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NETWORK PARAMETERS VIA EQUILIBRIUM MEASURES IN SCHRÖDINGER RANDOM WALKS*

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Abstract. In this study, we explore the concept of equilibrium measures within the framework of Schrödinger random walks on networks. Building on previous work, we demonstrate how these equilibrium measures can be leveraged to compute key network parameters, such as the mean first passage time and Kemeny's constant. By expressing these parameters in terms of generalized inverses of the associated M-matrix, we aim to provide new insights and efficient computational tools for network analysis, as demonstrated in the subsequent sections. The results are particularly applicable to both star and path networks, where we offer explicit formulations for these fundamental quantities. This work highlights the potential significance of equilibrium measures as a powerful tool for the study of complex networks.

Key words. Random walk, Equilibrium measure, Mean first passage time, Kemeny's constant, Combinatorial Laplacian, Group inverse.

AMS subject classifications. 05C81, 15A10, 31C20, 60J10.

1. Introduction.

Random walks on networks have long been a fundamental tool for modeling and analyzing complex systems across various disciplines, including physics, biology, computer science, and the social sciences. In contrast to traditional models, which often assume uniform transition probabilities or simple *lazy* walks with a fixed probability of remaining in place, real-world phenomena frequently demand more nuanced approaches. Schrödinger random walks provide a natural generalization of classical random walks on weighted graphs by incorporating vertex-dependent weights and a global potential parameter while preserving reversibility. This additional flexibility allows the modeling of heterogeneous networks in which both edge interactions and nodal effects play a role and leads naturally to the study of symmetric M-matrices beyond the classical graph Laplacian setting.

This study investigates how Schrödinger random walks can enhance the analysis of complex systems, see [13]. Several models appearing in the literature can be interpreted as particular instances of Schrödinger random walks. For example, locally biased random walks introduce node-dependent transition preferences determined by structural quantities such as node degrees, producing transition probabilities that depend on weights associated with each node [16]. Similarly, core-biased random walks use information about the degree hierarchy of the network (e.g., the number of connections to nodes of higher degree) to bias the movement of the walker. In both cases, the transition mechanism is modified through node-dependent weights, leading to reversible dynamics that extend the classical random walk on weighted graphs [22]. These examples illustrate how Schrödinger-type walks naturally arise when local structural information is incorporated into the transition probabilities.

Two key parameters in the study of random walks are the mean first passage time (MFPT) and Kemeny's constant, which provide crucial insights into the dynamics and structure of networks. The MFPT, also known

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as the *Hitting time*, which measures the expected number of steps for a random walker to reach a specific node for the first time, has been extensively studied owing to its relevance in understanding transport processes, information diffusion, and search strategies on networks [21]. Kemeny's constant, on the other hand, is a global parameter that represents the expected number of steps to reach any random target node, regardless of the starting point, and has gained increasing attention due to its applications in assessing network efficiency and robustness [23].

Building on our previous work on discrete potential theory and M-matrices, we continue to work on a generalized random walk model that associates transition probabilities with symmetric M-matrices. This approach, which we term *Schrödinger random walks* [13], eliminates the diagonally dominant hypothesis and provides a versatile tool for network analysis. As mentioned, our model extends the idea of biased random walks [16, 22] by considering both varied transition probabilities and the possibility of remaining at a node, with the lazy term treated as a function rather than a fixed parameter.

Central to our analysis is the concept of *equilibrium measures*, a potential theoretical topic introduced in the context of networks by the authors in [2]. By expressing these parameters in terms of generalized inverses of the associated M-matrix, we provide new insights and efficient computational tools for network analysis. Notably, M-matrices serve as a generalization of the graph Laplacian, while equilibrium measures, which have recently re-appeared in the literature as the so-called landscape function, act as an effective potential for localization [14]. The use of generalized inverses in the study of mean first passage times and related quantities has a substantial history in the Markov chain literature; see, for instance, Hunter [17] and the monograph by Campbell and Meyer [11].

We hypothesize that, within the class of reversible chains arising from symmetric M-matrices, equilibrium measures can provide a unifying and constructive framework for MFPT and Kemeny's constant, even for general symmetric M-matrices beyond the singular case.

Recent advancements in MFPT and Kemeny's constant analysis have focused on their relationship with network topology and the development of efficient computational methods [19]. The interplay between these parameters has been a subject of particular interest. In particular, Breen *et al.* [9] explored their relationship with the connectivity of graphs, providing new bounds and characterizations. Furthermore, Breen and Kirkland [10] made significant contributions by investigating the sensitivity of Kemeny's constant to perturbations in the network structure.

From a computational viewpoint, equilibrium measures can be obtained by solving linear systems associated with symmetric M-matrices, which can be interpreted variationally as energy minimization problems. While the worst-case complexity for dense matrices remains $O(n^3)$, this formulation is designed to avoid the explicit computation of generalized inverses and facilitate the use of efficient iterative solvers such as conjugate gradient or multigrid methods, particularly effective for large sparse networks.

This paper is structured as follows: We begin by introducing the necessary preliminaries and notation for Schrödinger random walks on networks. We then explore the properties of equilibrium measures and their relationship to the generalized inverses of M-matrices. Finally, we demonstrate the application of our method to compute the MFPT and Kemeny's constant, providing explicit expressions for star and path networks as illustrative examples.

This work highlights the potential significance of equilibrium measures as a powerful tool in the study of complex networks, offering a more accurate representation of real-world systems and opening new avenues for research in network science, statistical physics, and related fields.

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The main purpose of this work is not to extend the theory of mean first passage times or Kemeny's constant to arbitrary irreducible Markov chains, but rather to show that within the important class of reversible chains arising from symmetric M-matrices, equilibrium measures offer a unifying and constructive framework. In this setting, quantities such as mean first passage times, capacities, and Kemeny's constant admit explicit representations that are closely connected to discrete potential theory.

From an applied perspective, random walks and Markov processes on weighted graphs with additional nodal structures naturally arise in network-based models for signal processing and data analysis, including applications in biomedical signal analysis and graph-based filtering methods. Recent studies have highlighted the relevance of weighted graph structures and diffusion-type processes in practical contexts, motivating the development of mathematically transparent tools for extracting global network parameters from local equilibrium properties.

2. Preliminaries. A network, or *weighted graph*, $\Gamma=(V,E,c)$ consists of a set V of vertices –or nodes, or even states–, and a set E of edges between those vertices. These sets have cardinalities $|V|=n$ and $|E|=m$. Two different vertices $x,y\in V$ are adjacent, denoted as $x\sim y$, if $\{x,y\}\in E$. Additionally, we assign a *conductance* $c(x,y)>0$ to each adjacent pair of vertices, defined by the function $c: V\times V\longrightarrow[0,+\infty)$, which satisfies $c(x,y)=c(y,x)$ and $c(x,y)=0$ if and only if $\{x,y\}\notin E$. If we denote $\mathcal{C}(V)$ as the set of real functions on V, then $k\in\mathcal{C}(V)$, where $k(x)=\sum\limits_{y\in V}c(x,y)$ is the *degree* of x. When $c(x,y)=1$, for any $x\sim y$, Γ is called a *graph*. In this case, $k(x)$ denotes the number of states adjacent to x. Throughout this study, we focus on *connected* networks, ensuring that there exists a path in Γ between any two different vertices $x,y\in V$. Furthermore, we consider *simple* networks, so there are no loops and no multiple edges. A positive function $\omega\in\mathcal{C}(V)$ is called a *weight* if $\sum\limits_{x\in V}\omega(x)^2=1$. The set of weights on V is denoted by $\Omega(V)$.

Throughout this paper, we use the usual convention that empty sums equal 0.

If we label the vertex set $V=\big\{1,2,\dots,n\big\}$, then functions can be identified with vectors in $\mathbb{R}^n$, and operators can be identified with square n-matrices. For the reader's convenience, matrices will be in sans serif font, and vectors will be in either sans serif font or boldface. In particular, $\mathsf{k}=\big(k_1,k_2,\dots,k_n\big)^\top$ is the degree column vector for Γ, e_i is the i-th vector of the canonical basis, and $\boldsymbol{\omega}$ is the vector associated with the weight ω. Given a vector u, D_u denotes the diagonal matrix whose diagonal entries are given by the vector u and given a matrix X, we denote by X_d the diagonal matrix whose diagonal entries are given by the diagonal of X. Moreover, we denote $c_{ij}=c(x_i,x_j)$ and $\mathsf{A}_c=(c_{ij})$ as the *adjacency matrix* of Γ. Therefore, A_c is a symmetric nonnegative matrix, and the connectedness of Γ is equivalent to A_c being irreducible. Conversely, it is clear that any symmetric non-negative matrix with null diagonal entries A is the adjacency matrix of $\Gamma=(V,c)$, where the conductance c is given by the off-diagonal entries of A.

The *combinatorial Laplacian matrix* for Γ is the symmetric and irreducible matrix $\mathsf{L}_c=\mathsf{D}_\mathsf{k}-\mathsf{A}_c$; that is,

$$\mathsf{L}_c=\begin{bmatrix} k_1 & -c_{12} & \dots & -c_{1n}\\ -c_{12} & k_2 & \dots & -c_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ -c_{1n} & -c_{2n} & \dots & k_n\end{bmatrix}.$$

More generally, we call *Schrödinger matrix* for Γ with *potential* $\mathsf{q}\in\mathbb{R}^n$ to the symmetric and irreducible matrix of the form $\mathsf{L}_c+\mathsf{D}_\mathsf{q}$. So, the combinatorial Laplacian corresponds with the Schrödinger matrix with null potential.

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The combinatorial Laplacian matrix L_c is diagonally dominant (d.d.) and hence is positive semidefinite. Moreover, it is singular, and 0 is a simple eigenvalue whose associated eigenvector is constant, $\mathsf{L}_c\mathsf{1} = \mathsf{0}$, where $\mathsf{1}$ represents the all ones vector. In addition, L_c is a Z-matrix and, more specifically, a M-matrix.

Recall that a symmetric matrix $\mathsf{M} \in \mathcal{M}_n(\mathbb{R})$ is called a *Z-matrix* if and only if its off-diagonal entries are non-positive. Moreover, M is a symmetric M-matrix if and only if it is a positive semidefinite Z-matrix, see [8].

Any symmetric Z-matrix is of the form $\mathsf{D}_\mathsf{u} - \mathsf{A}$ where A is a symmetric nonnegative matrix with null diagonal entries and $\mathsf{u} \in \mathbb{R}^n$. Therefore, any irreducible Z-matrix is of the form $\mathsf{D}_\mathsf{u} - \mathsf{A}_c$, where c is a conductance on V such that $\Gamma = (V, c)$ is connected. Since $\mathsf{D}_\mathsf{u} - \mathsf{A}_c = \mathsf{L}_c + \mathsf{D}_\mathsf{q}$, where $\mathsf{q} = \mathsf{u} - \mathsf{k}$, any irreducible Z-matrix can be seen as a Schrödinger matrix for some connected network. In particular, for a fixed network $\Gamma = (V, c)$, the set $\{\mathsf{L}_c + \mathsf{D}_\mathsf{q} : \mathsf{q} \in \mathbb{R}^n\} \subset \mathcal{M}_n(\mathbb{R})$ describes all irreducible Z-matrices whose off-diagonal entries are given by the conductance c.

In the sequel, we consider a fixed network $\Gamma = (V, c)$, and for this reason, we omit the subindex c from the Laplacian and adjacency matrices. Therefore, L represents the combinatorial Laplacian and $\mathsf{L} + \mathsf{D}_\mathsf{q}$ the Schrödinger matrix with potential $\mathsf{q} \in \mathbb{R}^n$ for Γ. Clearly, when q is nonnegative, $\mathsf{L} + \mathsf{D}_\mathsf{q}$ is d.d. and hence a M-matrix. However, since a M-matrix whose off-diagonal entries are given by the conductance c is not necessarily d.d. to characterize this type of matrices, we must introduce some additional concepts.

Given a weight $\omega \in \Omega(V)$, we define q_ω, the potential whose entries are given by $(\mathsf{q}_\omega)_i = -\dfrac{(\mathsf{L}\boldsymbol{\omega})_i}{\omega_i}$, $i = 1, \ldots, n$.

Given a potential q, there exists a unique weight $\omega \in \Omega(V)$ and a unique $\lambda \in \mathbb{R}$ such that $\mathsf{q} = \mathsf{q}_\omega + \lambda\mathsf{1}$, see [5, 13]. Therefore, any irreducible Z-matrix whose off-diagonal entries are given by the conductance c; that is, any irreducible Schrödinger matrix for Γ, is of the form $\mathsf{L} + \mathsf{D}_{\mathsf{q}_\omega + \lambda\mathsf{1}} = \mathsf{D}_{\mathsf{k} + \mathsf{q}_\omega + \lambda\mathsf{1}} - \mathsf{A}$. In the sequel, we denote this matrix as $\mathsf{L}_{\lambda,\omega}$; that is,

$$\mathsf{L}_{\lambda,\omega} = \begin{bmatrix} k_1 + q_1 & -c_{12} & \dots & -c_{1n} \\ -c_{12} & k_2 + q_2 & \dots & -c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -c_{1n} & -c_{2n} & \dots & k_n + q_n \end{bmatrix}, \tag{2.1}$$

were $\mathsf{q} = \mathsf{q}_\omega + \lambda\mathsf{1}$. In [5], we proved that $\mathsf{L}_{\lambda,\omega}$ is a M-matrix if and only if $\lambda \geq 0$, so given a complete characterization of those M-matrices whose off-diagonal entries are given by c. Hereafter, we focus on such matrices.

