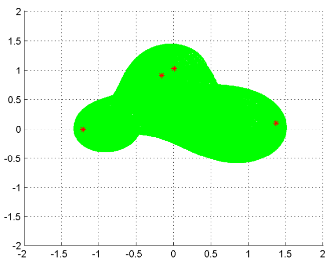

FIG. 3.8. $\mathcal{K}_1(A) = \bar{\mathcal{K}}(A) \bigcup \tilde{\mathcal{K}}(A)$

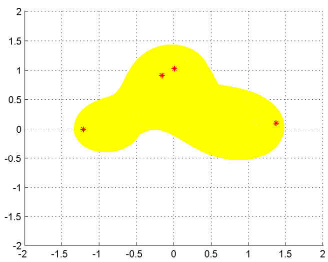

FIG. 3.9. $\mathcal{K}_2(A) = \bar{\mathcal{K}}(A) \bigcup \hat{\mathcal{K}}(A)$

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