Thus, we set $\lambda \geq 0$, a weight $\omega \in \Omega(V)$ and their associated potential $\mathsf{q} = \mathsf{q}_\omega + \lambda\mathsf{1}$ and the matrix $\mathsf{L}_{\lambda,\omega}$. Then, $\mathsf{L}_{\lambda,\omega}$ is symmetric, irreducible, positive semidefinite, and singular if and only if $\lambda = 0$. Moreover, λ is the smallest eigenvalue of $\mathsf{L}_{\lambda,\omega}$ whose associated eigenvectors are multiples of $\boldsymbol{\omega}$. When $\lambda = 0$ and $\boldsymbol{\omega}$ is constant, we recover the combinatorial Laplacian.

Observe that the diagonal entries of $\mathsf{L}_{\lambda,\omega}$ are $k_i + q_i = \lambda + \dfrac{1}{\omega_i}\sum_{j=1}^n c_{ij}\omega_j$, for any $i = 1, \ldots, n$.

In this paper, we consider *Schrödinger random walks* that were introduced in [7] by some of the authors and studied more deeply in [13]. The probability laws governing the evolution of the random walk are given by the (*one step*) *transition probability matrix with respect to* λ *and* ω, $\mathsf{P}_{\lambda,\omega} \in \mathcal{M}_n(\mathbb{R})$, that is defined as

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$$\mathsf{P}_{\lambda,\omega} = \mathsf{D}_{\mathsf{k}_\omega}^{-1}\big(\mathsf{A} + \lambda \boldsymbol{\omega}\boldsymbol{\omega}^\top\big)\mathsf{D}_{\boldsymbol{\omega}},$$

where we denote by $\mathsf{k}_\omega = (\mathsf{A} + \lambda \mathsf{I})\boldsymbol{\omega}$ the vector whose components are $(k_i + q_i)\omega_i$.

This definition retains the spirit of the effective resistance concerning a parameter and weight introduced by the authors in [5]. Observe that for any $i, j \in V$,

$$\big(\mathsf{P}_{\lambda,\omega}\big)_{ij} = \frac{(\,c_{ij} + \lambda\,\omega_i\omega_j)\omega_j}{(k_i + q_i)\omega_i}. \tag{2.2}$$

Therefore, in our model, the walker jumps to a neighbor with a probability depending on the value of the conductance c_{ij} times the factor ω_j representing the desired state property, but we also add the probability depending on $\lambda\omega_i\omega_j$. Moreover, as we can see, the main novelty in our definition is the consideration of a non-negative probability of remaining at vertex x_i,

$$\big(\mathsf{P}_{\lambda,\omega}\big)_{ii} = \frac{\lambda\,\omega_i^2}{k_i + q_i}, \tag{2.3}$$

see Fig. 1.

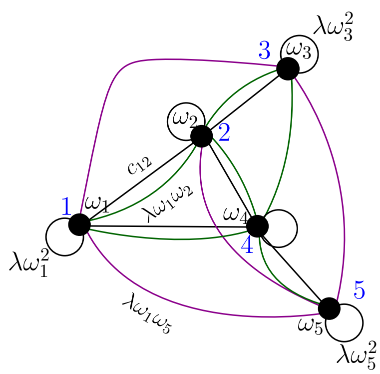

FIGURE 1. *An example of a Schrödinger random walk and some of the conductance parameters.*

Consider now $\boldsymbol{\pi}_{\lambda,\omega} \in \mathbb{R}^n$ defined for each $x_i \in V$ as

$$\big(\boldsymbol{\pi}_{\lambda,\omega}\big)_i = \frac{(k_i + q_i)\omega_i^2}{\lambda + \sum\limits_{j,\ell=1}^{n} c_{j\ell}\omega_j\omega_\ell}. \tag{2.4}$$

As shown in [13, Lemma 1], the transition probability matrix is Markovian, reversible, and has $\boldsymbol{\pi}_{\lambda,\omega}$ as a stationary distribution. We call the *volume of* Γ the value $\mathsf{vol}(\Gamma) = \lambda + \sum\limits_{j,\ell=1}^{n} c_{j\ell}\omega_j\omega_\ell$. Observe that $\mathsf{vol}(\Gamma) = \boldsymbol{\omega}^\top \mathsf{k}_\omega$ and $\big(\pi_{\lambda,\omega}\big)_i = \dfrac{(k_i + q_i)\omega_i^2}{\mathsf{vol}(\Gamma)}$.

The short-term behavior of a Schrödinger random walk is modeled by the *mean first passage time with respect to* λ *and* ω $\big(\mathsf{M}_{\lambda,\omega}\big)_{ij}$, for $i, j = 1, \ldots, n$, $i \neq j$, which gives the expected number of time steps before

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the system reaches node j, if it starts at node i. In addition, the *mean recurrence time for state* i, denoted by $\big(\mathsf{M}_{\lambda,\omega}\big)_{ii}$, is the expected number of time steps before returning to i for the first time, for any $i = 1,\dots,n$.

If we define J as the matrix of order n with all entries equal to 1, we obtain the mean first passage time as the solution of the following equation,

$$(2.5)\qquad \big(\mathsf{I}-\mathsf{P}_{\lambda,\omega}\big)\mathsf{M}_{\lambda,\omega} = \mathsf{J} - \mathsf{P}_{\lambda,\omega}(\mathsf{M}_{\lambda,\omega})_d.$$

The mean recurrence time for state i is $\dfrac{1}{\big(\pi_{\lambda,\omega}\big)_i}$, since multiplying both sides of (2.5) by $\boldsymbol{\pi}_{\lambda,\omega}^{\mathsf{T}}$, we obtain $\mathsf{0}^{\mathsf{T}} = \boldsymbol{\pi}_{\lambda,\omega}^{\mathsf{T}}\left(\mathsf{J} - \mathsf{P}_{\lambda,\omega}(\mathsf{M}_{\lambda,\omega})_d\right)$ or $\mathsf{0}^{\mathsf{T}} = \mathsf{1}^{\mathsf{T}} - \boldsymbol{\pi}_{\lambda,\omega}^{\mathsf{T}}(\mathsf{M}_{\lambda,\omega})_d$.

On the other hand, Kemeny's constant represents the time for reaching a random state j, starting from an initial state i according to the stationary distribution. In our case, we define the *Kemeny's constant (with respect to* λ *and* ω*)* as the value

$$K_{\lambda,\omega}(\Gamma) = \sum_{j=1}^{n} \big(\mathsf{M}_{\lambda,\omega}\big)_{ij}\big(\boldsymbol{\pi}_{\lambda,\omega}\big)_j.$$

In the standard case, it is known that K does not depend on i, and hence, the name *Kemeny's constant*. This is also true for $K_{\lambda,\omega}(\Gamma)$ and $\mathsf{M}_{\lambda,\omega}\boldsymbol{\pi}_{\lambda,\omega} = K_{\lambda,\omega}(\Gamma)\mathsf{1}$, see [13]. There, we also tackled how to solve Equation (2.5) using 1-inverses of the considered M-matrix. In particular, the so-called Green functions were of main interest. In this work, we approach MFPT and Kemeny's constant with one of those 1-inverses related to the equilibrium measure introduced in [2]. In the following, for $\lambda = 0$, we drop the sub-index λ in the above parameters and call it MFPT with respect to ω.

3. Green matrices and equilibrium measures. In this section, we present the concept of the *equilibrium matrix* and its relation to the Green matrices. This concept is crucial for obtaining a closed formula for the mean first passage time and Kemeny's constant.

We consider the matrix $\mathsf{F}_{\lambda,\omega} = \mathsf{L}_{\lambda,\omega} - \lambda\boldsymbol{\omega}\boldsymbol{\omega}^{\top}$, which is called the *generalized Schrödinger matrix with respect to* λ *and* ω and is a symmetric and positive semidefinite matrix; that is, $\mathsf{F}_{\lambda,\omega}$ is an M-matrix. Moreover, it is a singular matrix with 0 as a simple eigenvalue and $\boldsymbol{\omega}$ as the unique positive unitary vector that satisfies $\mathsf{F}_{\lambda,\omega}\boldsymbol{\omega} = 0$, see [13, p. 327]. Given these properties, our attention is directed toward determining $\mathsf{u} \in \mathbb{R}^n$ such that $\mathsf{F}_{\lambda,\omega}\mathsf{u} = \mathsf{f}$ for a given $\mathsf{f} \in \mathbb{R}^n$.

It is well known that the above equation has a solution if and only if $\mathsf{f} \in \boldsymbol{\omega}^{\perp}$, and the solution is unique, except for a multiple of $\boldsymbol{\omega}$. Therefore, we can define a *generalized inverse of* $\mathsf{F}_{\lambda,\omega}$, or 1-*inverse of* $\mathsf{F}_{\lambda,\omega}$, as any n-matrix, X, that provides a solution for $\mathsf{F}_{\lambda,\omega}\mathsf{u} = \mathsf{f}$, to any f such that $\boldsymbol{\omega}^{\top}\mathsf{f} = 0$. It is well known that there exist infinite generalized inverses of $\mathsf{F}_{\lambda,\omega}$, which are characterized by the identity

$$(3.6)\qquad \mathsf{F}_{\lambda,\omega}\mathsf{X}\mathsf{F}_{\lambda,\omega} = \mathsf{F}_{\lambda,\omega}.$$

This implies that any generalized inverse is either invertible or has 0 as a simple eigenvalue.

The idea of generalized inverses of $\mathsf{F}_{\lambda,\omega}$ includes a particular type of 1-inverses. We call *Green matrix* for $\mathsf{F}_{\lambda,\omega}$ any matrix G, such that

$$(3.7)\qquad \mathsf{F}_{\lambda,\omega}\mathsf{G} = \mathsf{I} - \boldsymbol{\omega}\boldsymbol{\omega}^{\top}.$$

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Because $(\mathsf{I}-\boldsymbol{\omega}\boldsymbol{\omega}^\top)\boldsymbol{\omega}=0$, there exist infinitely many Green matrices. Moreover, any Green matrix is a 1-inverse of $\mathsf{F}_{\lambda,\omega}$ because $\mathsf{F}_{\lambda,\omega}\mathsf{G}\mathsf{F}_{\lambda,\omega}=(\mathsf{I}-\boldsymbol{\omega}\boldsymbol{\omega}^\top)\mathsf{F}_{\lambda,\omega}=\mathsf{F}_{\lambda,\omega}$, where we have considered that $\boldsymbol{\omega}^\top\mathsf{F}_{\lambda,\omega}=0$ because $\mathsf{F}_{\lambda,\omega}$ is symmetric. The following lemma was presented in [6, Theorem 3.3]. We have included here the proof for completeness.

Lemma 3.1. *If G is a 1-inverse of $\mathsf{F}_{\lambda,\omega}$, then G is a Green matrix if and only if $\boldsymbol{\omega}$ is an eigenvector of G, that is, $\mathsf{G}\boldsymbol{\omega}=\alpha\boldsymbol{\omega}$ for some $\alpha\in\mathbb{R}$. Moreover, the unique Green matrix satisfying $\boldsymbol{\omega}^\top\mathsf{G}=0$ is symmetric and coincides with the group inverse of $\mathsf{F}_{\lambda,\omega}$, $\mathsf{F}^{\#}_{\lambda,\omega}$.*

Proof. If G is a Green matrix for $\mathsf{F}_{\lambda,\omega}$, then $\mathsf{F}_{\lambda,\omega}\mathsf{G}\boldsymbol{\omega}=(\mathsf{I}-\boldsymbol{\omega}\boldsymbol{\omega}^\top)\boldsymbol{\omega}=0$ and hence $\mathsf{G}\boldsymbol{\omega}=\alpha\boldsymbol{\omega}$ for some $\alpha\in\mathbb{R}$.

Conversely, if $\mathsf{F}_{\lambda,\omega}\mathsf{G}\mathsf{F}_{\lambda,\omega}=\mathsf{F}_{\lambda,\omega}$, then $(\mathsf{F}_{\lambda,\omega}\mathsf{G}-\mathsf{I})\mathsf{F}_{\lambda,\omega}=0$. Therefore, we have that $\mathsf{F}_{\lambda,\omega}(\mathsf{G}^\top\mathsf{F}_{\lambda,\omega}-\mathsf{I})=0$, which implies that $(\mathsf{G}^\top\mathsf{F}_{\lambda,\omega}-\mathsf{I})=\boldsymbol{\omega}\boldsymbol{\tau}^\top$ for certain $\boldsymbol{\tau}\in\mathbb{R}^n$. Then, $(\mathsf{F}_{\lambda,\omega}\mathsf{G}-\mathsf{I})=\boldsymbol{\tau}\boldsymbol{\omega}^\top$ and hence $\mathsf{F}_{\lambda,\omega}\mathsf{G}\boldsymbol{\omega}-\boldsymbol{\omega}=\boldsymbol{\tau}$. If in addition, $\mathsf{G}\boldsymbol{\omega}=\alpha\boldsymbol{\omega}$, the above identity implies that $-\boldsymbol{\omega}=\boldsymbol{\tau}$ and the result follows.

In addition, there is a unique Green matrix satisfying $\boldsymbol{\omega}^\top\mathsf{G}=0$. Then, $\mathsf{G}\boldsymbol{\omega}$=0 because $0=\boldsymbol{\omega}^\top\mathsf{G}\boldsymbol{\omega}=\alpha$.

On the other hand, $\mathsf{G}^\top$ is also a 1-inverse because $\mathsf{F}_{\lambda,\omega}$ is symmetric. Moreover, $\mathsf{G}^\top\boldsymbol{\omega}=0$ and hence, $\mathsf{G}^\top$ is a Green matrix. Now, $\boldsymbol{\omega}^\top\mathsf{G}^\top=0$ because $(\boldsymbol{\omega}^\top\mathsf{G}^\top)^\top=\mathsf{G}\boldsymbol{\omega}=0$ and hence $\mathsf{G}^\top=\mathsf{G}$. Moreover, it is fulfilled that $\mathsf{F}_{\lambda,\omega}\mathsf{G}=\mathsf{G}\mathsf{F}_{\lambda,\omega}=\mathsf{I}-\boldsymbol{\omega}\boldsymbol{\omega}^\top$ and $\mathsf{G}\mathsf{F}_{\lambda,\omega}\mathsf{G}=\mathsf{G}$, and therefore, $\mathsf{G}=\mathsf{F}^{\#}_{\lambda,\omega}$. ∎

In [7, 12], some of the authors established that for any $j=1,\dots,n$ there exists a unique vector $\nu^j_{\lambda,\omega}\in\mathbb{R}^n$ such that $(\nu^j_{\lambda,\omega})_j=0$ and $(\nu^j_{\lambda,\omega})_i>0$ for $i\neq j$ and verifies

$$\mathsf{L}_{\lambda,\omega}\nu^j_{\lambda,\omega}=\boldsymbol{\omega}-\left[1-\lambda\mathsf{cap}^j_{\lambda,\omega}\right]\frac{\mathsf{e}_j}{\omega_j},\quad j=1,\dots,n,$$

where the value $\mathsf{cap}^j_{\lambda,\omega}=\boldsymbol{\omega}^\top\nu^j_{\lambda,\omega}$ is called the *capacity of $V\setminus\{j\}$*, and $\nu^j_{\lambda,\omega}$ is called the *equilibrium measure of $V\setminus\{j\}$ with respect to λ and ω*. In [7, Corollary 2.5], we also proved that $\lambda\mathsf{cap}^j_{\lambda,\omega}<1$. These definitions are analogs to the classical concepts in potential theory; the interested reader can find more details and interpretations of equilibrium measures in [3] and references therein.

In the standard case corresponding to $\lambda=0$ and ω constant, the equilibrium measure was defined as the solution of

$$\mathsf{L}\nu^j=\mathsf{1}-n\mathsf{e}_j;$$

see [2]. In the literature, where the weights on the nodes are not considered, one can interpret that $\omega_i=1$ for any i; however, as we consider that the weights are normalized when we say that the weight is constant, we mean $\omega_i=\dfrac{1}{\sqrt{n}}$; hence, when comparing our results with the existing results in the literature for the standard case, an extra factor of n may appear. Note that $\dfrac{1}{\sqrt{n}}\nu^j=\nu^j_{0,\omega}$.

The relationship between the equilibrium measures and the group inverse of $\mathsf{F}_{\lambda,\omega}$ is given by the following lemma:

Lemma 3.2. *If for any $j=1,\dots,n$, $\nu^j_{\lambda,\omega}$ denotes the equilibrium measure for node j, it is satisfied that*

$$\left(\mathsf{F}^{\#}_{\lambda,\omega}\right)_{ij}=\frac{\omega_i\omega_j}{1-\lambda\mathsf{cap}^j_{\lambda,\omega}}\left[\mathsf{cap}^j_{\lambda,\omega}-\frac{(\nu^j_{\lambda,\omega})_i}{\omega_i}\right].$$

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Moreover,

$$\big(\mathsf{F}^{\#}_{\lambda,\omega}\big)_{ii} = \frac{\omega_i^2 \mathsf{cap}^i_{\lambda,\omega}}{1-\lambda\mathsf{cap}^i_{\lambda,\omega}},$$

and hence

$$\big(\mathsf{F}^{\#}_{\lambda,\omega}\big)_{ji} = \frac{\omega_i}{\omega_j}\big(\mathsf{F}^{\#}_{\lambda,\omega}\big)_{jj} - \frac{\omega_j(\nu^j_{\lambda,\omega})_i}{1-\lambda\mathsf{cap}^j_{\lambda,\omega}}.$$

Proof. Observe that $\big(\mathsf{F}^{\#}_{\lambda,\omega}\big)_j$ is the unique solution of

$$\mathsf{F}_{\lambda,\omega}\big(\mathsf{F}^{\#}_{\lambda,\omega}\big)_j = \mathsf{e}_j - \omega_j\boldsymbol{\omega},$$

which is orthogonal to $\boldsymbol{\omega}$. Moreover, for any f satisfying $\mathsf{f}\in\boldsymbol{\omega}^\perp$, $\mathsf{u}\in\mathbb{R}^n$ is a solution of $\mathsf{F}_{\lambda,\omega}\mathsf{u}=\mathsf{f}$ if and only if $\mathsf{L}_{\lambda,\omega}\mathsf{u}=\mathsf{f}$. Hence, we consider $a\nu^j_{\lambda,\omega}+b\boldsymbol{\omega}$, and we impose that

$$\mathsf{e}_j - \omega_j\boldsymbol{\omega} = \mathsf{L}_{\lambda,\omega}(a\nu^j_{\lambda,\omega}+b\boldsymbol{\omega}) = a\mathsf{L}_{\lambda,\omega}\nu^j_{\lambda,\omega} + b\mathsf{L}_{\lambda,\omega}\boldsymbol{\omega} = (a+\lambda b)\boldsymbol{\omega} - a\Big[1-\lambda\mathsf{cap}^j_{\lambda,\omega}\Big]\frac{\mathsf{e}_j}{\omega_j},$$

we obtain $a=\dfrac{\omega_j}{1-\lambda\mathsf{cap}^j_{\lambda,\omega}}$ and $b=\dfrac{\omega_j\mathsf{cap}^j_{\lambda,\omega}}{1-\lambda\mathsf{cap}^j_{\lambda,\omega}}$. Then, the results follow. ∎

In the standard case, we have that

$$\big(\mathsf{F}^{\#}\big)_{ij} = \frac{1}{n^2}\big(\mathsf{cap}^j - n\nu^j_i\big),$$

which corresponds to the group inverse for the combinatorial Laplacian L, as given in [2, 4].

From now on, we call *equilibrium matrix with respect to* λ *and* ω, the matrix $\mathsf{E}_{\lambda,\omega}$ whose columns are

$$\big(\mathsf{E}_{\lambda,\omega}\big)_j = -\frac{\omega_j}{1-\lambda\mathsf{cap}^j_{\lambda,\omega}}\nu^j_{\lambda,\omega},\quad j=1\ldots,n.$$

Using Lemma 3.2, we obtain the following result:

Lemma 3.3. *The equilibrium matrix with respect to* λ *and* ω *can be written in terms of* $\mathsf{F}^{\#}_{\lambda,\omega}$ *in the following way*

$$\big(\mathsf{E}_{\lambda,\omega}\big)_{ij} = \big(\mathsf{F}^{\#}_{\lambda,\omega}\big)_{ij} - \frac{\omega_i}{\omega_j}\big(\mathsf{F}^{\#}_{\lambda,\omega}\big)_{jj},$$

for any $i,j=1,\ldots,n$.

The following result is essential for writing the Kemeny constant in a closed form.

Theorem 3.4. *The equilibrium matrix with respect to* λ *and* ω *is a Green matrix for* $\mathsf{F}_{\lambda,\omega}$. *Moreover,* $\mathsf{E}_{\lambda,\omega}\boldsymbol{\omega} = -\mathsf{tr}(\mathsf{F}^{\#}_{\lambda,\omega})\boldsymbol{\omega}$, *it is invertible and*

$$\mathsf{E}^{-1}_{\lambda,\omega} = \mathsf{F}_{\lambda,\omega} - \frac{1}{\mathsf{tr}(\mathsf{F}^{\#}_{\lambda,\omega})}\boldsymbol{\omega}\,\boldsymbol{\theta}^\top,$$

where $\boldsymbol{\theta} = \boldsymbol{\omega} + \mathsf{F}_{\lambda,\omega}\big(\mathsf{F}^{\#}_{\lambda,\omega}\big)_d\boldsymbol{\omega}^{-1}$.

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Proof. The result in Lemma 3.3 can be written in matricial form as

$$\mathsf{E}_{\lambda,\omega} = \mathsf{F}^{\#}_{\lambda,\omega} - \mathsf{D}_{\boldsymbol{\omega}}\mathsf{J}(\mathsf{F}^{\#}_{\lambda,\omega})_d\mathsf{D}_{\boldsymbol{\omega}}^{-1}.$$

On the other hand,

$$\mathsf{F}_{\lambda,\omega}\mathsf{E}_{\lambda,\omega} = \mathsf{F}_{\lambda,\omega}\left(\mathsf{F}^{\#}_{\lambda,\omega} - \mathsf{D}_{\boldsymbol{\omega}}\mathsf{J}(\mathsf{F}^{\#}_{\lambda,\omega})_d\mathsf{D}_{\boldsymbol{\omega}}^{-1}\right) = \mathsf{I} - \boldsymbol{\omega}\boldsymbol{\omega}^\top.$$

Hence, $\mathsf{E}_{\lambda,\omega}$ is a Green matrix for $\mathsf{F}_{\lambda,\omega}$. Moreover,

$$\mathsf{E}_{\lambda,\omega}\boldsymbol{\omega} = -\mathsf{D}_{\boldsymbol{\omega}}\mathsf{J}(\mathsf{F}^{\#}_{\lambda,\omega})_d\mathsf{D}_{\boldsymbol{\omega}}^{-1}\boldsymbol{\omega} = -\mathsf{D}_{\boldsymbol{\omega}}\mathsf{J}(\mathsf{F}^{\#}_{\lambda,\omega})_d 1 = -\mathsf{tr}(\mathsf{F}^{\#}_{\lambda,\omega})\boldsymbol{\omega}.$$

Finally, from [6, Theorem 3.3 (iv)] taking $\sigma = \boldsymbol{\omega}$ and $\boldsymbol{\tau} = (\mathsf{F}^{\#}_{\lambda,\omega})_d\,\boldsymbol{\omega}^{-1}$,, we get that $\mathsf{E}_{\lambda,\omega}$ is invertible since $\mathsf{F}^{\#}_{\lambda,\omega}$ is positive semidefinite and hence

$$\langle(\mathsf{F}^{\#}_{\lambda,\omega})_d\,\boldsymbol{\omega}^{-1},\boldsymbol{\omega}\rangle = \mathsf{tr}(\mathsf{F}^{\#}_{\lambda,\omega}) > 0,$$

which also implies that

$$\mathsf{E}^{-1}_{\lambda,\omega} = \mathsf{F}_{\lambda,\omega} - \frac{1}{\mathsf{tr}(\mathsf{F}^{\#}_{\lambda,\omega})}\boldsymbol{\omega}\boldsymbol{\theta}^\top,$$

where $\boldsymbol{\theta} = \boldsymbol{\omega} + \mathsf{F}_{\lambda,\omega}\big(\mathsf{F}^{\#}_{\lambda,\omega}\big)_d\,\boldsymbol{\omega}^{-1}$. ◻

Next, we provide an example of the computation of the equilibrium matrix for the star network S_n, with $n+1$ vertices, weight ω, and parameter $\lambda \geq 0$. Suppose that the center of the star is labeled as 0, and each edge has been assigned a conductance $c_i = c_{i0}$, $i = 1,\dots,n$. Then, if we consider $c_0 = \dfrac{1}{\omega_0}\sum\limits_{i=1}^{n} c_i\omega_i$, we have $k_0 + q_0 = \dfrac{1}{\omega_0}(\lambda\omega_0 + \sum\limits_{i=1}^{n} c_i\omega_i) = \dfrac{1}{\omega_0}(\lambda\omega_0 + c_0\omega_0)$ and $k_i + q_i = \dfrac{1}{\omega_i}(\lambda\omega_i + c_i\omega_0)$. On the other hand, $(\mathsf{k}_\omega)_i = \lambda\omega_i + c_i\omega_0$, $\mathsf{vol}(S_n) = \lambda + 2c_0\omega_0^2$, and we consider the value $Q(\lambda,\omega) = \sum\limits_{j=1}^{n}\dfrac{\omega_j^3}{\lambda\omega_j + c_j\omega_0}$.

Proposition 3.5. *Let S_n be a star network, then the equilibrium measures with respect to λ and ω, for $i,j = 1,\dots,n,\ \ i \neq j$, are given by*

$$\begin{aligned}
(\nu^0_{\lambda,\omega})_i &= \frac{\omega_i^2}{(\mathsf{k}_\omega)_i},\\
(\nu^j_{\lambda,\omega})_0 &= \frac{\omega_0\big((\mathsf{k}_\omega)_j\big(1-\lambda Q(\lambda,\omega)\big) - c_j\omega_0\omega_j^2\big)}{\lambda(\mathsf{k}_\omega)_j(1-\lambda Q(\lambda,\omega)) + c_j^2\omega_0^2\omega_j},\\
(\nu^j_{\lambda,\omega})_i &= \frac{\omega_i}{(\mathsf{k}_\omega)_i}\left(\frac{(\mathsf{k}_\omega)_j(\mathsf{k}_\omega)_i\big(1-\lambda Q(\lambda,\omega)\big) + c_j\omega_0^2\omega_j(c_j\omega_i - c_i\omega_j)}{\lambda(\mathsf{k}_\omega)_j(1-\lambda Q(\lambda,\omega)) + c_j^2\omega_0^2\omega_j}\right).
\end{aligned}$$

And for any $j = 1,\dots,n$, the capacity is given by

$$\mathsf{cap}^0_{\lambda,\omega} = Q(\lambda,\omega),$$

$$\mathsf{cap}^j_{\lambda,\omega} = Q(\lambda,\omega) + (\nu^j_{\lambda,\omega})_0\left(\frac{1}{\omega_0}\big(1-\lambda Q(\lambda,\omega)\big) - \frac{\omega_j^2 c_j}{(\mathsf{k}_\omega)_j}\right) - \frac{\omega_j^3}{(\mathsf{k}_\omega)_j}.$$

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Proof. From the definition of the equilibrium measure, we get that

$$\omega_i = \big(\mathsf{L}_{\lambda,\omega}\nu^0_{\lambda,\omega}\big)_i = (k_i+q_i)\big(\nu^0_{\lambda,\omega}\big)_i = \frac{(\nu^0_{\lambda,\omega})_i}{\omega_i}(\mathsf{k}_\omega)_i,$$

and hence

$$\big(\nu^0_{\lambda,\omega}\big)_i = \frac{\omega_i^2}{(\mathsf{k}_\omega)_i}, \text{ for } i=1,\dots,n.$$

Moreover, $\mathsf{cap}^0_{\lambda,\omega} = \sum_{i=1}^n \frac{\omega_i^3}{(\mathsf{k}_\omega)_i} = Q(\lambda,\omega)$.

Consider now $j\neq 0$, then $\omega_0 = \big(\mathsf{L}_{\lambda,\omega}\nu^j_{\lambda,\omega}\big)_0 = (k_0+q_0)(\nu^j_{\lambda,\omega})_0 - \sum_{\ell=1}^n c_\ell(\nu^j_{\lambda,\omega})_\ell$ and hence

$$\big(\nu^j_{\lambda,\omega}\big)_0(\mathsf{k}_\omega)_0 = \omega_0^2 + \omega_0\sum_{\ell=1}^n c_\ell(\nu^j_{\lambda,\omega})_\ell. \tag{3.8}$$

Suppose now $i\neq j$,

$$\omega_i = \big(\mathsf{L}_{\lambda,\omega}\nu^j_{\lambda,\omega}\big)_i = -c_i(\nu^j_{\lambda,\omega})_0 + (k_i+q_i)(\nu^j_{\lambda,\omega})_i = -c_i(\nu^j_{\lambda,\omega})_0 + \frac{1}{\omega_i}(\mathsf{k}_\omega)_i(\nu^j_{\lambda,\omega})_i,$$

and hence

$$(\nu^j_{\lambda,\omega})_i(\mathsf{k}_\omega)_i = \omega_i^2 + \omega_i c_i(\nu^j_{\lambda,\omega})_0. \tag{3.9}$$

Therefore, by substituting (3.9) in equation (3.8), we get that

$$\begin{aligned}
\big(\nu^j_{\lambda,\omega}\big)_0(\mathsf{k}_\omega)_0 &= \omega_0^2 + \omega_0\sum_{\ell=1}^n \frac{c_\ell\big(\omega_\ell^2+\omega_\ell c_\ell\big(\nu^j_{\lambda,\omega}\big)_0\big)}{(\mathsf{k}_\omega)_\ell} - \omega_0\frac{c_j\big(\omega_j^2+\omega_j c_j\big(\nu^j_{\lambda,\omega}\big)_0\big)}{(\mathsf{k}_\omega)_j}\\
&= \omega_0^2 + \omega_0\sum_{\ell=1}^n c_\ell\frac{\omega_\ell^2}{(\mathsf{k}_\omega)_\ell} + \omega_0\big(\nu^j_{\lambda,\omega}\big)_0\sum_{\ell=1}^n\frac{\omega_\ell c_\ell^2}{(\mathsf{k}_\omega)_\ell} - \omega_0\frac{c_j\big(\omega_j^2+\omega_j c_j\big(\nu^j_{\lambda,\omega}\big)_0\big)}{(\mathsf{k}_\omega)_j}\\
&= 1-\lambda Q(\lambda,\omega) - \frac{c_j\omega_0\omega_j^2}{(\mathsf{k}_\omega)_j} + \left((\mathsf{k}_\omega)_0 - \frac{\lambda}{\omega_0}(1-\lambda Q(\lambda,\omega)) - \frac{c_j^2\omega_0\omega_j}{(\mathsf{k}_\omega)_j}\right)(\nu^j_{\lambda,\omega})_0,
\end{aligned}$$

and hence

$$(\nu^j_{\lambda,\omega})_0 = \frac{\left(1-\lambda Q(\lambda,\omega) - \dfrac{c_j\omega_0\omega_j^2}{(\mathsf{k}_\omega)_j}\right)}{\dfrac{\lambda}{\omega_0}(1-\lambda Q(\lambda,\omega)) + \dfrac{c_j^2\omega_0\omega_j}{(\mathsf{k}_\omega)_j}} = \frac{\omega_0\big((\mathsf{k}_\omega)_j\big(1-\lambda Q(\lambda,\omega)\big) - c_j\omega_0\omega_j^2\big)}{\lambda(\mathsf{k}_\omega)_j(1-\lambda Q(\lambda,\omega)) + c_j^2\omega_0^2\omega_j},$$

where we have used that

$$\omega_0\sum_{j=1}^n\frac{\omega_j c_j^2}{(\mathsf{k}_\omega)_j} = (\mathsf{k}_\omega)_0 - \frac{\lambda}{\omega_0}(1-\lambda Q(\lambda,\omega)).$$

Substituting the obtained value into Equation (3.9), we obtain the announced value for $(\nu^j_{\lambda,\omega})_i$. Moreover,

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for any $j=1,\dots,n$, we get that

$$\begin{aligned}
\mathsf{cap}^j_{\lambda,\omega} &= \omega_0(\nu^j_{\lambda,\omega})_0+\sum_{i=1}^n\frac{\omega_i^3}{(\mathsf{k}_\omega)_i}+(\nu^j_{\lambda,\omega})_0\sum_{i=1}^n\frac{\omega_i^2c_i}{(\mathsf{k}_\omega)_i}-\frac{\omega_j^3}{(\mathsf{k}_\omega)_j}-\frac{\omega_j^2c_j}{(\mathsf{k}_\omega)_j}(\nu^j_{\lambda,\omega})_0\\
&=(\nu^j_{\lambda,\omega})_0\left(\omega_0-\frac{\omega_j^2c_j}{(\mathsf{k}_\omega)_j}+\frac{1}{\omega_0}\big(1-\lambda Q(\lambda,\omega)\big)-\omega_0\right)+Q(\lambda,\omega)-\frac{\omega_j^3}{(\mathsf{k}_\omega)_j}\\
&=(\nu^j_{\lambda,\omega})_0\left(-\frac{\omega_j^2c_j}{(\mathsf{k}_\omega)_j}+\frac{1}{\omega_0}\big(1-\lambda Q(\lambda,\omega)\big)\right)+Q(\lambda,\omega)-\frac{\omega_j^3}{(\mathsf{k}_\omega)_j},
\end{aligned}$$

where we have used that

$$\frac{1}{\omega_0}\big(1-\lambda Q(\lambda,\omega)\big)=\omega_0+\sum_{i=1}^n\frac{\omega_i^2c_i}{(\mathsf{k}_\omega)_i}.$$

∎

Theorem 3.6. *The equilibrium matrix with respect to λ and ω for the star network is given by*

$$\begin{aligned}
\big(\mathsf{E}_{\lambda,\omega}\big)_{i0}&=-\frac{\omega_0\omega_i^2}{(\mathsf{k}_\omega)_i\big(1-\lambda Q(\lambda,\omega)\big)},\quad \textit{for}\ \ i=1,\dots,n,\\
\big(\mathsf{E}_{\lambda,\omega}\big)_{0j}&=\frac{c_j\omega_0^2\omega_j^2}{(1-\lambda Q(\lambda,\omega))(\mathsf{k}_\omega)_j^2}-\frac{\omega_0}{(\mathsf{k}_\omega)_j},\quad \textit{for}\ \ j=1,\dots,n,\\
\big(\mathsf{E}_{\lambda,\omega}\big)_{ij}&=\frac{\omega_0^2\omega_i\omega_jc_j(c_i\omega_j-c_j\omega_i)}{(\mathsf{k}_\omega)_i(\mathsf{k}_\omega)_j^2(1-\lambda Q(\lambda,\omega))}-\frac{\omega_i}{(\mathsf{k}_\omega)_j},\quad \textit{for}\ \ i,j=1,\dots,n,\ \ i\neq j.
\end{aligned}$$

Proof. According to the results of the above proposition, we get that the entries of the first column of the equilibrium matrix are

$$\big(\mathsf{E}_{\lambda,\omega}\big)_{i0}=-\frac{\omega_0}{1-\lambda\mathsf{cap}^0_{\lambda,\omega}}\big(\nu^0_{\lambda,\omega}\big)_i=-\frac{\omega_0\omega_i^2}{(1-\lambda Q(\lambda,\omega))(\mathsf{k}_\omega)_i}.$$

In order to compute the rest of the columns, we need to calculate first the value

$$\begin{aligned}
1-\lambda\mathsf{cap}^j_{\lambda,\omega}&=\lambda\frac{\big(\nu^j_{\lambda,\omega}\big)_0}{\omega_0(\mathsf{k}_\omega)_j}\left(\omega_0\omega_j^2c_j-(\mathsf{k}_\omega)_j\big(1-\lambda Q(\lambda,\omega)\big)\right)+1-\lambda Q(\lambda,\omega)+\lambda\frac{\omega_j^3}{(\mathsf{k}_\omega)_j}\\
&=\big(1-\lambda Q(\lambda,\omega)\big)\left(1-\lambda\frac{\big(\nu^j_{\lambda,\omega}\big)_0}{\omega_0}\right)+\lambda\frac{\omega_j^2}{(\mathsf{k}_\omega)_j}\left(\omega_j+\big(\nu^j_{\lambda,\omega}\big)_0c_j\right)\\
&=\big(1-\lambda Q(\lambda,\omega)\big)\frac{\omega_j(\mathsf{k}^2_\omega)_j}{\lambda(\mathsf{k}_\omega)_j(1-\lambda Q(\lambda,\omega))+c_j^2\omega_0^2\omega_j},
\end{aligned}$$

where we have kept in mind that

$$\begin{aligned}
\omega_0-\lambda\big(\nu^j_{\lambda,\omega}\big)_0&=\frac{\lambda\omega_0(\mathsf{k}_\omega)_j(1-\lambda Q(\lambda,\omega))+c_j^2\omega_0^3\omega_j-\lambda\omega_0(\mathsf{k}_\omega)_j\big(1-\lambda Q(\lambda,\omega)\big)+\lambda c_j\omega_0^2\omega_j^2}{\lambda(\mathsf{k}_\omega)_j(1-\lambda Q(\lambda,\omega))+c_j^2\omega_0^2\omega_j}\\
&=\frac{c_j\omega_0^2\omega_j(\mathsf{k}_\omega)_j\lambda(\mathsf{k}_\omega)_j(1-\lambda Q(\lambda,\omega))+c_j^2\omega_0^2\omega_j}{\lambda(\mathsf{k}_\omega)_j(1-\lambda Q(\lambda,\omega))+c_j^2\omega_0^2\omega_j},
\end{aligned}$$

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and

$$\omega_j + \left(\nu^j_{\lambda,\omega}\right)_0 c_j = \frac{\lambda\omega_j(\mathsf{k}_\omega)_j(1-\lambda Q(\lambda,\omega)) + c_j^2\omega_0^2\omega_j^2 + \omega_0(\mathsf{k}_\omega)_j c_j\big(1-\lambda Q(\lambda,\omega)\big) - c_j^2\omega_0^2\omega_j^2}{\lambda(\mathsf{k}_\omega)_j(1-\lambda Q(\lambda,\omega)) + c_j^2\omega_0^2\omega_j}$$
$$= \frac{(\mathsf{k}_\omega)_j^2(1-\lambda Q(\lambda,\omega))}{\lambda(\mathsf{k}_\omega)_j(1-\lambda Q(\lambda,\omega)) + c_j^2\omega_0^2\omega_j}.$$

Then, for any $i = 1,\dots,n$ we get

$$\left(\mathsf{E}_{\lambda,\omega}\right)_{ij} = -\frac{\omega_i}{(\mathsf{k}_\omega)_i}\left(\frac{(\mathsf{k}_\omega)_j(\mathsf{k}_\omega)_i\big(1-\lambda Q(\lambda,\omega)\big) + c_j\omega_0^2\omega_j(c_j\omega_i - c_i\omega_j)}{\mathsf{k}^2_{\omega_j}(1-\lambda Q(\lambda,\omega))}\right)$$
$$= -\frac{\omega_i}{(\mathsf{k}_\omega)_j} - \frac{\omega_0^2 c_j\omega_i\omega_j(c_j\omega_i - c_i\omega_j)}{(\mathsf{k}_\omega)_i(\mathsf{k}_\omega^2)_j(1-\lambda Q(\lambda,\omega))}.$$

Analogously, we get the result for $\left(\mathsf{E}_{\lambda,\omega}\right)_{0j}$. □

4. MFPT and Kemeny's constant in terms of the equilibrium matrix. In this section, we use the equilibrium matrix with respect to λ and ω to obtain an expression for the mean first passage time and Kemeny's constant that can be more easily computed in some cases. Generalized inverses have long been used to derive expressions for mean first passage times and related quantities in finite Markov chains; see, e.g., Hunter [17] and Campbell and Meyer [11]. In our setting, the reversible structure associated with symmetric M-matrices allows us to rewrite these formulas in terms of equilibrium measures.

Using the generalized Schrödinger matrix with respect to λ and ω, in [13], we obtained the following expression for equation (2.5)

$$\mathsf{F}_{\lambda,\omega}\mathsf{D}_{\boldsymbol{\omega}}\mathsf{M}_{\lambda,\omega} = \mathsf{D}_{\mathsf{k}_\omega}\left(\mathsf{J} - \mathsf{P}_{\lambda,\omega}(\mathsf{M}_{\lambda,\omega})_d\right). \tag{4.10}$$

Subsequently, we can use this last identity to obtain the following expression for the MFPT, see [13, Proposition 2] for the proof.

Proposition 4.1. *Let Γ be a network, then the mean first passage time matrix $\mathsf{M}_{\lambda,\omega}$ can be written as*

$$\mathsf{M}_{\lambda,\omega} = \mathsf{D}_{\boldsymbol{\omega}}^{-1}\mathsf{G}\mathsf{D}_{\mathsf{k}_\omega}\mathsf{J} - \mathsf{J}\mathsf{D}_{\boldsymbol{\omega}}^{-1}(\mathsf{G}\mathsf{D}_{\mathsf{k}_\omega}\mathsf{J})_d + \mathsf{vol}(\Gamma)\Big(\mathsf{D}_{\boldsymbol{\omega}}^{-1}\mathsf{D}_{\mathsf{k}_\omega}^{-1} - \mathsf{D}_{\boldsymbol{\omega}}^{-1}\mathsf{G}\mathsf{D}_{\boldsymbol{\omega}}^{-1} + \mathsf{J}\mathsf{D}_{\boldsymbol{\omega}}^{-1}\mathsf{G}_d\mathsf{D}_{\boldsymbol{\omega}}^{-1}\Big).$$

In addition, for 1-inverses such that $\mathsf{G}\mathsf{k}_\omega = g\boldsymbol{\omega}$, being g a constant, we obtain

$$\mathsf{M}_{\lambda,\omega} = \mathsf{vol}(\Gamma)\Big(\mathsf{D}_{\boldsymbol{\omega}}^{-1}\mathsf{D}_{\mathsf{k}_\omega}^{-1} - \mathsf{D}_{\boldsymbol{\omega}}^{-1}\mathsf{G}\mathsf{D}_{\boldsymbol{\omega}}^{-1} + \mathsf{J}\mathsf{D}_{\boldsymbol{\omega}}^{-1}\mathsf{G}_d\mathsf{D}_{\boldsymbol{\omega}}^{-1}\Big).$$

Note that if we conveniently define matrix $\mathsf{B} = \mathsf{D}_{\boldsymbol{\omega}}^{-1}\mathsf{G}\mathsf{D}_{\mathsf{k}_\omega}$, the equality $\big(\mathsf{BJ}\big)_{ij}^\top = \Big(\mathsf{J}\big(\mathsf{BJ}\big)_d\Big)_{ij}$ holds true, and then $\mathsf{J}\mathsf{D}_{\boldsymbol{\omega}}^{-1}(\mathsf{G}\mathsf{D}_{\mathsf{k}_\omega}\mathsf{J})_d = \big(\mathsf{D}_{\boldsymbol{\omega}}^{-1}\mathsf{G}\mathsf{D}_{\mathsf{k}_\omega}\mathsf{J}\big)^\top$. So Proposition 4.1 can be written as

$$\mathsf{M}_{\lambda,\omega} = \mathsf{D}_{\boldsymbol{\omega}}^{-1}\mathsf{G}\mathsf{D}_{\mathsf{k}_\omega}\mathsf{J} - \big(\mathsf{D}_{\boldsymbol{\omega}}^{-1}\mathsf{G}\mathsf{D}_{\mathsf{k}_\omega}\mathsf{J}\big)^\top + \mathsf{vol}(\Gamma)\Big(\mathsf{D}_{\boldsymbol{\omega}}^{-1}\mathsf{D}_{\mathsf{k}_\omega}^{-1} - \mathsf{D}_{\boldsymbol{\omega}}^{-1}\mathsf{G}\mathsf{D}_{\boldsymbol{\omega}}^{-1} + \mathsf{J}\mathsf{D}_{\boldsymbol{\omega}}^{-1}\mathsf{G}_d\mathsf{D}_{\boldsymbol{\omega}}^{-1}\Big).$$

Theorem 4.2. *Let Γ be a network. Then, the mean first passage time matrix $\mathsf{M}_{\lambda,\omega}$ can be written in terms of the equilibrium matrix as*

$$\left(\mathsf{M}_{\lambda,\omega}\right)_{ij} = \sum_{\ell=1}^{n}\left(\frac{\left(\mathsf{E}_{\lambda,\omega}\right)_{i\ell}}{\omega_i}(\mathsf{k}_\omega)_\ell - \frac{\left(\mathsf{E}_{\lambda,\omega}\right)_{j\ell}}{\omega_j}(\mathsf{k}_\omega)_\ell - \omega_\ell(\mathsf{k}_\omega)_\ell\frac{\left(\mathsf{E}_{\lambda,\omega}\right)_{ij}}{\omega_i\omega_j}\right) + \frac{\mathsf{vol}(\Gamma)}{\omega_i(\mathsf{k}_\omega)_i}\delta_{ij},$$

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where δ_{ij} denotes the Kronecker delta. Moreover, Kemeny's constant with respect to λ and ω can be expressed as

$$K_{\lambda,\omega}(\Gamma) = 1 - \frac{1}{\mathsf{vol}(\Gamma)}\mathsf{k}_\omega^\top \mathsf{E}_{\lambda,\omega}\mathsf{k}_\omega.$$

Proof. Taking into account Proposition 4.1 and the fact that $\mathsf{E}_{\lambda,\omega}$ is a Green matrix for $\mathsf{F}_{\lambda,\omega}$, with null diagonal entries, we obtain

$$\mathsf{M}_{\lambda,\omega}= \mathsf{D}_{\boldsymbol{\omega}}^{-1}\mathsf{E}_{\lambda,\omega}\mathsf{D}_{\mathsf{k}_\omega}\mathsf{J} - \left(\mathsf{D}_{\boldsymbol{\omega}}^{-1}\mathsf{E}_{\lambda,\omega}\mathsf{D}_{\mathsf{k}_\omega}\mathsf{J}\right)^\top + \mathsf{vol}(\Gamma)\Big(\mathsf{D}_{\boldsymbol{\omega}}^{-1}\mathsf{D}_{\mathsf{k}_\omega}^{-1} - \mathsf{D}_{\boldsymbol{\omega}}^{-1}\mathsf{E}_{\lambda,\omega}\mathsf{D}_{\boldsymbol{\omega}}^{-1}\Big).$$

Keeping in mind that the mean recurrence time for state i is $\left(\mathsf{M}_{\lambda,\omega}\right)_{ii} = \dfrac{1}{\left(\pi_{\lambda,\omega}\right)_i} = \dfrac{\mathsf{vol}(\Gamma)}{\omega_i(\mathsf{k}_\omega)_i}$, it is enough to calculate $\left(\mathsf{M}_{\lambda,\omega}\right)_{ij}$, for $i \neq j$, and then

$$\left(\mathsf{M}_{\lambda,\omega}\right)_{ij} = \left(\mathsf{D}_{\boldsymbol{\omega}}^{-1}\mathsf{E}_{\lambda,\omega}\mathsf{D}_{\mathsf{k}_\omega}\mathsf{J}\right)_{ij} - \left(\mathsf{D}_{\boldsymbol{\omega}}^{-1}\mathsf{E}_{\lambda,\omega}\mathsf{D}_{\mathsf{k}_\omega}\mathsf{J}\right)_{ji} - \mathsf{vol}(\Gamma)\Big(\mathsf{D}_{\boldsymbol{\omega}}^{-1}\mathsf{E}_{\lambda,\omega}\mathsf{D}_{\boldsymbol{\omega}}^{-1}\Big)_{ij},$$

where $\left(\mathsf{D}_{\boldsymbol{\omega}}^{-1}\mathsf{E}_{\lambda,\omega}\mathsf{D}_{\mathsf{k}_\omega}\mathsf{J}\right)_{ij} = \displaystyle\sum_{\ell=1}^n \frac{\left(\mathsf{E}_{\lambda,\omega}\right)_{i\ell}}{\omega_i}(\mathsf{k}_\omega)_\ell$ and $\left(\mathsf{D}_{\boldsymbol{\omega}}^{-1}\mathsf{E}_{\lambda,\omega}\mathsf{D}_{\boldsymbol{\omega}}^{-1}\right)_{ij} = \dfrac{\left(\mathsf{E}_{\lambda,\omega}\right)_{ij}}{\omega_i\omega_j}$. Hence, the result for matrix $\mathsf{M}_{\lambda,\omega}$ holds because $\mathsf{vol}(\Gamma) = \sum_{\ell=1}^n \omega_\ell(\mathsf{k}_\omega)_\ell$. To compute Kemeny's constant, we study the vector $\mathsf{M}_{\lambda,\omega}\boldsymbol{\pi}_{\lambda,\omega}$, since we know that $\mathsf{M}_{\lambda,\omega}\boldsymbol{\pi}_{\lambda,\omega} = K_{\lambda,\omega}(\Gamma)\mathsf{1}$, where $\boldsymbol{\pi}_{\lambda,\omega} = \dfrac{1}{\mathsf{vol}(\Gamma)}\mathsf{k}_\omega\boldsymbol{\omega}$. So, we have that

$$\begin{aligned}
\mathsf{M}_{\lambda,\omega}\mathsf{k}_\omega\boldsymbol{\omega} &= \mathsf{D}_{\boldsymbol{\omega}}^{-1}\mathsf{E}_{\lambda,\omega}\mathsf{D}_{\mathsf{k}_\omega}\mathsf{J}\mathsf{k}_\omega\omega - \mathsf{J}\mathsf{D}_{\boldsymbol{\omega}}^{-1}(\mathsf{E}_{\lambda,\omega}\mathsf{D}_{\mathsf{k}_\omega}\mathsf{J})_d\mathsf{k}_\omega\omega \\
&\quad + \mathsf{vol}(\Gamma)\Big(\mathsf{D}_{\boldsymbol{\omega}}^{-1}\mathsf{D}_{\mathsf{k}_\omega}^{-1}\mathsf{k}_\omega\omega - \mathsf{D}_{\boldsymbol{\omega}}^{-1}\mathsf{E}_{\lambda,\omega}\mathsf{D}_{\boldsymbol{\omega}}^{-1}\mathsf{k}_\omega\omega\Big) \\
&= \mathsf{vol}(\Gamma)\mathsf{D}_{\boldsymbol{\omega}}^{-1}\mathsf{E}_{\lambda,\omega}\mathsf{k}_\omega - \mathsf{J}(\mathsf{E}_{\lambda,\omega}\mathsf{D}_{\mathsf{k}_\omega}\mathsf{J})_d\mathsf{k}_\omega + \mathsf{vol}(\Gamma)\Big(\mathsf{1} - \mathsf{D}_{\boldsymbol{\omega}}^{-1}\mathsf{E}_{\lambda,\omega}\mathsf{k}_\omega\Big) \\
&= -\mathsf{J}(\mathsf{E}_{\lambda,\omega}\mathsf{D}_{\mathsf{k}_\omega}\mathsf{J})_d\mathsf{k}_\omega + \mathsf{vol}(\Gamma)\mathsf{1} = \left(\mathsf{vol}(\Gamma) - \mathsf{k}_\omega^\top\mathsf{E}_{\lambda,\omega}\mathsf{k}_\omega\right)\mathsf{1}.
\end{aligned}$$

Therefore,

$$K_{\lambda,\omega}(\Gamma) = 1 - \frac{1}{\mathsf{vol}(\Gamma)}\mathsf{k}_\omega^\top \mathsf{E}_{\lambda,\omega}\mathsf{k}_\omega.$$

∎

Observe that from the relation obtained in Lemma 3.3 between the equilibrium matrix and the group inverse, the above formula coincides with the one obtained in [13, Proposition 4] and with the formula obtained in [24, Theorem 4] for the standard case, that is, when $\lambda = 0$ and ω is constant.

Corollary 4.3. *Let Γ be a network, then the mean first passage time matrix $\mathsf{M}_{\lambda,\omega}$, and Kemeny's constant $K_{\lambda,\omega}(\Gamma)$ can be written in terms of equilibrium measures as*

$$\left(\mathsf{M}_{\lambda,\omega}\right)_{ij} = \sum_{\ell=1}^n \omega_\ell(\mathsf{k}_\omega)_\ell\left(\frac{(\nu^j_{\lambda,\omega})_i}{\omega_i(1-\lambda(\mathsf{cap}_{\lambda,\omega})_j)} + \frac{(\nu^\ell_{\lambda,\omega})_j}{\omega_j(1-\lambda(\mathsf{cap}_{\lambda,\omega})_\ell)} - \frac{(\nu^\ell_{\lambda,\omega})_i}{\omega_i(1-\lambda(\mathsf{cap}_{\lambda,\omega})_\ell)}\right),$$

for any $i \neq j$. Moreover,

$$K_{\lambda,\omega}(\Gamma) = 1 + \frac{1}{\mathsf{vol}(\Gamma)}\sum_{i,j=1}^n \frac{\omega_j(\mathsf{k}_\omega)_i(\mathsf{k}_\omega)_j}{1-\lambda(\mathsf{cap}_{\lambda,\omega})_j}\left(\nu^j_{\lambda,\omega}\right)_i.$$

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For the classical case, we have the following result, which was obtained by some of the authors in [3] using a different approach.

COROLLARY 4.4. *Let Γ be a network, then the mean first passage time matrix between i and j, for $i \neq j$, is given by*

$$\mathsf{M}_{ij} = \frac{1}{n}\sum_{\ell=1}^{n} k_\ell\Big(\nu_i^j + \nu_j^\ell - \nu_i^\ell\Big).$$

Moreover,

$$K(\Gamma) = 1 + \frac{1}{2m}\sum_{i,j=1}^{n} k_i k_j \nu_i^j.$$

5. MFPT and Kemeny's constant for the star network. In this section, we specify the results obtained previously for the star network S_n, see Fig. 2. We use the same notation as in Section 3.

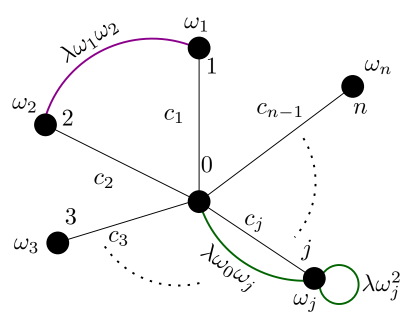

FIGURE 2. *Star network S_n. The node j is representative of each and every node in the network.*

PROPOSITION 5.1. *The mean first passage time with respect to λ and ω for a star network S_n is given for $i, j = 1, \dots, n$ by*

$$\big(\mathsf{M}_{\lambda,\omega}\big)_{ij} = \frac{\lambda\omega_0^2(c_i\omega_j - c_j\omega_i)}{\big(1-\lambda Q(\lambda,\omega)\big)(\mathsf{k}_\omega)_i \mathsf{k}_{\omega_j}^2}\big(\omega_0\omega_j(\lambda+2c_0) + c_j(\omega_0^2-1)\big) + \frac{(\lambda+2c_0\omega_0^2)}{\omega_j(\mathsf{k}_\omega)_j},$$

$$\big(\mathsf{M}_{\lambda,\omega}\big)_{0j} = \frac{\lambda\omega_0\omega_j}{(1-\lambda Q(\lambda,\omega))\mathsf{k}_{\omega_j}^2}(\omega_0\omega_j(\lambda+2c_0) + c_j(\omega_0^2-1)) + \frac{(\lambda+2c_0\omega_0^2)}{\omega_j(\mathsf{k}_\omega)_j} - 1,$$

$$\big(\mathsf{M}_{\lambda,\omega}\big)_{i0} = \frac{\lambda\omega_i(1-\omega_0^2)}{(\mathsf{k}_\omega)_i(1-\lambda Q(\lambda,\omega))} + 1, \quad \big(\mathsf{M}_{\lambda,\omega}\big)_{00} = \frac{(\lambda+2c_0\omega_0^2)}{\omega_0(\mathsf{k}_\omega)_0}.$$

Proof. The result for $\big(\mathsf{M}_{\lambda,\omega}\big)_{jj}, j = 0,\dots,n$ follows directly from Theorem 4.2 taking into account that $\big(\mathsf{E}_{\lambda,\omega}\big)_{ii} = 0$, $i = 0,\dots,n$. Moreover, for $i \neq j$ and $i \neq 0$, $j \neq 0$, from the expression for the equilibrium matrix for the star network S_n given in Proposition 3.6, we get

$$\begin{aligned}\big(\mathsf{M}_{\lambda,\omega}\big)_{ij} &= -\frac{\omega_0\omega_i(\mathsf{k}_\omega)_0}{(\mathsf{k}_\omega)_i\big(1-\lambda Q(\lambda,\omega)\big)} - \frac{\omega_0\omega_j(\mathsf{k}_\omega)_0}{(\mathsf{k}_\omega)_j\big(1-\lambda Q(\lambda,\omega)\big)}\\ &\quad + \sum_{\ell=1}^{n}\left[\frac{\omega_0^2\omega_\ell c_\ell(c_i\omega_\ell - c_\ell\omega_i)}{(\mathsf{k}_\omega)_i(\mathsf{k}_\omega)_\ell(1-\lambda Q(\lambda,\omega))} - \frac{\omega_0^2\omega_\ell c_\ell(c_j\omega_\ell - c_\ell\omega_j)}{(\mathsf{k}_\omega)_j(\mathsf{k}_\omega)_\ell(1-\lambda Q(\lambda,\omega))}\right]\\ &\quad - \left[\frac{\omega_0^2 c_j(c_i\omega_j - c_j\omega_i)}{(\mathsf{k}_\omega)_i(\mathsf{k}_\omega)_j^2(1-\lambda Q(\lambda,\omega))} - \frac{1}{\omega_j(\mathsf{k}_\omega)_j}\right]\mathrm{vol}(S_n),\end{aligned}$$

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and therefore

$$
\begin{aligned}
\big(\mathsf{M}_{\lambda,\omega}\big)_{ij} &= -\frac{\omega_0(\mathsf{k}_\omega)_0}{(\mathsf{k}_\omega)_i(\mathsf{k}_\omega)_j(1-\lambda Q(\lambda,\omega))}((\mathsf{k}_\omega)_j\omega_i-(\mathsf{k}_\omega)_i\omega_j)\\
&+ \left[\frac{\omega_0^2 c_i}{(\mathsf{k}_\omega)_i(1-\lambda Q(\lambda,\omega))}-\frac{\omega_0^2 c_j}{(\mathsf{k}_\omega)_j(1-\lambda Q(\lambda,\omega))}\right]\sum_{\ell=1}^n\frac{\omega_\ell^2 c_\ell}{(\mathsf{k}_\omega)_\ell}\\
&- \left[\frac{\omega_0^2 \omega_i}{(\mathsf{k}_\omega)_i(1-\lambda Q(\lambda,\omega))}-\frac{\omega_0^2 \omega_j}{(\mathsf{k}_\omega)_j(1-\lambda Q(\lambda,\omega))}\right]\sum_{\ell=1}^n\frac{\omega_\ell c_\ell^2}{(\mathsf{k}_\omega)_\ell}\\
&- \left[\frac{\omega_0^2 c_j(c_i\omega_j-c_j\omega_i)}{(\mathsf{k}_\omega)_i(\mathsf{k}_\omega)_j^2(1-\lambda Q(\lambda,\omega))}-\frac{1}{\omega_j(\mathsf{k}_\omega)_j}\right]\mathrm{vol}(S_n)\\
&= -\frac{\omega_0(\mathsf{k}_\omega)_0((\mathsf{k}_\omega)_j\omega_i-(\mathsf{k}_\omega)_i\omega_j)}{(\mathsf{k}_\omega)_i(\mathsf{k}_\omega)_j(1-\lambda Q(\lambda,\omega))}+\frac{(\mathsf{k}_\omega)_j\omega_0 c_i-(\mathsf{k}_\omega)_i\omega_0 c_j}{(\mathsf{k}_\omega)_i(\mathsf{k}_\omega)_j(1-\lambda Q(\lambda,\omega))}\left[1-\lambda Q(\lambda,\omega)-\omega_0^2\right]\\
&- \frac{(\mathsf{k}_\omega)_j\omega_i-(\mathsf{k}_\omega)_i\omega_j}{(\mathsf{k}_\omega)_i(\mathsf{k}_\omega)_j(1-\lambda Q(\lambda,\omega))}\Big[\omega_0(\mathsf{k}_\omega)_0-\lambda\left(1-\lambda Q(\lambda,\omega)\right)\Big]\\
&- \left[\frac{\omega_0^2 c_j(c_i\omega_j-c_j\omega_i)}{(\mathsf{k}_\omega)_i(\mathsf{k}_\omega)_j^2(1-\lambda Q(\lambda,\omega))}-\frac{1}{\omega_j(\mathsf{k}_\omega)_j}\right]\mathrm{vol}(S_n)\\
&= -\frac{\omega_0(\mathsf{k}_\omega)_0((\mathsf{k}_\omega)_j\omega_i-(\mathsf{k}_\omega)_i\omega_j)}{(\mathsf{k}_\omega)_i(\mathsf{k}_\omega)_j(1-\lambda Q(\lambda,\omega))}-\frac{\omega_0^2 c_0\left((\mathsf{k}_\omega)_j\omega_i-(\mathsf{k}_\omega)_i\omega_j\right)}{(\mathsf{k}_\omega)_i(\mathsf{k}_\omega)_j(1-\lambda Q(\lambda,\omega))}\\
&- \left[\frac{\omega_0^2 c_j(c_i\omega_j-c_j\omega_i)}{(\mathsf{k}_\omega)_i(\mathsf{k}_\omega)_j^2(1-\lambda Q(\lambda,\omega))}-\frac{1}{\omega_j(\mathsf{k}_\omega)_j}\right]\mathrm{vol}(S_n)\\
&= \frac{\omega_0^3(c_i\omega_j-c_j\omega_i)(\lambda+2c_0)}{(\mathsf{k}_\omega)_i(\mathsf{k}_\omega)_j(1-\lambda Q(\lambda,\omega))}-\left[\frac{\omega_0^2 c_j(c_i\omega_j-c_j\omega_i)}{(\mathsf{k}_\omega)_i(\mathsf{k}_\omega)_j^2(1-\lambda Q(\lambda,\omega))}-\frac{1}{\omega_j(\mathsf{k}_\omega)_j}\right]\mathrm{vol}(S_n)\\
&= \frac{\lambda\omega_0^2(c_i\omega_j-c_j\omega_i)}{\big(1-\lambda Q(\lambda,\omega)\big)(\mathsf{k}_\omega)_i\mathsf{k}_{\omega_j}^2}\big(\omega_0\omega_j(\lambda+2c_0)+c_j(\omega_0^2-1)\big)+\frac{(\lambda+2c_0\omega_0^2)}{\omega_j(\mathsf{k}_\omega)_j},
\end{aligned}
$$

because $(\mathsf{k}_\omega)_0=\lambda\omega_0+c_0\omega_0$ and $(\mathsf{k}_\omega)_j\omega_0 c_i-(\mathsf{k}_\omega)_i\omega_0 c_j+\lambda(\mathsf{k}_\omega)_j\omega_i-\lambda(\mathsf{k}_\omega)_i\omega_j=0$.

Now, consider $i=0$ and $j\neq 0$. Then,

$$
\begin{aligned}
\big(\mathsf{M}_{\lambda,\omega}\big)_{0j} =& \quad \frac{\omega_0\omega_j(\mathsf{k}_\omega)_0}{(\mathsf{k}_\omega)_j\big(1-\lambda Q(\lambda,\omega)\big)}+\sum_{\ell=1}^n\frac{\omega_0\omega_\ell^2 c_\ell}{(\mathsf{k}_\omega)_\ell\big(1-\lambda Q(\lambda,\omega)\big)}\\
&- \sum_{\ell=1}^n\frac{\omega_0^2\omega_\ell c_\ell(c_j\omega_\ell-c_\ell\omega_j)}{(\mathsf{k}_\omega)_j(\mathsf{k}_\omega)_\ell(1-\lambda Q(\lambda,\omega))}-1\\
&- \left[\frac{c_j\omega_0\omega_j}{(\mathsf{k}_\omega)_j^2(1-\lambda Q(\lambda,\omega))}-\frac{1}{\omega_j(\mathsf{k}_\omega)_j}\right]\mathrm{vol}(S_n)\\
&= \frac{\omega_0\omega_j(\mathsf{k}_\omega)_0}{(\mathsf{k}_\omega)_j\big(1-\lambda Q(\lambda,\omega)\big)}+\frac{\omega_0^2\omega_j}{(\mathsf{k}_\omega)_j(1-\lambda Q(\lambda,\omega))}\sum_{\ell=1}^n\frac{\omega_\ell c_\ell^2}{(\mathsf{k}_\omega)_\ell}\\
&+ \left[\frac{\omega_0}{1-\lambda Q(\lambda,\omega)}-\frac{\omega_0^2 c_j}{(\mathsf{k}_\omega)_j(1-\lambda Q(\lambda,\omega))}\right]\sum_{\ell=1}^n\frac{\omega_\ell^2 c_\ell}{(\mathsf{k}_\omega)_\ell}\\
&- \left[\frac{c_j\omega_0\omega_j}{(\mathsf{k}_\omega)_j^2(1-\lambda Q(\lambda,\omega))}-\frac{1}{\omega_j(\mathsf{k}_\omega)_j}\right]\mathrm{vol}(S_n)-1,
\end{aligned}
$$

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that is,

$$
\begin{aligned}
\big(\mathsf{M}_{\lambda,\omega}\big)_{0j} = \;& \frac{\omega_0\omega_j(\mathsf{k}_\omega)_0}{(\mathsf{k}_\omega)_j\big(1-\lambda Q(\lambda,\omega)\big)} + \frac{\omega_j}{(\mathsf{k}_\omega)_j(1-\lambda Q(\lambda,\omega))}\left[\omega_0(\mathsf{k}_\omega)_0 - \lambda\left(1-\lambda Q(\lambda,\omega)\right)\right]\\
+\;& \left[\frac{1}{1-\lambda Q(\lambda,\omega)} - \frac{\omega_0 c_j}{(\mathsf{k}_\omega)_j(1-\lambda Q(\lambda,\omega))}\right]\left[1-\lambda Q(\lambda,\omega)-\omega_0^2\right]\\
-\;& \left[\frac{c_j\omega_0\omega_j}{(\mathsf{k}_\omega)_j^2(1-\lambda Q(\lambda,\omega))} - \frac{1}{\omega_j(\mathsf{k}_\omega)_j}\right]\mathrm{vol}(S_n) - 1\\
=\;& \frac{2\omega_0\omega_j(\lambda\omega_0 + c_0\omega_0) - \lambda\omega_0^2\omega_j}{(\mathsf{k}_\omega)_j(1-\lambda Q(\lambda,\omega))} - \left[\frac{c_j\omega_0\omega_j}{(\mathsf{k}_\omega)_j^2(1-\lambda Q(\lambda,\omega))} - \frac{1}{\omega_j(\mathsf{k}_\omega)_j}\right]\mathrm{vol}(S_n) - 1\\
=\;& \frac{\lambda\omega_0\omega_j}{(1-\lambda Q(\lambda,\omega))(\mathsf{k}_\omega)_j^2}\left(\omega_0\omega_j(\lambda+2c_0) + c_j(\omega_0^2-1)\right) + \frac{(\lambda+2c_0\omega_0^2)}{\omega_j(\mathsf{k}_\omega)_j} - 1.
\end{aligned}
$$

Finally, for $j=0$ and $i\neq 0$,

$$
\begin{aligned}
\big(\mathsf{M}_{\lambda,\omega}\big)_{i0} =\;& -\frac{\omega_0\omega_i(\mathsf{k}_\omega)_0}{(\mathsf{k}_\omega)_i\big(1-\lambda Q(\lambda,\omega)\big)} + \sum_{\ell=1}^{n}\frac{\omega_0^2\omega_\ell c_\ell(c_i\omega_\ell - c_\ell\omega_i)}{(\mathsf{k}_\omega)_i(\mathsf{k}_\omega)_\ell(1-\lambda Q(\lambda,\omega))} + 1\\
-\;& \sum_{\ell=1}^{n}\frac{\omega_0\omega_\ell^2 c_\ell}{(\mathsf{k}_\omega)_\ell\big(1-\lambda Q(\lambda,\omega)\big)} + \frac{\omega_i}{(\mathsf{k}_\omega)_i(1-\lambda Q(\lambda,\omega))}\mathrm{vol}(S_n)\\
=\;& -\frac{\omega_i}{(\mathsf{k}_\omega)_i\big(1-\lambda Q(\lambda,\omega)\big)}\left(\omega_0(\mathsf{k}_\omega)_0 - \mathrm{vol}(S_n)\right) - \frac{\omega_0^2\omega_i}{(\mathsf{k}_\omega)_i(1-\lambda Q(\lambda,\omega))}\sum_{\ell=1}^{n}\frac{\omega_\ell c_\ell^2}{(\mathsf{k}_\omega)_\ell}\\
+\;& \left[\frac{\omega_0^2 c_i}{(\mathsf{k}_\omega)_i(1-\lambda Q(\lambda,\omega))} - \frac{\omega_0}{1-\lambda Q(\lambda,\omega)}\right]\sum_{\ell=1}^{n}\frac{\omega_\ell^2 c_\ell}{(\mathsf{k}_\omega)_\ell} + 1\\
=\;& \frac{\omega_i}{(\mathsf{k}_\omega)_i\big(1-\lambda Q(\lambda,\omega)\big)}\left(\lambda + 2c_0\omega_0^2 - \omega_0(\mathsf{k}_\omega)_0\right)\\
-\;& \frac{\omega_i}{(\mathsf{k}_\omega)_i(1-\lambda Q(\lambda,\omega))}\left[\omega_0(\mathsf{k}_\omega)_0 - \lambda\left(1-\lambda Q(\lambda,\omega)\right)\right]\\
+\;& \left[\frac{\omega_0 c_i}{(\mathsf{k}_\omega)_i(1-\lambda Q(\lambda,\omega))} - \frac{1}{1-\lambda Q(\lambda,\omega)}\right]\left[1-\lambda Q(\lambda,\omega)-\omega_0^2\right] + 1\\
=\;& \frac{\lambda\omega_i(1-\omega_0^2)}{(\mathsf{k}_\omega)_i(1-\lambda Q(\lambda,\omega))} + 1.
\end{aligned}
$$

∎

Proposition 5.2. *Kemeny's constant for the star network, S_n, is given by*

$$
K_{\lambda,\omega}(S_n) = n + \frac{(c_0+\lambda)\omega_0^2}{\lambda + 2c_0\omega_0^2} + \frac{\lambda Q(\lambda,\omega)}{1-\lambda Q(\lambda,\omega)} - \frac{\lambda(1-\omega_0^2)^2}{(\lambda+2c_0\omega_0^2)(1-\lambda Q(\lambda,\omega))}.
$$

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Proof. From Theorem 4.2, we must compute the quadratic form $\mathsf{k}_\omega^\top \mathsf{E}_{\lambda,\omega}\mathsf{k}_\omega$ for the star.

$$\begin{aligned}
\mathsf{k}_\omega^\top \mathsf{E}_{\lambda,\omega}\mathsf{k}_\omega &= -(\mathsf{k}_\omega)_0\omega_0\sum_{i=1}^n \frac{\omega_i^2}{1-\lambda Q(\lambda,\omega)} + \frac{(\mathsf{k}_\omega)_0\omega_0^2}{1-\lambda Q(\lambda,\omega)}\sum_{i=1}^n \frac{c_i\omega_i^2}{(\mathsf{k}_\omega)_i} - n(\mathsf{k}_\omega)_0\omega_0 \\
&+ \frac{\omega_0^2}{1-\lambda Q(\lambda,\omega)}\sum_{i,j=1}^n \frac{\omega_i\omega_j c_j(c_i\omega_j - c_j\omega_i)}{(\mathsf{k}_\omega)_j} - (n-1)(\mathrm{vol}(S_n) - (\mathsf{k}_\omega)_0\omega_0) \\
&= -\frac{(\mathsf{k}_\omega)_0\omega_0}{1-\lambda Q(\lambda,\omega)}(1-\omega_0^2) + \frac{(\mathsf{k}_\omega)_0\omega_0}{1-\lambda Q(\lambda,\omega)}(1-\omega_0^2-\lambda Q(\lambda,\omega)) \\
&+ \frac{\omega_0^2}{1-\lambda Q(\lambda,\omega)}\left(c_0\omega_0\sum_{j=1}^n \frac{c_j\omega_j^2}{(\mathsf{k}_\omega)_j} - (1-\omega_0^2)\sum_{j=1}^n \frac{c_j^2\omega_j}{(\mathsf{k}_\omega)_j}\right) \\
&- (n-1)\mathrm{vol}(S_n) - (\mathsf{k}_\omega)_0\omega_0 \\
&= -\frac{(\mathsf{k}_\omega)_0\omega_0\lambda Q(\lambda,\omega)}{1-\lambda Q(\lambda,\omega)} - (\mathsf{k}_\omega)_0\omega_0 - (n-1)\mathrm{vol}(S_n) \\
&+ \frac{\omega_0^2 c_0(1-\lambda Q(\lambda,\omega)-\omega_0^2) - (1-\omega_0^2)\big(\omega_0(\mathsf{k}_\omega)_0 - \lambda\,(1-\lambda Q(\lambda,\omega))\big)}{1-\lambda Q(\lambda,\omega)} \\
&= -\Big[(n-1) + \frac{\lambda Q(\lambda,\omega)}{1-\lambda Q(\lambda,\omega)}\Big]\mathrm{vol}(S_n) + \frac{\lambda(1-\omega_0^2)^2}{1-\lambda Q(\lambda,\omega)} - (c_0+\lambda)\omega_0^2,
\end{aligned}$$

and the result follows, keeping in mind that $\mathrm{vol}(S_n) = \lambda + 2c_0\omega_0^2$. ◻

The next result encompasses the standard case (ω constant and $\lambda = 0$) for a star network S_n and coincides with that obtained in [15, 18].

Corollary 5.3. *The mean first passage time for a star network S_n with respect to ω is given for $i,j = 1,\dots,n$ by*

$$\big(\mathsf{M}_\omega\big)_{ij} = \frac{2c_0\omega_0^2}{\omega_j(\mathsf{k}_\omega)_j},\quad \big(\mathsf{M}_\omega\big)_{0j} = \frac{2c_0\omega_0^2}{\omega_j(\mathsf{k}_\omega)_j} - 1,\quad \big(\mathsf{M}_\omega\big)_{i0} = 1,\quad \big(\mathsf{M}_\omega\big)_{00} = 2.$$

Moreover, Kemeny's constant is $K_\omega(S_n) = n + \dfrac{1}{2}$.

6. MFPT and Kemeny's constant with respect to ω for a path. In this section, we consider an n-path, P_n, with vertex set $\{1,\dots,n\}$, conductances $c_j = c_{jj+1}$, $j = 1,\dots,n-1$ and arbitrary weights $\omega_1,\dots,\omega_n$; see Fig. 3. With this notation, the equilibrium measure for a n-path was computed in [6, Proposition 5.1] for $\lambda = 0$, and is given by

$$\nu_i^j = \omega_i\left(\sum_{k=i}^{j-1}\frac{W_k}{C_k} + \sum_{k=j}^{i-1}\frac{1-W_k}{C_k}\right),$$

where

$$W_k = \sum_{\ell=1}^k \omega_\ell^2,\ \text{and}\ C_k = c_k\omega_k\omega_{k+1},\ \ k = 1,\dots,n-1.$$

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On the other hand,

$$(\mathsf{k}_\omega)_i = c_{i-1}\omega_{i-1} + c_i\omega_{i+1} \text{ for } i = 2, \dots, n-1,\ (\mathsf{k}_\omega)_1 = c_1\omega_2 \quad \text{and} \quad (\mathsf{k}_\omega)_n = c_{n-1}\omega_{n-1}.$$

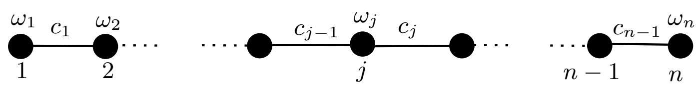

FIGURE 3. *Representation of a path P_n. The node j is representative of each and every node in the network.*

We can use these expressions and the results of the preceding section to compute the mean first passage time on a path.

PROPOSITION 6.1. *Let P_n be a n-path, then the mean first passage time matrix is given by*

$$(\mathsf{M}_\omega)_{ij} = \begin{cases} j - i + 2\displaystyle\sum_{k=i}^{j-1}\sum_{\ell=1}^{k-1}\frac{C_\ell}{C_k} & \text{if } i < j, \\ j - i + 2\displaystyle\sum_{k=j}^{i-1}\sum_{\ell=k}^{n-1}\frac{C_\ell}{C_k} & \text{if } i > j. \end{cases}$$

Proof. From Proposition 4.3, the mean first passage time matrix M_ω can be written in terms of equilibrium measures of P_n as

$$\begin{aligned}
(\mathsf{M}_\omega)_{ij} = & \sum_{\ell=1}^{n}\omega_\ell(\mathsf{k}_\omega)_\ell\left(\frac{\nu_i^j}{\omega_i} + \frac{\nu_j^\ell}{\omega_j} - \frac{\nu_i^\ell}{\omega_i}\right) \\
= & \frac{\nu_i^j}{\omega_i}\sum_{\ell=1}^{n}\omega_\ell(\mathsf{k}_\omega)_\ell + \sum_{\ell=2}^{n-1}\omega_\ell(\mathsf{k}_\omega)_\ell\frac{\nu_j^\ell}{\omega_j} + \omega_1(\mathsf{k}_\omega)_1\frac{\nu_j^1}{\omega_j} + \omega_n(\mathsf{k}_\omega)_n\frac{\nu_j^n}{\omega_j} \\
- & \sum_{\ell=2}^{n-1}\omega_\ell(\mathsf{k}_\omega)_\ell\frac{\nu_i^\ell}{\omega_i} - \omega_1(\mathsf{k}_\omega)_1\frac{\nu_i^1}{\omega_i} - \omega_n(\mathsf{k}_\omega)_n\frac{\nu_i^n}{\omega_i},
\end{aligned}$$

for any $i \neq j$.

Let us compute S_j, the partial sum of the above expression:

$$\begin{aligned}
S_j = & \sum_{\ell=2}^{n-1}\omega_\ell(\mathsf{k}_\omega)_\ell\frac{\nu_j^\ell}{\omega_j} + \omega_1(\mathsf{k}_\omega)_1\frac{\nu_j^1}{\omega_j} + \omega_n(\mathsf{k}_\omega)_n\frac{\nu_j^n}{\omega_j} \\
= & \sum_{\ell=2}^{j-1}\omega_\ell(\mathsf{k}_\omega)_\ell\frac{\nu_j^\ell}{\omega_j} + \sum_{\ell=j}^{n-1}\omega_\ell(\mathsf{k}_\omega)_\ell\frac{\nu_j^\ell}{\omega_j} + \omega_1(\mathsf{k}_\omega)_1\frac{\nu_j^1}{\omega_j} + \omega_n(\mathsf{k}_\omega)_n\frac{\nu_j^n}{\omega_j} \\
= & \sum_{\ell=2}^{j-1}(C_{\ell-1} + C_\ell)\sum_{k=\ell}^{j-1}\frac{(1-W_k)}{C_k} + \sum_{\ell=j}^{n-1}(C_{\ell-1} + C_\ell)\sum_{k=j}^{\ell-1}\frac{W_k}{C_k} + C_1\sum_{k=1}^{j-1}\frac{1-W_k}{C_k} + C_{n-1}\sum_{k=j}^{n-1}\frac{W_k}{C_k},
\end{aligned}$$

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that is,

$$\begin{aligned}
S_j = & \sum_{k=2}^{j-1}\sum_{\ell=2}^{k}(C_{\ell-1}+C_\ell)\frac{(1-W_k)}{C_k}+\sum_{k=j}^{n-1}\sum_{\ell=k+1}^{n-1}(C_{\ell-1}+C_\ell)\frac{W_k}{C_k}+C_1\sum_{k=1}^{j-1}\frac{(1-W_k)}{C_k}+C_{n-1}\sum_{k=j}^{n-1}\frac{W_k}{C_k}\\
= & \sum_{k=2}^{j-1}\frac{(1-W_k)}{C_k}\Big(C_1+C_k+2\sum_{\ell=2}^{k-1}C_\ell\Big)+\sum_{k=j}^{n-1}\frac{W_k}{C_k}\Big(C_{n-1}+C_k+2\sum_{\ell=k+1}^{n-2}C_\ell\Big)\\
+ & C_1\sum_{k=1}^{j-1}\frac{1-W_k}{C_k}+C_{n-1}\sum_{k=j}^{n-1}\frac{W_k}{C_k}.
\end{aligned}$$

Therefore, for $i<j$, we have that

$$\begin{aligned}
\big(\mathsf{M}_\omega\big)_{ij} = & \sum_{k=i}^{j-1}\left[\frac{(1-W_k)}{C_k}\Big(C_1+C_k+2\sum_{\ell=2}^{k-1}C_\ell\Big)-\frac{W_k}{C_k}\Big(C_{n-1}+C_k+2\sum_{\ell=k+1}^{n-2}C_\ell\Big)\right]\\
+ & \sum_{k=i}^{j-1}\left(C_1\frac{(1-W_k)}{C_k}-C_{n-1}\frac{W_k}{C_k}\right)+2\sum_{k=i}^{j-1}\frac{W_k}{C_k}\sum_{\ell=1}^{n-1}C_\ell\\
= & \sum_{k=i}^{j-1}\left(\frac{1}{C_k}\Big(C_1+C_k+2\sum_{\ell=2}^{k-1}C_\ell\Big)\right)+\sum_{k=i}^{j-1}\frac{C_1}{C_k}\\
= & (j-i)+2\sum_{k=i}^{j-1}\sum_{\ell=1}^{k-1}\frac{C_\ell}{C_k},
\end{aligned}$$

whereas for $i>j$, we have that

$$\begin{aligned}
\big(\mathsf{M}_\omega\big)_{ij} = & -\sum_{k=j}^{i-1}\left[\frac{(1-W_k)}{C_k}\Big(C_1+C_k+2\sum_{\ell=2}^{k-1}C_\ell\Big)-\frac{W_k}{C_k}\Big(C_{n-1}+C_k+2\sum_{\ell=k+1}^{n-2}C_\ell\Big)\right]\\
- & \sum_{k=j}^{i-1}\left(C_1\frac{(1-W_k)}{C_k}-C_{n-1}\frac{W_k}{C_k}\right)+2\sum_{k=j}^{i-1}\frac{(1-W_k)}{C_k}\sum_{\ell=1}^{n-1}C_\ell\\
= & -\sum_{k=j}^{i-1}\left(\frac{1}{C_k}\Big(C_1+C_k+2\sum_{\ell=2}^{k-1}C_\ell\Big)\right)-\sum_{k=j}^{i-1}C_1\frac{1}{C_k}+2\sum_{k=j}^{i-1}\frac{1}{C_k}\sum_{\ell=1}^{n-1}C_\ell\\
= & -\sum_{k=j}^{i-1}\frac{1}{C_k}\Big(C_1+C_k+2\sum_{\ell=2}^{k-1}C_\ell+C_1-2\sum_{\ell=1}^{n-1}C_\ell\Big)\\
= & (j-i)+2\sum_{k=j}^{i-1}\sum_{\ell=k}^{n-1}\frac{C_\ell}{C_k}.
\end{aligned}$$

∎

THEOREM 6.2. *Kemeny's constant with respect to ω for a n-path is given by*

$$K_\omega(P_n)=1+\frac{1}{2\sum\limits_{j=1}^{n-1}C_j}\sum_{i=2}^{n}\sum_{j=1}^{i-1}(C_{i-1}+C_i)(C_{j-1}+C_j)\Bigg(\sum_{k=j}^{i-1}\frac{1}{C_k}\Bigg).$$

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Proof. From Proposition 4.3, $K_\omega(\Gamma) = 1 + \dfrac{1}{\mathsf{vol}(\Gamma)} \displaystyle\sum_{i,j=1}^{n} \omega_j (\mathsf{k}_\omega)_i (\mathsf{k}_\omega)_j \big(\nu^j_{0,\omega}\big)_i$. Then, we must compute the value of $\alpha = \displaystyle\sum_{i,j=1}^{n} \omega_j (\mathsf{k}_\omega)_i (\mathsf{k}_\omega)_j \nu^j_i$.

$$
\begin{aligned}
\alpha = & \sum_{i,j=1}^{n} \omega_i\omega_j \left(\sum_{k=i}^{j-1} \frac{W_k}{C_k} + \sum_{k=j}^{i-1} \frac{(1-W_k)}{C_k} \right) (c_{i-1}\omega_{i-1} + c_i\omega_{i+1})(c_{j-1}\omega_{j-1} + c_j\omega_{j+1}) \\
= & \sum_{i=1}^{n} \sum_{j=1}^{i-1} \left(\sum_{k=j}^{i-1} \frac{(1-W_k)}{C_k} \right) (C_{i-1} + C_i)(C_{j-1} + C_j) \\
+ & \sum_{i=1}^{n} \sum_{j=i+1}^{n} \left(\sum_{k=i}^{j-1} \frac{W_k}{C_k} \right) (C_{i-1} + C_i)(C_{j-1} + C_j) \\
= & \sum_{i=1}^{n} \sum_{j=1}^{i-1} \left(\sum_{k=j}^{i-1} \frac{1-W_k}{C_k} \right) (C_{i-1} + C_i)(C_{j-1} + C_j) \\
+ & \sum_{i=1}^{n} \sum_{j=1}^{i-1} \left(\sum_{k=j}^{i-1} \frac{W_k}{C_k} \right) (C_{i-1} + C_i)(C_{j-1} + C_j) \\
= & \sum_{i=2}^{n} \sum_{j=1}^{i-1} (C_{i-1} + C_i)(C_{j-1} + C_j) \left(\sum_{k=j}^{i-1} \frac{1}{C_k} \right).
\end{aligned}
$$

Finally, the result follows by considering that $\mathrm{vol}(P_n) = 2 \displaystyle\sum_{j=1}^{n-1} C_j$. ◻

As a particular case of the above result, we recover the mean first passage time and the value of Kemeny's constant for an n-path with constant weight and constant conductances. In this case, $C_k = c_k\omega_k\omega_{k+1} = \dfrac{c}{n}$, $k = 1, \ldots, n-1$. However, we can have C_k constant without c_k or ω_k being constant.

COROLLARY 6.3. *Let P_n be a n-path, with $C_k = C$. Then, the mean first passage time matrix is given by*

$$
\mathsf{M}_{ij} = \begin{cases} (j-i)(i+j-2) & \text{if } i < j \\[2ex] (i-j)(2n-i-j) & \text{if } i > j. \end{cases}
$$

Moreover, Kemeny's constant is

$$
K(P_n) = \frac{1}{3}n^2 - \frac{2}{3}n + \frac{3}{2}.
$$

The above result can also be deduced from [1, Theorem 7]. In the mentioned paper, the author obtained the mean first passage time between vertices of a tree and hence by using the value of the distances in a path, we recover our result. There are other references that present the above result but in a different way; see, for instance, [18, 20].

It is observed that the mean first passage time on a path varies based on the vertex positions and direction of travel. The formula depends on whether starting vertex i is before or after the target vertex j in the path. Moreover, the MFPT incorporates both the distance between vertices $d(i,j)$ and their positions

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relative to the ends of the path. We can rewrite the formula as

$$\mathsf{M}_{ij} = \begin{cases} d(i,j)\big(d(i,1)+d(j,1)\big) & \text{if } \ i<j \\ \\ d(i,j)\big(d(i,n)+d(j,n)\big) & \text{if } \ i>j. \end{cases}$$

We can interpret this as follows: for $i<j$ (moving forward in the path), the MFPT increases with both the distance and the position of the vertices from the start. However, for $i>j$ (moving backward in the path), the MFPT increases with distance and is affected by how close the vertices are to the end of the path.

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